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Computation of the Characteristic Variety and the Singular Locus of a System of Differential Equations with Polynomial Coefficients

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It is proved that for a system of linear partial differential equations with polynomial coefficients, the Gröbner basis in the Weyl algebra is sufficient for the computation of the characteristic variety. In particular, this yields a correct algorithm of computing the singular locus of a holonomic system with polynomial coefficients. The characteristic variety is defined analytically, i.e. by using the ring of power series, and it has not been obvious that it can be computed by purely algebraic procedure. Thus the algorithm of computing the characteristic variety and the singular locus of a system of differential equations with polynomial coefficients can be readily implemented on a computer algebra system.

Key words: system of partial differential equations, characteristic variety, Weyl algebra, Gröbner basis, computer algebra

Introduction

In this paper, we consider a system $\mathcal{M}$ of linear partial differential equations with polynomial coefficients. More concretely, $\mathcal{M}$ is given by the system of equations

$$\sum_{j=1}^{r} P_{ij} u_j = 0 \qquad (i = 1, \ldots, s)$$

for unknown functions $u_1, \ldots, u_r$, where P_{ij} are linear partial differential operators of n variables $x_1, \ldots, x_n$ with polynomial coefficients (i.e. elements of the Weyl algebra A_n).

The characteristic variety $\mathrm{Char}(\mathcal{M})$ of $\mathcal{M}$ is by definition an analytic subset of the complex cotangent bundle $T^*\mathbf{C}^n$ and it represents the analytic nature of the system $\mathcal{M}$. For example, the holonomicity, ellipticity, and hyperbolicity of $\mathcal{M}$ are all defined through $\mathrm{Char}(\mathcal{M})$. In particular, $\mathcal{M}$ is called a holonomic system if the dimension of $\mathrm{Char}(\mathcal{M})$ as an analytic set is minimal; i.e., equal to n. The projection $\mathrm{Sing}(\mathcal{M})$ of $\mathrm{Char}(\mathcal{M}) \setminus 0$ to the x-space $\mathbf{C}^n$ is an analytic subset of $\mathbf{C}^n$, where 0 denotes the zero section of $T^*\mathbf{C}^n$. We call $\mathrm{Sing}(\mathcal{M})$ the singular locus of $\mathcal{M}$. When $\mathcal{M}$ is holonomic, $\mathrm{Sing}(\mathcal{M})$ is a proper analytic subset, and it was proved by Kashiwara [K1] that any local analytic solution of $\mathcal{M}$ is continued to an analytic solution on the universal covering space of $\mathbf{C}^n \setminus \mathrm{Sing}(\mathcal{M})$.

The characteristic variety is defined analytically, i.e., through the sheaf of rings $\mathcal{D}$ of linear partial differential operators with analytic coefficients. Hence even for a system of equations with algebraic (i.e. polynomial) coefficients, it is not obvious that its characteristic variety can be computed purely algebraically, i.e. without any computation in the ring of the power series.

The aim of this paper is to show that there is a correct and purely algebraic algorithm which, given a system $\mathcal{M}$ with polynomial coefficients, computes Char ($\mathcal{M}$) and Sing ($\mathcal{M}$).

The key point of our argument is the Theorem in Section 2, which asserts that for a system $\mathcal{M}$ with polynomial coefficients, the Gröbner basis in the Weyl algebra with an appropriate monomial order gives a so-called involutive basis. Then by standard methods, we can compute the characteristic variety and, if $\mathcal{M}$ is holonomic, the singular locus and the rank of $\mathcal{M}$ almost immediately.

We note that the Gröbner basis algorithm in the Weyl algebra was initiated by Galligo [G] and has been extended and applied to actual computation by Takayama [Tak1], [Tak2].

1. Gröbner Bases for Modules over the Weyl Algebra

In this paper, we use the following three kinds of rings of linear partial differential operators (we use the notation $\partial_i = \partial/\partial x_i$):

(i) The ring of differential operators with polynomial coefficients (the Weyl algebra)

$$A_n := \mathbf{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle,$$

(ii) The ring of differential operators with rational function coefficients

$$R_n := \mathbf{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle,$$

(iii) The ring of differential operators with convergent power series coefficients

$$\mathcal{D}_0 := \mathbf{C}\{x_1, \dots, x_n\}\langle \partial_1, \dots, \partial_n \rangle.$$

These are non-commutative $\mathbf{C}$-algebras with fundamental relations

$$x_i x_j = x_j x_i, \qquad \partial_i \partial_j = \partial_j \partial_i,$$
$$x_i \partial_j - \partial_j x_i = -\delta_{ij} \quad \text{for} \quad 1 \le i, j \le n,$$

where $\mathbf{C}$ denotes the field of the complex numbers and δ_{ij} is the Kronecker delta. The first two rings are algebraic (cf. [Bj]), and the Gröbner basis algorithm is applied effectively as was shown in [G], [C], [N], [Tak1], [Tak2]. On the other hand, the ring $\mathcal{D}_0$ (more precisely, the sheaf $\mathcal{D}$ of differential operators with analytic coefficients whose stalks are isomorphic to $\mathcal{D}_0$) is used as a fundamental tool in the theory of the system of linear partial differential equations (cf. [K1], [K2], [SKK]). Our motivation is to find relations among modules over these three rings.

Let us review the Gröbner basis theory for modules over the Weyl algebra. First we define a lexicographic order $\prec$ in $\mathbf{N}^n$ by

$$\alpha \prec \beta \quad \text{if and only if there is some } k \text{ with } 1 \le k \le n \text{ such that}$$
$$\alpha_k < \beta_k \quad \text{and} \quad \alpha_i = \beta_i \quad \text{for any} \quad i < k$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Here we write $\mathbf{N} = \{0, 1, 2, \dots\}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Then we define a total order (a monomial order) in $\mathbf{N}^{2n}$ by

$$\begin{aligned}(\alpha, \beta) \prec (\alpha', \beta') \quad \text{if and only if} \quad & (|\beta| < |\beta'|) \\ \text{or} \quad & (|\beta| = |\beta'| \quad \text{and} \quad \beta \prec \beta') \\ \text{or} \quad & (\beta = \beta' \quad \text{and} \quad |\alpha| < |\alpha'|) \\ \text{or} \quad & (\beta = \beta' \quad \text{and} \quad |\alpha| = |\alpha'| \quad \text{and} \quad \alpha \prec \alpha')\end{aligned}$$

for $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$.

An element P of A_n is written as a finite sum

$$P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$$

with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial^\beta = \partial_1{}^{\beta_1} \dots \partial_n{}^{\beta_n}$, $a_{\alpha\beta} \in \mathbf{C}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then we define the leading exponent lexp(P), the order ord(P), and the leading coefficient lcoef(P) of P by

$$\begin{aligned}\mathrm{lexp}(P) &= \max{}_{\prec}\{(\alpha, \beta) \in \mathbf{N}^{2n} \mid a_{\alpha\beta} \ne 0\}, \\ \mathrm{ord}(P) &= \max\{|\beta| \mid a_{\alpha\beta} \ne 0\}, \\ \mathrm{lcoef}(P) &= a_{\alpha\beta} \quad \text{with} \quad (\alpha, \beta) = \mathrm{lexp}(P),\end{aligned}$$

where $\max_{\prec}$ denotes the maximum element with respect to the monomial order $\prec$ in $\mathbf{N}^{2n}$. When $\mathrm{ord}(P) \le m$ we write

$$\sigma_m(P) = \sum_{\alpha, |\beta| = m} a_{\alpha\beta} x^\alpha \xi^\beta$$

with $\xi = (\xi_1, \dots, \xi_n)$. If $\mathrm{ord}(P) = m$, we write simply $\sigma(P) = \sigma_m(P)$ and call it the principal symbol of P.

Moreover, for an r-vector $\vec{P} = (P_1, \dots, P_r) \in (A_n)^r$, we define its order, the leading point lp($\vec{P}$), the leading exponent and the leading coefficient by

$$\begin{aligned}\mathrm{ord}(\vec{P}) &= \max\{\mathrm{ord}(P_\nu) \mid \nu = 1, \dots, r\}, \\ \mathrm{lp}(\vec{P}) &= \max\{\nu \in \{1, \dots, r\} \mid \mathrm{ord}(P_\nu) = \mathrm{ord}(\vec{P})\}, \\ \mathrm{lexp}(\vec{P}) &= (\mathrm{lexp}(P_\nu), \mathrm{lp}(\vec{P})) \quad \text{with} \quad \nu = \mathrm{lp}(\vec{P}), \\ \mathrm{lcoef}(\vec{P}) &= \mathrm{lcoef}(P_\nu) \quad \text{with} \quad \nu = \mathrm{lp}(\vec{P}).\end{aligned}$$

Let N be a left A_n-submodule of $(A_n)^r$. Then the set $E(N)$ of leading exponents of N is defined by

$$E(N) = \{\mathrm{lexp}(\vec{P}) \mid \vec{P} \in N, \ \vec{P} \ne 0\} \subset \mathbf{N}^{2n} \times \{1, \dots, r\}.$$

We introduce a total order $\prec$ in the set $\mathbf{N}^{2n} \times \{1, \dots, r\}$ by

$$\begin{aligned}(\alpha, \beta, \nu) \prec (\alpha', \beta', \nu') \quad \text{if and only if} \quad & (|\beta| < |\beta'|) \\ \text{or} \quad & (|\beta| = |\beta'| \quad \text{and} \quad \nu < \nu') \\ \text{or} \quad & (|\beta| = |\beta'| \quad \text{and} \quad \nu = \nu' \quad \text{and} \quad (\alpha, \beta) \prec (\alpha', \beta'))\end{aligned}$$

for $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$ and $\nu, \nu' \in \{1, \dots, r\}$.

DEFINITION. A finite subset $\mathbf{G}$ of a left A_n-submodule N of $(A_n)^r$ is called a Gröbner basis of N if

$$E(N) = \bigcup_{\vec{P} \in \mathbf{G}} (\mathrm{lexp}(\vec{P}) + \mathbf{N}^{2n})$$

holds, where we put

$$(\alpha, \beta, \nu) + \mathbf{N}^{2n} = \{(\alpha + \alpha', \beta + \beta', \nu) \mid \alpha', \beta' \in \mathbf{N}^n\}.$$

The algorithm of constructing a Gröbner basis from a given set of generators of N is similar to the Buchberger algorithm for ideals of polynomial rings ([Bu], [BW], [CLO]) and is described in [Tak1] in a more general setting.

2. Gröbner Bases and Involutive Bases

Let N be an A_n-submodule of $(A_n)^r$. Then N is generated by a finite set $\{\vec{P}_1, \dots, \vec{P}_s\}$. Write $\vec{P}_i = (P_{i1}, \dots, P_{ir})$ for $i = 1, \dots, s$. Then in the theory of systems of linear partial differential equations, it is natural to regard the system

$$\sum_{j=1}^{r} P_{ij} u_j = 0 \qquad (i = 1, \dots, s)$$

as a sheaf of $\mathcal{D}$-modules $\mathcal{M} := (\mathcal{D})^r / \mathcal{N}$ with $\mathcal{N} := \mathcal{D}\vec{P}_1 + \cdots + \mathcal{D}\vec{P}_s$.

Let us denote by $\mathcal{D}^{(m)}$ the subsheaf of $\mathcal{D}$ consisting of operators of order at most m. Define a filtration $\{\mathcal{N}^{(m)}\}$ of $\mathcal{N}$ by $\mathcal{N}^{(m)} = \mathcal{N} \cap (\mathcal{D}^{(m)})^r$ and let

$$\overline{\mathcal{N}} := \bigoplus_{m \geq 0} \mathcal{N}^{(m)} / \mathcal{N}^{(m-1)}$$

be the graded module associated with the filtration. Then $\overline{\mathcal{N}}$ is a sheaf of $\mathcal{O}[\xi]$-submodules of $(\mathcal{O}[\xi])^r$, where $\mathcal{O}$ denotes the sheaf of holomorphic functions of x. Put $\overline{\mathcal{M}} = (\mathcal{O}[\xi])^r / \overline{\mathcal{N}}$. It is obvious by definition that

$$\overline{\mathcal{N}} = \bigoplus_{m \geq 0} \{\sigma(\vec{P}) \mid \vec{P} \in \mathcal{N}^{(m)}\},$$

where $\sigma(\vec{P}) = (\sigma_m(P_1), \dots, \sigma_m(P_s))$ for $\vec{P} = (P_1, \dots, P_s)$ and $m = \mathrm{ord}(\vec{P})$.

The characteristic variety $\mathrm{Char}(\mathcal{M})$ of $\mathcal{M}$ is the subset of the cotangent bundle $T^*\mathbf{C}^n = \mathbf{C}^n \times \mathbf{C}^n$ defined as the support of the sheaf $\mathcal{O}_{\mathbf{C}^n \times \mathbf{C}^n} \otimes_{\mathcal{O}[\xi]} \overline{\mathcal{M}}$ on $T^*\mathbf{C}^n$, where $\mathcal{O}_{\mathbf{C}^n \times \mathbf{C}^n}$ denotes the sheaf of holomorphic functions in (x, ξ). For a sheaf $\mathcal{F}$ on $\mathbf{C}^n$ and a point $p \in \mathbf{C}^n$ we denote by $\mathcal{F}_p$ the stalk (i.e. the set of germs) of $\mathcal{F}$ at p. For $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$ we set

$$\alpha \vee \beta = (\max\{\alpha_1, \beta_1\}, \dots, \max\{\alpha_n, \beta_n\}).$$

THEOREM. *Let N be an A_n-submodule of $(A_n)^r$ and assume that $\mathbf{G}$ is a Gröbner basis of N. Then for any point p of $\mathbf{C}^n$, the module $\overline{\mathcal{N}}_p$ is generated by the set $\{\sigma(\vec{P}) \mid \vec{P} \in \mathbf{G}\}$ over $\mathcal{O}_p[\xi]$; i.e., $\mathbf{G}$ is an involutive basis.*

Proof. Put $\mathbf{G} = \{\vec{P}_1, \dots, \vec{P}_s\}$ and set $\mathrm{lexp}(\vec{P}_i) = (\alpha_i, \beta_i, \nu_i)$ for $i = 1, \dots, s$. We may assume $\mathrm{lcoef}(\vec{P}_i) = 1$ for any i without loss of generality. For two distinct $i, j \in \{1, \dots, s\}$ we put

$$S_{ij} = \begin{cases} x^{\alpha_{ij}} \partial^{\beta_{ij}}, & \text{if } \nu_i = \nu_j; \\ 0, & \text{otherwise} \end{cases}$$

with $\alpha_{ij} := \alpha_i \vee \alpha_j - \alpha_i$ and $\beta_{ij} := \beta_i \vee \beta_j - \beta_i$. Then since $\mathbf{G}$ is a Gröbner basis we have

$$S_{ij}\vec{P}_i - S_{ji}\vec{P}_j = \sum_{k=1}^{s} Q_{ijk}\vec{P}_k \tag{1}$$

with some $Q_{ijk} \in A_n$ such that

$$\mathrm{lexp}(Q_{ijk}\vec{P}_k) \prec (\alpha_i \vee \alpha_j, \beta_i \vee \beta_j, \nu_i) \tag{2}$$

if $\nu_i = \nu_j$ (cf. [Tak1]). Put

$$\vec{p}_i = \sigma_{m_i}(\vec{P}_i), \qquad s_{ij} = \sigma_{m_{ij} - m_i}(S_{ij}), \qquad q_{ijk} = \sigma_{m_{ij} - m_k}(Q_{ijk})$$

with $m_i = |\beta_i|$ and $m_{ij} = |\beta_i \vee \beta_j|$. Then it follows from (1) and (2) that if $\nu_i = \nu_j$,

$$s_{ij}\vec{p}_i - s_{ji}\vec{p}_j = \sum_{k=1}^{s} q_{ijk}\vec{p}_k, \qquad \mathrm{lexp}(q_{ijk}\vec{p}_k) \prec (\alpha_i \vee \alpha_j, \beta_i \vee \beta_j, \nu_i),$$

where lexp is defined in the same way as for $(A_n)^r$ with ∂ replaced by ξ. This implies that $\sigma(\mathbf{G}) := \{\sigma(\vec{P}) \mid \vec{P} \in \mathbf{G}\}$ is a Gröbner basis of the $\mathbf{C}[x, \xi]$-submodule

$$\overline{N} := \mathbf{C}[x, \xi]\sigma(\vec{P}_1) + \cdots + \mathbf{C}[x, \xi]\sigma(\vec{P}_s)$$

of $(\mathbf{C}[x, \xi])^r$. In view of the theory of the Gröbner basis for the polynomial ring, the first syzygy module for $\sigma(\mathbf{G})$:

$$\left\{(f_1, \dots, f_s) \in (\mathbf{C}[x, \xi])^s \,\middle|\, \sum_{k=1}^{s} f_k \vec{p}_k = 0\right\}$$

is generated by the set $\{\vec{v}_{ij} \mid 1 \leq i < j \leq s\}$ with

$$\vec{v}_{ij} := (0, \dots, \overbrace{s_{ij}}^{(i)}, \dots, \overbrace{-s_{ji}}^{(j)}, \dots, 0) - (q_{ij1}, \dots, q_{ijs}) \in (\mathbf{C}[x, \xi])^s.$$

(See e.g. [BW], [CLO] for the proof for the case $r = 1$. The proof for the case $r > 1$ is similar.) Hence we have an exact sequence of $\mathbf{C}[x, \xi]$-modules:

$$(\mathbf{C}[x, \xi])^{\binom{s}{2}} \xrightarrow{\psi} (\mathbf{C}[x, \xi])^s \xrightarrow{\varphi} (\mathbf{C}[x, \xi])^r, \tag{3}$$

where the homomorphisms φ and ψ are defined by

$$\varphi((f_1,\dots,f_s))=\sum_{k=1}^{s} f_k\vec{p}_k, \qquad \psi((f_{ij})_{i<j})=\sum_{i<j} f_{ij}\vec{v}_{ij}$$

for $f_k, f_{ij} \in \mathbf{C}[x,\xi]$. Since $\mathcal{O}_p \simeq \mathbf{C}\{x\}$ is flat over $\mathbf{C}[x]$ (cf. [Bou, Ch. 3]) and

$$\mathcal{O}_p \otimes_{\mathbf{C}[x]} \mathbf{C}[x,\xi] = \mathcal{O}_p[\xi],$$

we get from (3) an exact sequence of $\mathcal{O}_p[\xi]$-modules:

$$(\mathcal{O}_p[\xi])^{\binom{s}{2}} \xrightarrow{\tilde{\psi}} (\mathcal{O}_p[\xi])^s \xrightarrow{\tilde{\varphi}} (\mathcal{O}_p[\xi])^r, \tag{4}$$

where the homomorphisms $\tilde{\varphi}$ and $\tilde{\psi}$ are defined by

$$\tilde{\varphi}((f_1,\dots,f_s))=\sum_{k=1}^{s} f_k\vec{p}_k, \qquad \tilde{\psi}((f_{ij})_{i<j})=\sum_{i<j} f_{ij}\vec{v}_{ij}$$

for $f_k, f_{ij} \in \mathcal{O}_p[\xi]$.

Now let $\vec{P}$ be an arbitrary element of $\mathcal{N}_p$. Our aim is to show that there exist $Q_1,\dots,Q_s \in \mathcal{D}_p$ which satisfy

$$\vec{P} = Q_1\vec{P}_1 + \cdots + Q_s\vec{P}_s \tag{5}$$

and

$$\mathrm{ord}(Q_k\vec{P}_k) \le \mathrm{ord}(\vec{P}) \quad \text{for any} \quad k \in \{1,\dots,s\}. \tag{6}$$

For this purpose, let us take an expression (5) which minimizes the quantity

$$m := \max\{\mathrm{ord}(Q_k\vec{P}_k) \mid k=1,\dots,s\}.$$

Assume $m > \mathrm{ord}(\vec{P})$. Then taking the principal symbol of order m of both sides of (5), we get

$$\sum_{k=1}^{s} \sigma_{m'_k}(Q_k)\sigma_{m_k}(\vec{P}_k) = 0$$

with $m_k := \mathrm{ord}(\vec{P}_k)$ and $m'_k := m - m_k$. In view of the exact sequence (4), there exist $f_{ij} \in \mathcal{O}_p[\xi]$ which are homogeneous of degree $m - m_{ij}$ in ξ so that

$$(\sigma_{m'_1}(Q_1),\dots,\sigma_{m'_s}(Q_s)) = \sum_{i<j} f_{ij}\vec{v}_{ij}.$$

Put

$$\vec{V}_{ij} := (0,\dots,\overbrace{S_{ij}}^{(i)},\dots,\overbrace{-S_{ji}}^{(j)},\dots,0) - (Q_{ij1},\dots,Q_{ijs}) \in (A_n)^s.$$

Take $F_{ij} \in \mathcal{D}_p$ satisfying $\sigma_{m-m_{ij}}(F_{ij}) = f_{ij}$ and define $Q'_1,\dots,Q'_s$ by

$$(Q'_1,\dots .Q'_s) = (Q_1,\dots,Q_s) - \sum_{i<j} F_{ij}\vec{V}_{ij}.$$

Then it follows

$$\vec{P} = \sum_{k=1}^{s} Q'_k\vec{P}_k + \sum_{i<j} F_{ij}\vec{V}_{ij}\mathrm{mat}(\vec{P}_1,\dots,\vec{P}_s) = \sum_{k=1}^{s} Q'_k\vec{P}_k,$$

where $\mathrm{mat}(\vec{P}_1,\dots,\vec{P}_s)$ denotes the $s\times r$ matrix with $\vec{P}_1,\dots,\vec{P}_s$ as row vectors. This contradicts the definition of m since $\mathrm{ord}(Q'_k\vec{P}_k) \le m-1$. Hence we have proved that there exists an expression (5) with the condition (6). This implies

$$\sigma(\vec{P}) \in \mathcal{O}_p[\xi]\sigma(\vec{P}_1) + \cdots + \mathcal{O}_p[\xi]\sigma(\vec{P}_s).$$

This completes the proof.

3. Characteristic Varieties and Singular Loci

PROPOSITION 1. *Under the same assumptions as in the Theorem, put*

$$\mathbf{G}_\nu = \{\vec{P} \in \mathbf{G} \mid \mathrm{lp}(\vec{P}) = \nu\}$$

for each $\nu \in \{1,\dots,r\}$. *Then the characteristic variety of* $\mathcal{M}$ *is given by* $\mathrm{Char}(\mathcal{M}) = \bigcup_{\nu=1}^{r} V_\nu$ *with*

$$V_\nu = \{(x,\xi) \in T^*\mathbf{C}^n \mid \sigma(\vec{P})_\nu(x,\xi) = 0 \quad \text{for any} \quad \vec{P} \in \mathbf{G}_\nu\},$$

where $\sigma(\vec{P})_\nu$ *denotes the* ν*-th component of the vector* $\sigma(\vec{P})$.

Proof. We use the same notation as in the proof of the Theorem of Section 2. For each $\nu \in \{1,\dots,r\}$, define a sheaf $\mathcal{L}^{(\nu)}$ of $\mathcal{O}[\xi]$-modules by

$$\mathcal{L}^{(\nu)} = \{(f_1,\dots,f_r) \in \overline{\mathcal{N}} \mid f_\mu = 0 \quad \text{for } \mu > \nu\}.$$

Then $\mathcal{L}^{(\nu)}/\mathcal{L}^{(\nu-1)}$ can be regarded as a subsheaf of $\mathcal{O}[\xi]$ and we have $\mathcal{L}^{(r)} = \overline{\mathcal{N}}$. By the definition of the characteristic variety, we have

$$\mathrm{Char}(\mathcal{M}) = \{p^* = (p,q) \in T^*\mathbf{C}^n \mid (\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*} \otimes_{\mathcal{O}_p[\xi]} \overline{\mathcal{N}}_p = (\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*}{}^r\}.$$

It is easy to see that

$$(\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*} \otimes_{\mathcal{O}_p[\xi]} \overline{\mathcal{N}}_p = (\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*}{}^r$$

holds if and only if

$$(\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*} \otimes_{\mathcal{O}_p[\xi]} (\mathcal{L}^{(\nu)}/\mathcal{L}^{(\nu-1)})_p = (\mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n})_{p^*} \quad \text{for} \quad \nu = 1,\dots,r.$$

Hence in order to prove Proposition 1, it suffices to show that $(\mathcal{L}^{(\nu)}/\mathcal{L}^{(\nu-1)})_p$ is generated by $\{\sigma(\vec{P})_\nu \mid \vec{P} \in \mathbf{G}_\nu\}$. To prove this fact, let $\vec{f} = (f_1, \dots, f_r)$ be an arbitrary element of $(\mathcal{L}^{(\nu)})_p$. We put $\mathrm{lp}(\vec{f}) = \max\{\mu \mid f_\mu \neq 0\}$. Then we may assume $\mathrm{lp}(\vec{f}) = \nu$. In view of the Theorem, there exist $q_1, \dots, q_r \in \mathcal{O}_p[\xi]$ such that

$$\vec{f} = q_1\vec{p}_1 + \cdots + q_s\vec{p}_s. \tag{7}$$

Put $\mu = \max\{\mathrm{lp}(q_k\vec{p}_k) \mid k = 1, \dots, s\}$ and assume $\mu > \nu$. Let us denote by $(\vec{f})_\mu$ the μ-th component of the vector $\vec{f}$. Put $S_\mu = \{k \in \{1, \dots, s\} \mid \mathrm{lp}(\vec{p}_k) = \mu\}$. We can now assume $S_\mu = \{1, \dots, s'\}$ by a permutation of the elements of $\mathbf{G}$. Moreover, since $\{\sigma(\vec{P}) \mid \vec{P} \in \mathbf{G},\ \mathrm{lp}(\vec{P}) \leq \mu\}$ also constitutes a Gröbner basis of the module which it generates, we may assume $\mu = r$ without loss of generality. Note that (7) implies

$$\sum_{k=1}^{s'} q_k(\vec{p}_k)_r = 0.$$

Since $\{\vec{p}_1, \dots, \vec{p}_s\}$ is a Gröbner basis in $(\mathbf{C}[x, \xi])^r$, so is $\{(\vec{p}_k)_r \mid \mathrm{lp}(\vec{p}_k) = r\}$ in $\mathbf{C}[x, \xi]$. Put

$$\vec{v}_{ij}{}' := (0, \dots, \overbrace{s_{ij}}^{(i)}, \dots, \overbrace{-s_{ji}}^{(j)}, \dots, 0) - (q_{ij1}, \dots, q_{ijs'}) \in (\mathbf{C}[x, \xi])^{s'}.$$

Then by the same argument as that in the proof of the Theorem, there exist $f_{ij} \in \mathcal{O}_p[\xi]$ such that

$$(q_1, \dots, q_{s'}) = \sum_{1 \leq i < j \leq s'} f_{ij}\vec{v}_{ij}{}'.$$

Since $\vec{v}_{ij}\mathrm{mat}(\vec{p}_1, \dots, \vec{p}_s) = 0$, we get

$$\begin{aligned}
\vec{f} &= \sum_{k=1}^{s'} q_k\vec{p}_k + \sum_{k=s'+1}^{s} q_k\vec{p}_k \\
&= \sum_{1 \leq i < j \leq s'} f_{ij}\Big(s_{ij}\vec{p}_i - s_{ji}\vec{p}_j - \sum_{k=1}^{s'} q_{ijk}\vec{p}_k\Big) + \sum_{k=s'+1}^{s} q_k\vec{p}_k \\
&= \sum_{1 \leq i < j \leq s'} f_{ij} \sum_{k=s'+1}^{s} q_{ijk}\vec{p}_k + \sum_{k=s'+1}^{s} q_k\vec{p}_k.
\end{aligned}$$

Thus $\vec{f}$ is represented as a linear combination of elements of $\{\sigma(\vec{P}) \mid \vec{P} \in \mathbf{G},\ \mathrm{lp}(\vec{P}) \leq r - 1\}$ over $\mathcal{O}_p[\xi]$. By induction, we can show that $\vec{f}$ is represented as a linear combination of elements of $\{\sigma(\vec{P}) \mid \vec{P} \in \mathbf{G},\ \mathrm{lp}(\vec{P}) \leq \nu\}$. Hence we have proved that $(\mathcal{L}^{(\nu)}/\mathcal{L}^{(\nu-1)})_p$ is generated by $\{\sigma(\vec{P})_\nu \mid \vec{P} \in \mathbf{G}_\nu\}$. This completes the proof.

PROPOSITION 2. *Under the same assumptions as in the Theorem, let I_ν be the ideal of $\mathbf{C}[x, \xi]$ generated by $\{\sigma(\vec{P})_\nu \mid \vec{P} \in \mathbf{G}_\nu\}$. Put*

$$J_{\nu i} = \{f(x) \in \mathbf{C}[x] \mid f(x)\xi_i{}^{\beta_i} \in I_\nu \quad \textit{for some} \quad \beta_i \in \mathbf{N}\}.$$

Then the singular locus of $\mathcal{M}$ is given by

$$\mathrm{Sing}\,(\mathcal{M}) = \bigcup_{\nu=1}^{r}\bigcup_{i=1}^{n}\{x \in \mathbf{C}^n \mid f(x) = 0 \quad \textit{for any} \quad f \in J_{\nu i}\}.$$

Proof. Let $\pi : T^*\mathbf{C}^n \longrightarrow \mathbf{C}^n$ be the projection defined by $\pi(x, \xi) = x$. Then the singular locus is by definition the union of the algebraic sets $\pi(V_\nu \setminus 0)$, where 0 denotes the zero section of $T^*\mathbf{C}^n$. On the other hand, $\pi(V_\nu \setminus 0)$ is the zeros of the ideal $\bigcap_{i=1}^n J_{\nu i}$ in view of the projective extension theorem of [CLO, Chap. 8]. This completes the proof.

4. Rank of a Holonomic System

Suppose $\mathcal{M}$ is holonomic on $U := \mathbf{C}^n \setminus \mathrm{Sing}\,(\mathcal{M}) \neq \emptyset$. Then by virtue of a theorem of Kashiwara ([K1],[K2]), there exists an integer r_0 so that $\mathcal{M}$ is locally isomorphic to $\mathcal{O}^{r_0}$ as $\mathcal{O}$-module on a neighborhood of each point of U. This implies that the holomorphic solutions of $\mathcal{M}$ on the universal covering space of U constitute an r_0-dimensional vector space over $\mathbf{C}$. This integer r_0 is called the rank of $\mathcal{M}$. We define the projection

$$\varpi : \mathbf{N}^{2n} \times \{1, \dots, r\} \to \mathbf{N}^n \times \{1, \dots, r\}$$

by $\varpi(\alpha, \beta, \nu) = (\beta, \nu)$.

PROPOSITION 3. *Let $\mathcal{M}$, N, $\mathbf{G}$ be as in the Theorem. Put*

$$\begin{aligned}
E = \{(\alpha, \nu) \in \mathbf{N}^n \times \{1, \dots, r\} \mid\ & (\alpha, \beta, \nu) \in \mathrm{lexp}(\vec{P}) + \mathbf{N}^{2n} \\
& \textit{for some } \vec{P} \in \mathbf{G},\ \beta \in \mathbf{N}^n\}, \\
r_0 = \sharp(\mathbf{N}^n \times \{1, \dots, r\} \setminus E),&
\end{aligned}$$

where $\sharp$ denotes the cardinality of a set. Then, if r_0 is finite, $\mathcal{M}$ is holonomic on $\mathbf{C}^n \setminus \mathrm{Sing}\,(\mathcal{M}) \neq \emptyset$ and its rank is given by r_0.

Proof. Put $N_R = R_n\vec{P}_1 + \cdots + R_n\vec{P}_s \subset (R_n)^r$. It is known that $\mathcal{M}$ is holonomic on a Zariski open set of $\mathbf{C}^n$ if $(R_n)^r/N_R$ is finite dimensional over $\mathbf{C}(x)$, and its dimension is equal to the rank of $\mathcal{M}$ (see e.g. [OS] for the proof). A vector $\vec{P}$ in $(R_n)^r$ can be written

$$\vec{P} = \frac{1}{a(x)}\vec{Q}$$

with $a(x) \in \mathbf{C}[x]$ and $\vec{Q} \in (A_n)^r$. Then we define $\mathrm{lp}_R(\vec{P}) \in \{1, \dots, r\}$ and $\mathrm{lexp}_R(\vec{P}) \in \mathbf{N}^n \times \{1, \dots, r\}$ by

$$\begin{aligned}
\mathrm{lp}_R(\vec{P}) &= \mathrm{lp}(\vec{Q}), \\
\mathrm{lexp}_R(\vec{P}) &= \varpi(\mathrm{lexp}(\vec{Q})).
\end{aligned}$$

Let us show

$$E = E_R(N) := \{\mathrm{lexp}_R(\vec{P}) \mid \vec{P} \in N_R,\ \vec{P} \neq 0\}.$$

Since $\mathbf{G}$ is a Gröbner basis of N, we have $E = \varpi(E(N))$. Hence the inclusion $E \subset E_R(N)$ follows from $N \subset N_R$. In order to prove the converse inclusion, assume $\vec{P} \in N_R$. Then there exists a polynomial $a(x)$ such that $a(x)P \in N$. Hence we get $\mathrm{lexp}_R(P) = \mathrm{lexp}_R(a(x)P) \in E$. This completes the proof since the dimension of $(R_n)^r/N_R$ over $\mathbf{C}(x)$ is equal to $\sharp(\mathbf{N}^n \times \{1,\dots,r\} \setminus E_R(N))$.

5. Algorithm and Examples

Let N and $\mathcal{M}$ be as in Section 2. For a polynomial $f(x,\xi) \in \mathbf{C}[x,\xi]$ we write $\mathrm{subst}(f,\xi_i,1)$ the result of the substitution $\xi_i = 1$ in f.

Algorithm.
Input: A set $\mathbf{G}$ of generators of N;
$\mathbf{G}$:="a Gröbner basis of N";
$E := \bigcup_{\vec{P}\in\mathbf{G}} \varpi(\mathrm{lexp}(\vec{P})) + \mathbf{N}^n$;
for $\nu := 1$ **to** r **do** {
  $\mathbf{G}_\nu := \{\vec{P} \in \mathbf{G} \mid \mathrm{lp}(\vec{P}) = \nu\}$;
  $I_\nu := \{\sigma(\vec{P})_\nu \mid \vec{P} \in \mathbf{G}_\nu\}$;
}
$r_0 := \sharp(\mathbf{N}^n \times \{1,\dots,r\} \setminus E)$;
if $r_0 < \infty$ **then**
  for $\nu := 1$ **to** r **do**
    for $i := 1$ **to** n **do** {
      $I_{\nu i} := \{\mathrm{subst}(f,\xi_i,1) \mid f \in I_\nu\}$;
      $\mathbf{G}_{\nu i}$:= "a Gröbner basis of the ideal of $\mathbf{C}[x,\xi]$ generated by $I_{\nu i}$ with respect to the monomial order $\prec$";
      $J_{\nu i} := \mathbf{G}_{\nu i} \cap \mathbf{C}[x]$;
    }
Output: $\{I_\nu\}$, r_0, $\{J_{\nu i}\}$;

From the output of this algorithm we get

$$\mathrm{Char}\,(\mathcal{M}) = \bigcup_{\nu=1}^{r} \{(x,\xi) \mid f(x,\xi) = 0 \quad \text{for any} \quad f \in I_\nu\},$$

and if r_0 is finite, $\mathcal{M}$ is holonomic of rank r_0 on $\mathbf{C}^n \setminus \mathrm{Sing}\,(\mathcal{M}) \neq \emptyset$ with

$$\mathrm{Sing}\,(\mathcal{M}) = \bigcup_{\nu=1}^{r}\bigcup_{i=1}^{n} \{x \mid f(x) = 0 \quad \text{for any} \quad f \in J_{\nu i}\}.$$

The correctness of the computation of $\mathrm{Sing}\,(\mathcal{M})$ follows from the arguments in [CLO, Chap. 8].

The following computation has been performed by using our implementation of the above algorithm on a computer algebra system Risa/Asir (cf. [NT]).

Example 1. Let us consider the system of Maxwell's equations in the vacuum. This is the system for the vector $\vec{u} = (\vec{E}, \vec{H})$ of 6 unknown functions:

$$\mathcal{M}: \quad \vec{P}_i \cdot \vec{u} = 0 \qquad (i = 1,\dots,8),$$

where in the coordinate (t,x,y,z) instead of (x_1,x_2,x_3,x_4), we set

$$\begin{aligned}
&\vec{P}_1 = (\partial_x,\partial_y,\partial_z,0,0,0), && \vec{P}_2 = (0,0,0,\partial_x,\partial_y,\partial_z),\\
&\vec{P}_3 = (0,-\partial_z,\partial_y,\mu\partial_t,0,0), && \vec{P}_4 = (\partial_z,0,-\partial_x,0,\mu\partial_t,0),\\
&\vec{P}_5 = (-\partial_y,\partial_x,0,0,0,\mu\partial_t), && \vec{P}_6 = (-\varepsilon\partial_t,0,0,0,-\partial_z,\partial_y),\\
&\vec{P}_7 = (0,-\varepsilon\partial_t,0,\partial_z,0,-\partial_x), && \vec{P}_8 = (0,0,-\varepsilon\partial_t,-\partial_y,\partial_x,0).
\end{aligned}$$

Hence this system is apparently overdetermined. Let N be the submodule of $(A_4)^6$ generated by $\{\vec{P}_1,\dots,\vec{P}_8\}$. Then we get as a Gröbner basis of N, $\mathbf{G} = \{\vec{P}_1,\dots,\vec{P}_8, \vec{P}_9,\dots,\vec{P}_{13}\}$ with

$$\begin{aligned}
\vec{P}_9 &= (\varepsilon\mu{\partial_t}^2 - {\partial_x}^2 - {\partial_y}^2 - {\partial_z}^2,0,0,0,0,0),\\
\vec{P}_{10} &= (0,\varepsilon\mu{\partial_t}^2 - {\partial_x}^2 - {\partial_y}^2 - {\partial_z}^2,0,0,0,0),\\
\vec{P}_{11} &= (\varepsilon\partial_z\partial_t,0,0,\partial_y\partial_x,{\partial_y}^2 + {\partial_z}^2,0),\\
\vec{P}_{12} &= (\partial_z\partial_x,\partial_z\partial_y,\varepsilon\mu{\partial_t}^2 - {\partial_x}^2 - {\partial_y}^2,0,0,0),\\
\vec{P}_{13} &= (0,\varepsilon\partial_z\partial_t,-\varepsilon\partial_y\partial_t,-{\partial_x}^2 - {\partial_y}^2 - {\partial_z}^2,0,0).
\end{aligned}$$

In fact, this computation is the Gröbner basis algorithm for polynomials since the above vectors of operators are with constant coefficients. Thus, in the notation of Propositions 1 and 2, we get

$$\begin{aligned}
V_1 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_9)_1(\tau,\xi,\eta,\zeta) = 0\},\\
V_2 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_{10})_2 = 0\},\\
V_3 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_1)_3 = \sigma(\vec{P}_{12})_3 = 0\},\\
V_4 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_3)_4 = \sigma(\vec{P}_{13})_4 = 0\},\\
V_5 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_4)_5 = \sigma(\vec{P}_8)_5 = \sigma(\vec{P}_{11})_5 = 0\},\\
V_6 &= \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \sigma(\vec{P}_2)_6 = \sigma(\vec{P}_5)_6 = \sigma(\vec{P}_6)_6 = \sigma(\vec{P}_7)_6 = 0\},
\end{aligned}$$

where we write (τ,ξ,η,ζ) instead of $(\xi_1,\xi_2,\xi_3,\xi_4)$. Hence we have

$$\mathrm{Char}\,(\mathcal{M}) = \{(t,x,y,z,\tau,\xi,\eta,\zeta) \mid \varepsilon\mu\tau^2 - \xi^2 - \eta^2 - \zeta^2 = 0\}.$$

Example 2. Using the coordinate (x,y) instead of (x_1,x_2), let us consider the system for Appell's F_3

$$\mathcal{M}: \quad P_1 u = P_2 u = 0$$

with

$$\begin{aligned}
P_1 &:= x(1-x){\partial_x}^2 + y\partial_x\partial_y + \{\gamma - (\alpha+\beta+1)x\}\partial_x - \alpha\beta,\\
P_2 &:= y(1-y){\partial_y}^2 + x\partial_x\partial_y + \{\gamma - (\alpha'+\beta'+1)y\}\partial_y - \alpha'\beta'.
\end{aligned}$$

Then we get $\{P_1, P_2, P_3, P_4\}$ as a Gröbner basis of the left ideal $I := A_2P_1 + A_2P_2$ with

$$\mathrm{lexp}(P_1) = (2,0,2,0), \qquad \mathrm{lexp}(P_2) = (1,0,1,1),$$
$$\mathrm{lexp}(P_3) = (0,2,1,2), \qquad \mathrm{lexp}(P_4) = (1,4,0,3).$$

Hence the rank of $\mathcal{M}$ on $\mathbf{C}^2 \setminus \mathrm{Sing}\,(\mathcal{M})$ is 4. Moreover we have

$$\begin{aligned} I_1 &= \{\xi((-x^2+x)\xi + y\eta),\ \eta(x\xi + (-y^2+y)\eta),\ y^2\eta^2(\xi - (y-1)^2\eta), \\ &\quad y^2(y-1)(xy-x-y)\eta^3\}, \\ J_{11} &= \{x^2(x-1)(xy-x-y)\}, \qquad J_{12} = \{y^2(y-1)(xy-x-y)\}. \end{aligned}$$

Hence the characteristic variety and the singular locus are given by

$$\begin{aligned} \mathrm{Char}\,(\mathcal{M}) &= \{x=y=0\} \cup \{\xi=\eta=0\} \cup \{x=\eta=0\} \cup \{x-1=\eta=0\} \\ &\quad \cup \{y=\xi=0\} \cup \{y-1=\xi=0\} \cup \{xy-x-y=\xi-(y-1)^2\eta=0\}, \\ \mathrm{Sing}\,(\mathcal{M}) &= \{(x,y) \in \mathbf{C}^2 \mid xy(x-1)(y-1)(xy-x-y)=0\}. \end{aligned}$$

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References

[BW] T. Becker and V. Weispfenning, Gröbner Bases —A Computational Approach to Commutative Algebra. Graduate Texts in Mathematics, Springer, Berlin-Heidelberg-New York, 1993.

[Bj] J.-E. Björk, Rings of Differential Operators. North-Holland, Amsterdam/Oxford/New York, 1979.

[Bou] N. Bourbaki, Éléments de Mathématique. Algèbre Commutative. Hermann, Paris, 1961.

[Bu] B. Buchberger, Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. Aequationes Math., **4** (1970), 374–383.

[C] F. Castro, Calculs effectifs pour les idéaux d'opérateurs différentiels. Travaux en Cours, 24 (eds. J.-M. Aroca et al.), Hermann, Paris, 1987, 1–19.

[CLO] D. Cox, J. Little, D. O'Shea, Ideals, varieties, and Algorithms. Undergraduate Texts in Mathematics, Springer, Berlin-Heidelberg-New York, 1992.

[G] A. Galligo, Some algorithmic questions on ideals of differential operators. Lecture Notes in Comput. Sci., 204 (ed. B.F. Caviness), Springer, Berlin-Heidelberg-New York, 1985, 413–421.

[K1] M. Kashiwara, On the maximally overdetermined system of linear differential equations, I. Publ. RIMS, Kyoto Univ., **10** (1975), 563–579.

[K2] M. Kashiwara, Systems of Microdifferential Equations. Birkhäuser, Boston-Basel-Stuttgart, 1983.

[NT] M. Noro and T. Takeshima, Risa/Asir —a computer algebra system. Proceedings of International Symposium on Symbolic and Algebraic Computation, ACM Press, New York, 1992, 387–396.

[N] M. Noumi, Wronskian determinants and the Gröbner representation of a linear differential equation: an approach to nonlinear integrable systems. Algebraic Analysis (eds. M. Kashiwara and T. Kawai), Academic Press, Boston-San Diego-New York, 1988, 549–569.

[OS] T. Oaku and T. Shimoyama, A Gröbner basis method for modules over rings of differential operators. To appear in J. Symbolic Comput.

[SKK] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations. Lecture Notes in Math. 287 (ed. H. Komatsu), Springer, Berlin-Heidelberg-New York, 1973, 265–529.

[Tak1] N. Takayama, Gröbner basis and the problem of contiguous relations. Japan J. Indust. Appl. Math., **6** (1989), 147–160.

[Tak2] N. Takayama, An approach to the zero recognition problem by Buchberger algorithm. J. Symbolic Comput., **14** (1992), 265–282.