ON THE C_n-DISTANCE AND VASSILIEV INVARIANTS

SUMIKO HORIUCHI

Department of Mathematics, School of Arts and Sciences
Tokyo Woman's Christian University
2-6-1, Zempukuji, Suginami-ku, Tokyo, 167-8585, Japan
horiuchi@cis.twcu.ac.jp

YOSHIYUKI OHYAMA

Department of Mathematics, School of Arts and Sciences
Tokyo Woman's Christian University
2-6-1, Zempukuji, Suginami-ku, Tokyo, 167-8585, Japan
ohyama@lab.twcu.ac.jp

ABSTRACT

A local move called a C_n-move is closely related to Vassiliev invariants. A C_n-distance between two knots K and L, denoted by $d_{C_n}(K, L)$, is the minimum number of C_n-moves needed to transform K into L. Let p and q be natural numbers with $p > q \geq 1$. In this paper, we show that for any pair of knots K_1 and K_2 with $d_{C_n}(K_1, K_2) = p$ and for any given natural number m, there exist infinitely many knots J_i $(i = 1, 2, \ldots)$ such that $d_{C_n}(K_1, J_i) = q$ and $d_{C_n}(J_i, K_2) = p - q$, and they have the same Vassiliev invariants of order less than or equal to m. In the case that $n = 1$ or 2, the knots J_i $(i = 1, 2, \ldots)$ satisfy more conditions.

Keywords: Vassiliev invariant; C_n-move; C_n-distance.

Mathematics Subject Classification 2000: 57M25

1. Introduction

When we have a knot invariant v which takes values in some abelian group, we can define an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(K_-),$$

where *a singular knot* is an immersion of a circle into R^3 whose singularities are transversal double points only and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 1.

An invariant v is called *a Vassiliev invariant of order n* and denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points([5]).

A standard C_n-move is a local move depicted in Fig. 2. *A C_1-move* is defined as a usual crossing change and a C_2-move is the same move as a Delta move([10], [11]).
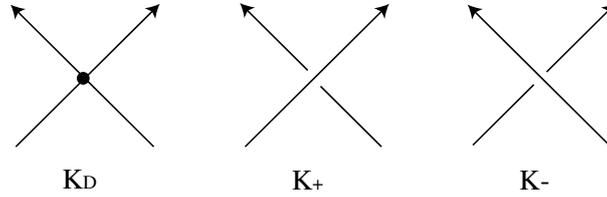
Fig. 1.

Two knots are called C_n-equivalent if they can be transformed into each other by a finite sequence of standard C_n-moves. M. N. Goussarov([6]) and K. Habiro([8]) showed the following theorem independently.

Theorem 1.1[6,8]. *Two knots are C_n-equivalent if and only if they have the same Vassiliev invariants of order less than n.*
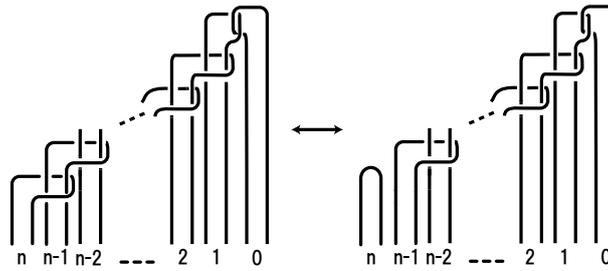
Fig. 2

C_n-moves are originally defined by Habiro in [7]. In [15] and [19], they are defined as a family of local moves. It is known that any kind of C_n-move can be realized by a finite sequence of standard C_n-moves.

If a knot K can be transformed into L by standard C_n-moves, we denote the minimum number of C_n-moves needed to transform K into L by $d_{C_n}(K,L)$ and call it *the C_n-distance between K and L.* In this paper, we use the notation d_G and d_Δ instead of d_{C_1} and d_{C_2}, respectively.

Let Γ_i $(i \in N)$ be a C_n-equivalence class of knots in R^3, then (Γ_i, d_{C_n}) is a metric space. We note that in the case that $n = 1$ and 2 we have only one C_n-equivalence

class. Let ℓ be a natural number and K a knot in Γ_i. Let $B_\ell^{C_n}(K) = \{K' \in \Gamma_i | d_{C_n}(K, K') \le \ell\}$. It is a ball whose center is K. We consider the intersection of two balls.

In [1], Baader showed the following.

Theorem 1.2 [1]. *For any pair of oriented knots K_1 and K_2 with $d_G(K_1, K_2) = 2$, there are infinitely many knots J_j $(j = 1, 2, \ldots)$ such that $d_G(K_1, J_j) = d_G(J_j, K_2) = 1$.*

From here p and q denote natural numbers with $p > q \ge 1$. Theorem 1.2 can be extended to the case that $d_G(K_1, K_2) = p$, $d_G(K_1, J_j) = q$ and $d_G(J_j, K_2) = p - q$.

In the case that $n = 2$, the first author shows Theorem 1.3.

Theorem 1.3 [9]. *For any pair of oriented knots K_1 and K_2 with $d_\Delta(K_1, K_2) = p$, there are infinitely many knots J_j $(j = 1, 2, \ldots)$ such that $d_\Delta(K_1, J_j) = q$ and $d_\Delta(J_j, K_2) = p - q$.*

In the proofs of Theorem 1.2 and 1.3, it is shown that the Conway polynomials of J_j and J_k are different if $j \ne k$. Taniyama extended Theorem 1.2 and the case $p = 2$ in Theorem 1.3 to the following.

Theorem 1.4[18]. *Let m and n be non-negative integers. Suppose oriented knots $K_0, K_1, \ldots, K_m, K_{m+1}, \ldots K_{m+n}$ satisfy $d_G(K_0, K_i) = 1$ $(i = 1, 2, \ldots, m)$ and $d_\Delta(K_0, K_i) = 1$ $(i = m + 1, m + 2, \ldots, m + n)$. Then there are infinitely many knots J_j $(j = 1, 2, \ldots)$ such that $d_G(J_j, K_i) = 1$ $(i = 0, 1, \ldots, m, j = 1, 2, \ldots)$ and $d_\Delta(J_j, K_i) = 1$ $(i = m + 1, m + 2, \ldots, m + n, j = 1, 2, \ldots)$.*

Recently, Baader shows Theorem 1.5.

Theorem 1.5 [2]. *Let m be a natural number and K a knot. For any pair of oriented knots K_1 and K_2 with $d_G(K_1, K_2) = 2$, there exists a knot K' which satisfies the following*:
(1) $d_G(K_1, K') = d_G(K', K_2) = 1$ and
(2) for any v_i $(i = 1, 2, \ldots, m)$, $v_i(K') = v_i(K)$.

We have Theorem 1.6 as a generalization of Theorem 1.5.

Theorem 1.6. *Let m be a natural number and K a knot. For any pair of oriented knots K_1 and K_2 with $d_G(K_1, K_2) = p$, there are infinitely many knots J_j $(j = 1, 2, \ldots)$ which satisfy the following*:
(1) $d_G(K_1, J_j) = q$ and $d_G(J_j, K_2) = p - q$ $(j = 1, 2, \ldots)$,

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(2) *for any v_i $(i = 1, 2, \ldots, m)$, $v_i(J_j) = v_i(K)$ $(j = 1, 2, \ldots)$ and*
(3) $\nabla_{J_j}(z) = \nabla_{J_k}(z)$ $(j \neq k, \ j, k = 1, 2, \ldots)$, where $\nabla_J(z)$ is the Conway polynomial
of J.

Theorem 1.7 is a generalization of Theorem 1.3.

Theorem 1.7. *Let m be a natural number. For any pair of oriented knots K_1 and K_2 with $d_\Delta(K_1, K_2) = p$, there are infinitely many knots J_j $(j = 1, 2, \ldots)$ which satisfy the following:*
(1) $d_\Delta(K_1, J_j) = q$ and $d_\Delta(J_j, K_2) = p - q$ $(j = 1, 2, \ldots)$,
(2) *for any v_i $(i = 1, 2, \ldots, m)$, $v_i(J_j) = v_i(J_k)$ $(j \neq k, \ j, k = 1, 2, \ldots)$ and*
(3) $\nabla_{J_j}(z) = \nabla_{J_k}(z)$ $(j \neq k, \ j, k = 1, 2, \ldots)$.

In the case that $n \geq 3$, we obtain Theorem 1.8.

Theorem 1.8. *Let m be a natural number. For any pair of oriented knots K_1 and K_2 with $d_{C_n}(K_1, K_2) = p$, there are infinitely many knots J_j $(j = 1, 2, \ldots)$ which satisfy the following:*
(1) $d_{C_n}(K_1, J_j) = q$ and $d_{C_n}(J_j, K_2) = p - q$ $(j = 1, 2, \ldots)$ and
(2) *for any v_i $(i = 1, 2, \ldots, m)$, $v_i(J_j) = v_i(J_k)$ $(j \neq k, \ j, k = 1, 2, \ldots)$.*

In the proof of Theorem 1.8, we will show that the Conway polynomials of J_j and J_k are different if $j \neq k$.

2. C_n-moves and Jacobi diagrams

A tangle T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. A tangle T is *trivial* if there exists a properly embedded disk in B^3 that contains T. A *local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$. Two local moves (T_1, T_2) and (U_1, U_2) are *equivalent*, if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(T_i)$ and U_i are ambient isotopic in B^3 relative to ∂B^3 for $i = 1, 2$.
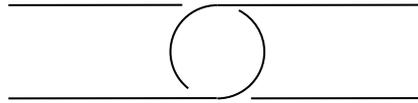
Fig. 3

Let (T_1, T_2) be a local move, t_1 a component of T_1 and t_2 a component of T_2
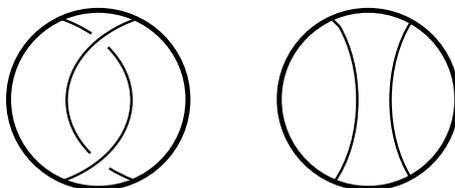
Fig. 4

such that $\partial t_1 = \partial t_2$. Replacing t_1 and t_2 by hooked arcs in Fig. 3, we obtain a new kind of local move. This local move is called a *double of (T_1, T_2) with respect to the components t_1 and t_2*. A *C_1-move* is a local move as illustrated in Fig. 4. A *C_{k+1}-move* is a double of a C_k-move. Then, there exist some kinds of C_n-move and any kind of C_n-move is realized by a finite sequence of standard C_n-moves.

A C_n-move is represented by the band sum of the link called the C_n-link model. The move in Fig. 5 is equivalent to the standard C_n-move. The link in Fig. 6 is the C_n-link model for the standard C_n-move. For details, refer to [15] or [19].
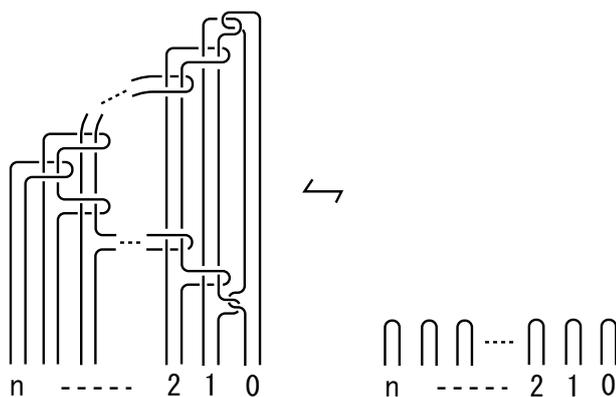
Fig. 5

By K^m, we denote a singular knot with m double points. From the definition of the Vassiliev invariant, $v_m(K^m)$ does not change by a crossing change and it is determined by the positions of the double points on K^m. To show the positions of double points, the notion of a chord diagram is introduced in [5]. A *chord diagram of order n* is an oriented circle with n chords. By connecting the preimages of each double point by a chord, we may associate the chord diagram to a singular knot. The value of v_n for a chord diagram of order n is defined as the value of it for a singular knot with n double points that is associated with the chord diagram. In the
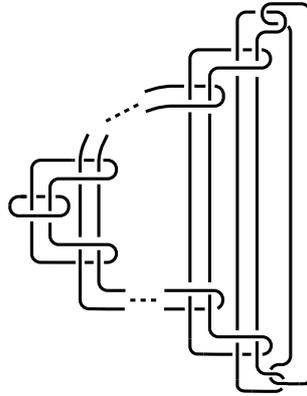
Fig. 6

additive group generated by the chord diagrams of order n, the relation in Fig. 7 is called *the 4T relation.*
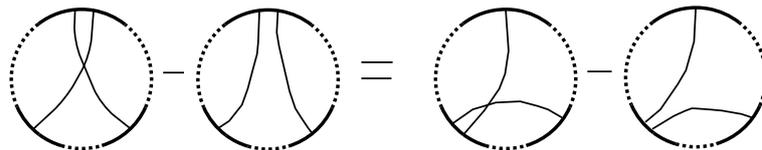
Fig. 7

Chord diagrams are generalized to Jacobi diagrams in [3]. *A Jacobi diagram of order n* is a trivalent graph with $2n$ vertices. It is a union of a circle and an internal graph G. The circle is oriented and the other edges are all unoriented. Each trivalent vertex on G has an orientation, that is a cyclic ordering of the edges incident to it. In the additive group generated by the Jacobi diagrams of order n, the relation in Fig. 8 is called *the STU relation. The IHX relation* in Fig. 9 and *the antisymmetry relation* in Fig. 10 can be obtained as a consequence of STU relations.

Let A_n be the additive group generated by the chord diagrams of order n modulo the 4T relation and B_n the additive group generated by the Jacobi diagrams of order n modulo the STU relation. Then the isomorphism between A_n and B_n is induced by the inclusion of chord diagrams into Jacobi diagrams [3].

A one-branch tree diagram T is a special kind of Jacobi diagram whose internal graph G is isomorphic to a standard n-tree in Fig. 11 preserving the vertex orientations([14]). Label the branches of the standard n-tree as in Fig. 11. We may label the branches of G under the isomorophism between G and the standard n-tree.
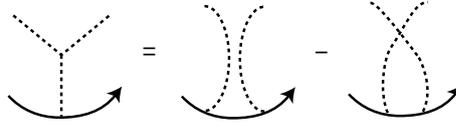
Fig. 8
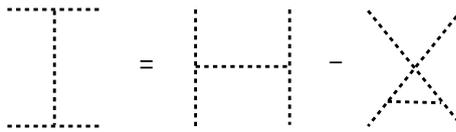
Fig. 9

Fig. 10

And number the vertices of the circle of T by $0, 1, 2, \ldots, n$ in the counterclockwise direction such that the vertex on the circle which corresponds to the branch 0 of G is numbered by 0. The correspondence between the labels of branches of G and the numbers of the corresponding vertices on the circle determines a permutation $\sigma \in S_n$. Conversely, if a permutation $\sigma \in S_n$ is given, a unique one-branch tree diagram T can be constructed. Then we denote a one-branch tree diagram by T_σ. By STU relations, a one-branch tree diagram can be expressed as a linear combination of chord diagrams. The value of v_n for a one-branch tree diagram of order n is defined as the linear combination of the values for the chord diagrams.

A one-branch tree diagram is closely related to a standard C_n-move.

Theorem 2.1[16]. *If a knot K' is obtained from a knot K by a single standard C_n-move, then*

$$v_n(K') - v_n(K) = \pm v_n(T_\sigma),$$

where T_σ is a one-branch tree diagram of order n.

In Theorem 2.1, the one-branch tree diagram T_σ is determined by the positions

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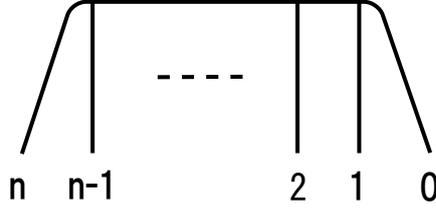
Fig. 11

of arcs on a knot K in the performed C_n-move and the sign of the formula depends only on the orientations of arcs in the C_n-move.

For a singular knot, Theorem 2.1' holds.

Theorem 2.1'[13]. *If a singular knot L^k with k double points is obtained from a singular knot K^k by a single standard C_n-move, then*

$$v_{k+n}(L^k) - v_{k+n}(K^k) = \pm v_{k+n}(T_\sigma),$$

where T_σ is a Jacobi diagram of order $k + n$ whose internal graph is isomorphic to the union of k chords and a one-branch tree diagram of order n.

Here, we consider a new kind of C_{m+1}-move. By a $C_{m+1}^{(i)}$-move, we denote a special kind of C_{m+1}-move which is obtained from the standard C_m-move by changing the arc labelled i ($2 \leq i \leq m - 2$) to hooked arcs. Fig. 12 shows a $C_{m+1}^{(m-3)}$-move which is used for the proof of Theorem 2.5. A $C_{m+1}^{(m-2)}$-move is used for the proof of Theorem 2.6.
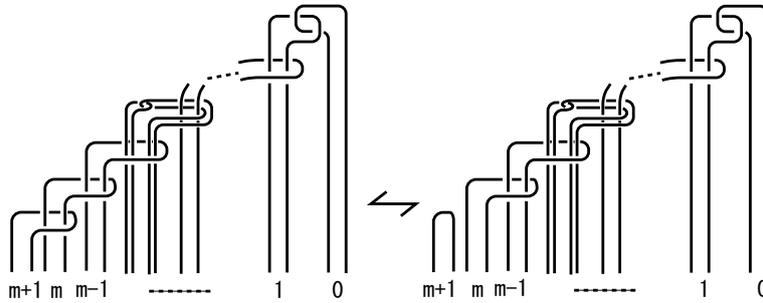
Fig. 12

By the results of [17], we have following two lemmas for a $C_{m+1}^{(i)}$-move.

Lemma 2.2[17]. *If a knot K' is obtained from a knot K by a single $C_{m+1}^{(i)}$-move, then*

$$v_{m+1}(K) - v_{m+1}(K') = \pm v_{m+1}(T_\sigma^i),$$

where T_σ^i is the one-branch tree diagram of order $m+1$ whose internal graph is isomorphic to the graph in Fig. 13 and $\sigma \in S_{m+1}$.
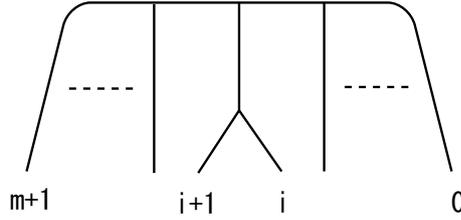
Fig. 13

In the case that the Jacobi diagram has the internal graph which is isomorphic to the graph in Fig. 13, the correspondence between the labels of branches and the numbers of the vertices on the circle determines a permutation $\sigma \in S_{m+1}$ as in a one-branch tree diagram.

Lemma 2.3[17]. *If a knot K' is obtained from a knot K by a single $C_{m+1}^{(i)}$-move, then*

$$\nabla_K(z) = \nabla_{K'}(z),$$

where $\nabla_K(z)$ is the Conway polynomial of K.

By the same way of the proof of Lemma 3.2 in [13], we obtain Lemma 2.4.

Lemma 2.4. *Let T_{id}^i be the Jacobi diagram of order $m+1$ in Lemma 2.2 whose permutation $\sigma \in S_{m+1}$ is the identity, then*

$$V^{(m+1)}(T_{id}^i) = 3(-2)^{m-1}(m+1)!,$$

where $V^{(m+1)}(K)$ is the $m+1$th derivative of the Jones polynomial of a knot K evaluated at 1.

In the standard C_n-move in Fig. 2, let $c_1, c_{21}, c_{22}, \ldots, c_{n1}, c_{n2}$ be the crossing points in Fig. 14.
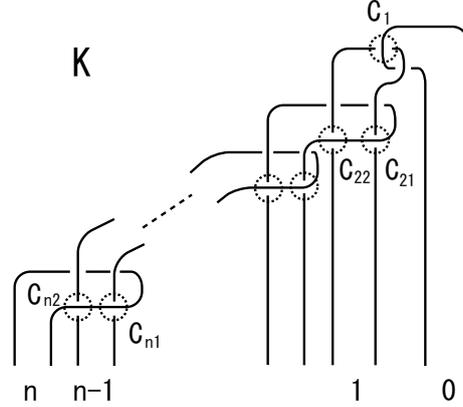
Fig. 14

By $K \begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}$ $(i_j = \pm 1, j = 2, 3, \ldots, n)$, we denote the singular knot which is obtained from K by the following:

Collapse C_1 to a double point. If $i_j = 1$, collapse c_{j1} to a double point. If $i_j = -1$, change a crossing at c_{j1} and collapse c_{j2} to a double point.

To show Theorem 2.1, Lemma 2.5 is proved in [16].

Lemma 2.5[16]. *Lf a knot L is obtained from a knot K by a single standard C_n-move, then*

$$v_n(K) - v_n(L) = \pm \sum_{\substack{i_j = \pm 1 \\ j = 2, 3, \ldots, n}} \prod i_j \, v_n(K \begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}).$$

We note that Lemma 2.5 holds for Vassiliev invariants of any order. By making $\displaystyle\sum_{\substack{i_j = \pm 1 \\ j = 2, 3, \ldots, n}} \prod i_j \, K \begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}$ to a one-branch tree diagram by STU relations, Theorem 2.1 is obtained. From Theorem 2.1' and Lemma 2.5, we have Theorem 2.6.

Theorem 2.6. *If a knot L is obtained from a knot K by a local move in Fig. 15, then*

$$v_{m+n+1}(K) - v_{m+n+1}(L) = \pm v_{m+n+1}(T_\sigma),$$

where T_σ is a one-branch tree diagram of order $m + n + 1$.
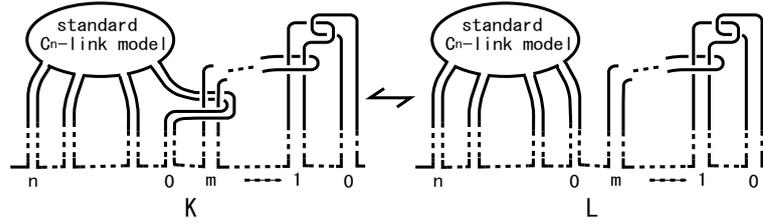
Fig. 15

Proof. By K' and L', we denote the knots that is obtained from K and L in Fig. 15 by performing C_n-moves and deleting the C_n-link models, respectively. By Lemma 2.5,

$$v_{m+n+1}(K) - v_{m+n+1}(K') = \pm \sum_{\substack{i_j = \pm 1 \\ j = 2,3,\ldots,n}} \prod i_j \; v_{m+n+1}\!\left(K\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}\right). \tag{2.1}$$

$$v_{m+n+1}(L) - v_{m+n+1}(L') = \pm \sum_{\substack{i_j = \pm 1 \\ j = 2,3,\ldots,n}} \prod i_j \; v_{m+n+1}\!\left(L\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}\right). \tag{2.2}$$

Here, $K\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}$ and $L\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}$ are singular knots that is obtained from K and L by making crossing points to double points in the C_n-link models, respectively. Since K' and L' are same knots, by (2.1) and (2.2)

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$$
v_{m+n+1}(K) - v_{m+n+1}(L)
$$
$$
= \pm \sum_{\substack{i_j = \pm 1 \\ j = 2, 3, \ldots, n}} \prod i_j \; \{ v_{m+n+1}(K \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix})
$$
$$
- v_{m+n+1}(L \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}) \}. \tag{2.3}
$$

If we perform C_{m+1}-moves on $K \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}$ two times, we have $L \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}$. Let $M \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}$ be the singular knot that is obtained from $K \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}$ by a single C_{m+1}-move as shown in Fig. 16.
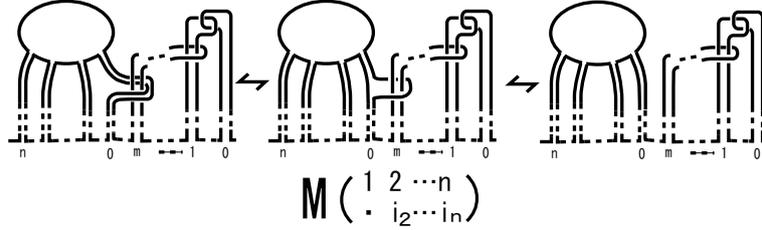
Fig. 16

By Theorem 2.1',

$$
v_{m+n+1}(K \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}) - v_{m+n+1}(M \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix})
$$
$$
= \pm v_{m+n+1}(T'_\sigma(i_2, \ldots, i_n). \tag{2.4}
$$

$$
v_{m+n+1}(M \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix}) - v_{m+n+1}(L \begin{pmatrix} 1 \; 2 \; \ldots \; n \\ \cdot \; i_2 \; \ldots \; i_n \end{pmatrix})
$$
$$
= \pm v_{m+n+1}(T''_\sigma(i_2, \ldots, i_n). \tag{2.5}
$$

Here, $T'_\sigma(i_2 \ldots, i_n)$ and $T''_\sigma(i_2, \ldots, i_n)$ are Jacobi diagrams of order $m + n + 1$ whose internal graphs are the union of n chords and a one-branch tree diagram of order $m + 1$. Since the orientations of $m + 1$th arcs are different in the first and the second performed $C_m + 1$-moves, the signs of (2.4) and (2.5) are opposite. By

Fig. 17

considering the positions of the double points, the internal graph in $T'_\sigma(i_2 \ldots, i_n)$ is (a) or (b) in Fig. 17 and that in $T''_\sigma(i_2, \ldots, i_n)$ is the other.

By (2.4) and (2.5),

$$
v_{m+n+1}(K\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix})) - v_{m+n+1}(L\begin{pmatrix} 1 & 2 & \ldots & n \\ \cdot & i_2 & \ldots & i_n \end{pmatrix}
$$
$$
= \pm\{v_{m+n+1}(T'_\sigma(i_2,\ldots,i_n)) - v_{m+n+1}(T''_\sigma(i_2,\ldots,i_n))\}. \tag{2.6}
$$

From (2.3) and (2.6),

$$
v_{m+n+1}(K) - v_{m+n+1}(L)
$$
$$
= \pm \sum_{\substack{i_j = \pm 1 \\ j = 2,3,\ldots,n}} \prod i_j \; \{v_{m+n+1}(T'_\sigma(i_2,\ldots,i_n))
$$
$$
- v_{m+n+1}(T''_\sigma(i_2,\ldots,i_n))\}. \tag{2.7}
$$

By the same way of the proof in Theorem 2.1 in [16], we have

$$
v_{m+n+1}(K) - v_{m+n+1}(L) = \pm\{v_{m+n+1}(T'_\sigma) - v_{m+n+1}(T''_\sigma)\}. \tag{2.8}
$$

In (2.8), T'_σ is the Jacobi diagram of order $m + n + 1$ whose internal graph is one of (a) and (b) in Fig. 18 and T''_σ is one whose internal graph is the other.

(a)   (b)
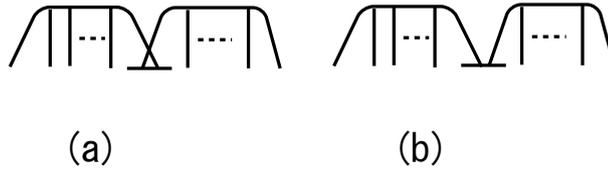
Fig. 18

By a STU relation, we have

$$
v_{m+n+1}(K) - v_{m+n+1}(L) = \pm v_{m+n+1}(T_\sigma).
$$

This completes the proof of Theorem 2.6.   □

Remark 2.7. The local move in Theorem 2.6 is equivalent to a band crossing change between a standard C_n-link model and a standard C_m-link model in Fig. 19.
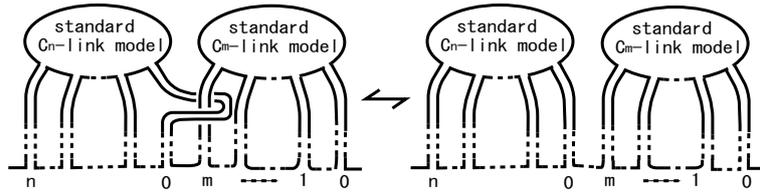
Fig. 19

3. Proof of Theorem 1.6

For a pair of knots K_1 and K_2 with $d_G(K_1, K_2) = p$, we consider the following sequence between K_1 and K_2, where adjacent knots are transformed into each other by a crossing change. In the sequence, $d_G(K_1, K') = q$. We direct our attention to the part from K'_1 to K'_2 where $d_G(K'_1, K'_2) = 2$ and $d_G(K'_1, K') = d_G(K', K'_2) = 1$.

$$\underbrace{K_1 \leftrightarrow \cdots \leftrightarrow K'_1 \leftrightarrow K'}_{q \text{ times crossing changes}} \leftrightarrow K'_2 \leftrightarrow \cdots \leftrightarrow K_2$$

By Theorem 1.5, we may assume that a knot K' satisfies the following:
(1) K' has a diagram as in Fig. 20 such that if we change a crossing at A on the diagram of K', we have a diagram of K'_1 and we change a crossing at B, we have a diagram of K'_2, and
(2) for any v_i $(i = 1, 2, \ldots m)$, $v_i(K') = v_i(K)$.
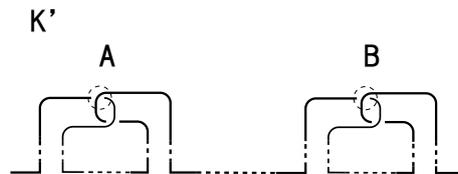
Fig. 20

The move in Fig. 21 is equivalent to a $C_{m+1}^{(m-3)}$-move in Fig. 12.
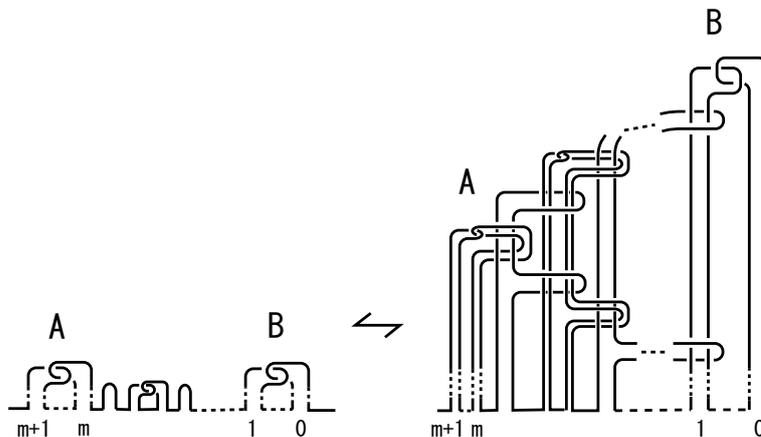
Fig. 21

We consider the move in Fig. 21 such that the permutation of the Jacobi diagram corresponding to the move is identity. By J_j, we denote the knot that is obtained from K' by performing the above move j times as is shown in Fig. 22.
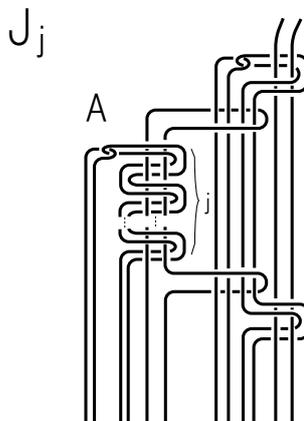
Fig. 22

If we change a crossing at A, J_j becomes a diagram of K'_1 and if we change a

crossing at B, J_j becomes a diagram of K_2'. Then, $d_G(K_1, J_j) = q$ and $d_G(J_j, K_2) = p - q$. A $C_{m+1}^{(m-3)}$-move is a kind of C_{m+1}-move. By Theorem 1.1, we have

$$v_i(J_j) = v_i(K') = v_i(K) \ (i = 1, 2, \ldots, m).$$

By Lemma 2.3,

$$\nabla_{J_j}(z) = \nabla_{J_k}(z) \ (j \neq k, \ j, k = 1, 2, \ldots).$$

In Lemma 2.2, the sign of the formula is determined only by the orientations of arcs in the performed $C_{m+1}^{(m-3)}$-move as in Theorem 2.1. Since we repeat the same $C_{m+1}^{(m-3)}$-move on the knot K', we have

$$V^{(m+1)}(J_j) - V^{(m+1)}(J_{j+1}) = V^{(m+1)}(J_{j+1}) - V^{(m+1)}(J_{j+2}).$$

By Lemma 2.4,

$$V^{(m+1)}(J_i) \neq V^{(m+1)}(J_k) \ (i \neq k, \ i, k = 1, 2, \ldots).$$

This completes the proof of Theorem 1.6.

4. Proof of Theorem 1.7

As in the proof of Theorem 1.6, for a pair of knots K_1 and K_2 with $d_\Delta(K_1, K_2) = p$, we consider the following sequence between K_1 and K_2, where adjacent knots are transformed into each other by a Delta move. We direct our attention to the part from K_1' to K_2' where $d_\Delta(K_1', K_2') = 2$ and $d_\Delta(K_1', K') = d_\Delta(K', K_2') = 1$.

$$\underbrace{K_1 \leftrightarrow \cdots \leftrightarrow K_1' \leftrightarrow K'}_{q \text{ times Delta moves}} \leftrightarrow K_2' \leftrightarrow \cdots \leftrightarrow K_2$$

A Delta move is represented by the band sum of a copy of Borromean rings[11]. A knot K' has a diagram such that if we delete a copy of Borromean rings A, we have a diagram of K_1' and if we delete a copy of Borromean rings B, we have a diagram of K_2'. We can arrange the bands such that the bands b_i $(i = 0, 1, 2)$ incident to the Borromean rings B appear in front of the bands b_k' $(k = 0, 1, 2)$ incident to the Borromean rings A along the knot K' by sliding the bands on K'. And by the symmetry of Borromean rings, we may suppose that bands b_0, b_1, b_2, b_0', b_1' and b_2' appear in this order along the orientation of K' as in Fig. 23.

Fig. 23 is deformed into Fig. 24. By performing $C_{m+1}^{(m-2)}$-moves for the knot in Fig. 24 two times, we obtain the knot in Fig. 25. We consider the move from Fig. 24 to Fig. 25 such that the permutation of the Jacobi diagram for the corresponding $C_{m+1}^{(m-2)}$-move is identity.
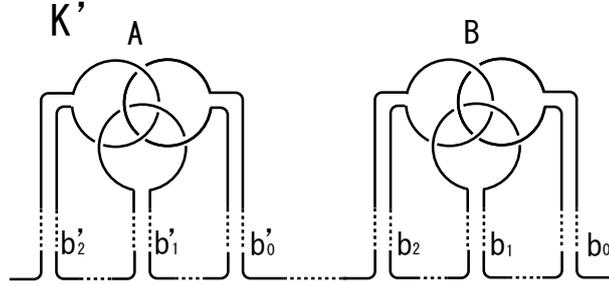
Fig. 23
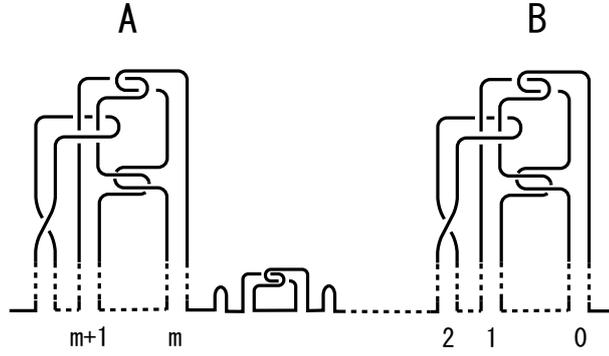
Fig. 24

By J_j, we denote the knot that is obtained from K' by performing the above move j times. By the same way of the proof of Theorem 2.6, we have $d_\Delta(K_1, J_j) = q$ and $d_\Delta(J_j, K_2) = p - q$. And we obtain

$$v_i(J_j) = v_i(J_k) = v_i(K')$$

and

$$\nabla_{J_j}(z) = \nabla_{J_k}(z) = \nabla_{K'}(z) \ (j \neq k, j, k = 1, 2, \cdots,)$$

By the same way of the proof of Theorem 3.6, we have

$$v_{m+3}(J_j) - v_{m+3}(J_{j+1}) = \pm v_{m+3}(T_\sigma),$$

where T_σ is the Jacobi diagram of order $m + 3$ whose internal graph is isomorphic to the graph in Fig. 26.

By Lemma 3.4,

$$V^{m+3}(T_\sigma) = 3(-2)^{m+1}(m + 3)!.$$

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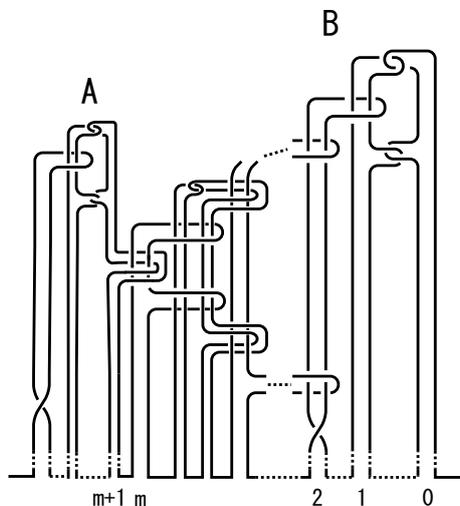
Fig. 25
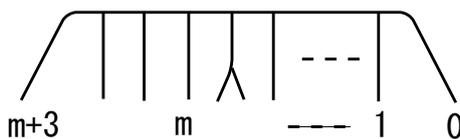
Fig. 26

Therefore,

$$V^{m+3}(J_j) \neq V^{m+3}(J_k) \ (j \neq k, j, k = 1, 2, \cdots).$$

5. Proof of Theorem 1.8

There exisits a C_n-move which changes the Conway polynomial. As to the one-branch tree diagram corresponding to the move, we have the following lemma by the proof of Theorem 1.3 in [17].

Lemma 4.1. *Let m' be an even natural number. By T_σ, we denote a one-branch tree diagram of order m' whose permutation σ satisfies that $\sigma(m' - 1) < \sigma(1) < \sigma(m')$ as in Fig. 27. Then*

$$a_{m'}(T_\sigma) = \pm 2,$$

where $a_{m'}$ is the coefficient of $z^{m'}$ of the Conway polynomial.
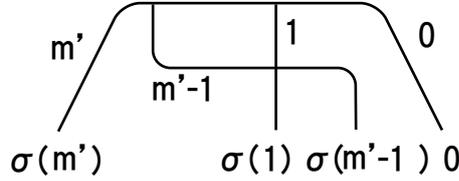
Fig. 27

As in sections 3 and 4, for a pair of knots K_1 and K_2 with $d_{C_n}(K_1, K_2) = p$, we consider the following sequence, where adjacent knots are transformed into each other by a C_n-move. We direct our attention to the part from K_1' to K_2'.

$$\underbrace{K_1 \leftrightarrow \cdots \leftrightarrow K_1' \leftrightarrow K'}_{q \text{ times } C_n\text{-moves}} \leftrightarrow K_2' \leftrightarrow \cdots \leftrightarrow K_2$$

A knot K' has a diagram such that if we delete a C_n-link model with bands A, we have a diagram of K_1' and if we delete a C_n-link model with bands B, we have a diagram of K_2' as in Fig. 28.
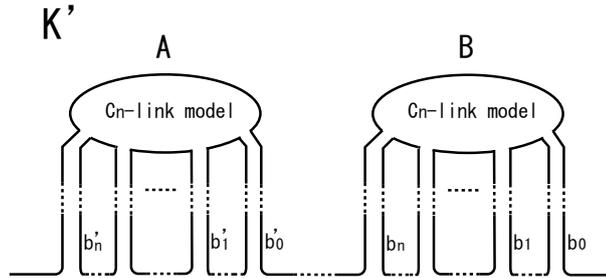
Fig. 28

We cannnot change the order of bands incident to the same C_n-link model on K'. However we can change the order of the band b_k' and the band b_i $(i, k = 0, 1, \ldots, n)$ in Fig. 28 by sliding the bands on K'. Then we may suppose that the band b_{n-1}' exists and the band b_n' does not exist between the bands b_0 and b_1.

Let m' be an even natural number more than $m + n$. We transform K' into the knot in Fig. 29. Fig. 29 shows the case $n = 3$.

The knot in Fig. 29 is considered to be obtained from K' by performing $C_{m'-n}$-moves two times. By J_j, we denote the knot that is obtained from K' by performing the moves above j times as shown in Fig. 30.

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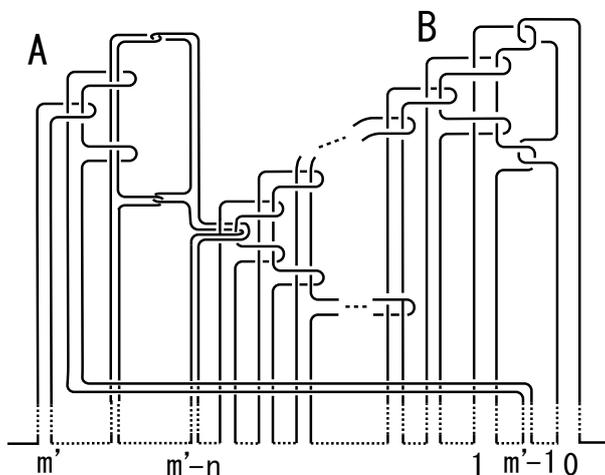
Fig. 29
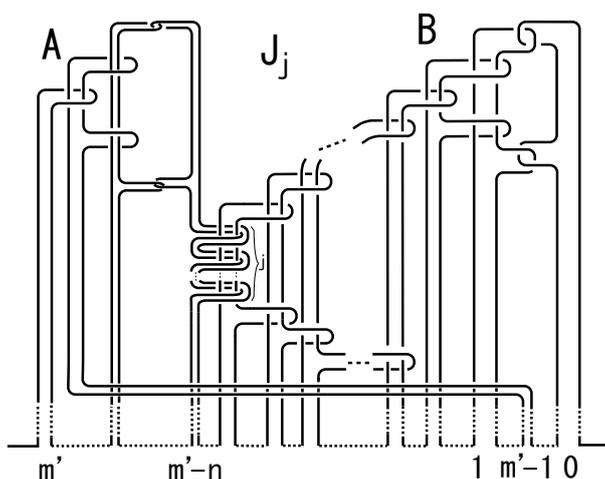
Fig. 30

The knot J_j satisfies that if we delete a C_n-link model with bands A, we have a diagram of K_1' and if we delete a C_n-link model with bands B, we have a diagram of K_2'. Then we have $d_{C_n}(K_1, J_j) = q$ and $d_{C_n}(J_j, K_2) = p - q$. Since $m' - n > m$ and by Theorem 1.1,

$$v_i(J_j) = v_i(J_k) \ (i = 1, 2, \ldots, m, \ j \neq k, \ j, k = 1, 2, \ldots).$$

By Theorem 3.6, we have

$$v_{m'}(J_j) - v_{m'}(J_{j+1}) = \pm v_{m'}(T_\sigma),$$

where T_σ is a one-branch tree diagram of order m' that satisfies the condition in Lemma 4.1. By Lemma 4.1, we have

$$a_{m'}(J_j) \neq a_{m'}(J_k) \ (j \neq k, \ j, k = 1, 2, \ldots).$$

This completes the proof of Theorem 1.8.

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