Plan of the course

1st lecture **Introduction:** Aim and an example
   **Chapter 1:** Basics of $D$-modules

2nd lecture **Chapter 2:** Gröbner bases in the ring of differential operators
   **Chapter 3:** Distributions as generalized functions

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   **Chapter 5:** Integration over the domain defined by polynomial inequalities
4. $D$-module theoretic integration algorithm

4.1. Integration as an operation on $D$-modules

Let $D_{n+p}$ be the ring of differential operators in the variables $(x, t)$ with $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_d)$. Let

$$\pi : K^{n+d} \ni (x, t) \mapsto x \in K^n$$

be the projection.

The integral of a left $D_{n+d}$-module $M$ along the fibers of $\pi$, or the direct image by $\pi$, is defined by

$$\pi_* M := M/(\partial_{t_1} M + \cdots + \partial_{t_d} M).$$

This is a left $D_n$-module since any element of $D_n$ commutes with $\partial_{t_j}$. 
For the sake of simplicity, let us assume that $M$ is generated by a single element $u \in M$ as left $D_{n+d}$-module. Let $[u]$ be the residue class of $u$ in $\pi_* M$. Then $\pi_* M$ is generated by $\{ t^\gamma [u] \mid \gamma \in \mathbb{N}^d \}$ over $D_n$.

Now assume $K = \mathbb{C}$ and let $\varphi$ be an element of $\text{Hom}_{D_{n+d}}(M, \mathcal{D}'\mathcal{E}'(U))$. Then $f := \varphi(u)$ belongs to $\mathcal{D}'\mathcal{E}'(U)$.

Define a $\mathbb{C}$-homomorphism $\varphi' : M \to \mathcal{D}'(U)$ by

$$\varphi'(Pu) = \int_{\mathbb{R}^d} Pf(x, t) \, dt \quad (\forall P \in D_{n+d}).$$

This is well-defined since $Pu = 0$ in $M$ implies $Pf = 0$ in $\mathcal{D}'\mathcal{E}'(U)$.

Note that $\varphi'$ is $D_n$-linear by differentiation under the integral sign.
For any $P \in D_n$, $\varphi \in C_0^\infty(U)$, and $1 \leq j \leq d$, we have

$$\left< \int_{\mathbb{R}^d} \partial_t^j Pf(x, t) \, dt, \, \varphi(x) \right> = \left< \partial_t^j Pf(x, t), \, \varphi(x)1(t) \right>$$

$$= -\left< Pf(x, t), \, \partial_t^j (\varphi(x)1(t)) \right> = 0.$$ 

Hence $\varphi'$ induces

$$\pi_*(\varphi) \in \text{Hom}_{D_n}(\pi_* M, \mathcal{D}'(U)).$$

In conclusion, we have a $\mathbb{C}$-linear map

$$\pi_* : \text{Hom}_{D_{n+d}}(M, \mathcal{D}'\mathcal{E}'(U)) \longrightarrow \text{Hom}_{D_n}(\pi_* M, \mathcal{D}'(U)).$$
The argument so far works with $\mathcal{D}'\mathcal{E}'(U)$ replaced by $\mathcal{E}\mathcal{S}(U)$, and also by $\mathcal{E}\mathcal{S}(U) + \mathcal{D}'\mathcal{E}'(U)$ giving a $\mathbb{C}$-linear map

$$\pi_* : \text{Hom}_{D_n}(M, \mathcal{E}\mathcal{S}(U) + \mathcal{D}'\mathcal{E}'(U)) \longrightarrow \text{Hom}_{D_{n-d}}(\pi_*(M), \mathcal{D}'(U)).$$

This means that for a solution in $\mathcal{D}\mathcal{S}(U) + \mathcal{D}'\mathcal{E}'(U)$ of a system $M$ of differential equations, its integral with respect to $t$ satisfies the system $\pi_* M$.

The generators $t^\gamma[u]$ of $\pi_* M$ with $\gamma' \in \mathbb{N}^d$ are sent by $\pi_*(\varphi)$ to

$$\pi_*(\varphi)(t^\gamma[u]) = \int_{\mathbb{R}^d} t^\gamma f(x, t) \, dt \in \mathcal{D}'(U).$$
Theorem (Bernstein, Kashiwara)

If $M$ is a holonomic $D_n$-module, then $\pi_* M$ is a holonomic $D_n$-module. In particular, $\pi_* M$ is finitely generated over $D_n$.

Hence $\pi_* M$ is generated by a finite subset of $\{t^\gamma[u] \mid \gamma \in \mathbb{N}^d\}$, and $\pi_* M$ represents the relations among these generators.
Example

Set \( n = d = 1 \) and write \( x = x_1, \ t = t_1 \). Consider

\[
\mathcal{M} := \frac{D_2}{D_2 t(t - 1)(\partial_t - x) + D_2 (\partial_x - t)}.
\]

Set \( u = 1 \in \mathcal{M} \). Then

\[
t(t - 1)(\partial_t - x)u = (\partial_x - t)u = 0 \text{ in } \mathcal{M}.
\]

Let \([u]\) be the residue class of \( u \) in \( \pi_* \mathcal{M} = \mathcal{M}/\partial_t \mathcal{M} \).

\( \pi_* \mathcal{M} \) is generated by \([u]\) since \( tu = \partial_x u \) by the 2nd equation for \( u \).

From \( \partial_t t(t - 1) = t(t - 1)\partial_t + 2t - 1 \), it follows

\[
\{ t(t - 1)\partial_t + 2t - 1 \}[u] = 0.
\]

This gives, combined with \( t(t - 1)\partial_t u = xt(t - 1)u = x\partial_x (\partial_x - 1)u \),

\[
\{ x\partial_x^2 - (x - 2)\partial_x - 1 \}[u] = 0.
\]
4.2 An algorithm for integration

Let $M$ be a left $D_{n+d}$-module generated by $u \in M$. Set

$$D_{\pi} := D_{n+d}/(\partial_{t_1} D_{n+d} + \cdots + \partial_{t_d} D_{n+d}).$$

Then $D_{\pi}$ has a structure of $(D_n, D_{n+d})$-bimodule and we have

$$\pi_* M = M/(\partial_{t_1} M + \cdots + \partial_{t_d} M) = D_{\pi} \otimes_{D_{n+d}} M$$

as left $D_n$-module. Set

$$\theta := \partial_{t_1} t_1 + \cdots + \partial_{t_d} t_d = t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d} + d.$$
Now let us fix the weight vector

\[ w := (0, \ldots, 0, 1, \ldots, 1; 0, \ldots, 0, -1, \ldots, -1) \in \mathbb{Z}^{2(n+d)}. \]

That is, we define the weight of \( x_i \) and \( \partial_{x_i} \) to be 0, while the weight of \( t_j \) and \( \partial_{t_j} \) are 1 and \(-1\) respectively. Set

\[ F_k(M) := F_k^w(D_{n+d})u, \quad \text{gr}^k(M) := F_k(M)/F_{k-1}(M) \quad (k \in \mathbb{Z}). \]

Then \( \{F_k(M)\} \) is a good \( w \)-filtration of \( M \).
**Theorem**

If $M = D_{n+d}$ is holonomic, then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ in $s$ such that $b(\theta)\text{gr}_0(M) = 0$. Such $b(s)$ of minimum degree is called the $b$-function of $M$ with respect to the weight vector $w$.

The $b$-function can be computed by using a $w$-involutive basis of $I := \text{Ann}_{D_{n+d}} u$.

**Lemma**

Let $b(s)$ be the $b$-function of $M$ with respect to $w$. Then $b(\theta - k)\text{gr}_k(M) = 0$ holds for any $k \in \mathbb{Z}$. 
Proof: Note that $\theta \cdot t_j = t_j(\theta + 1)$ and $\theta \cdot \partial_j = \partial_j(\theta - 1)$. Hence $b(\theta)P = P b(\theta + k)$ holds if $P$ is homogeneous of order $k$ with respect to $w$. This proves the lemma.

The proof of the following proposition also provides us with an algorithm to compute the $D_n$-module structure of $\pi_* M$.

**Proposition**

Suppose that a left $D_{n+d}$-module $M = D_{n+d}u = D_{n+d}/I$ has a $b$-function $b(s)$. Let $-k_1$ be the smallest integral root, if any, of $b(s)$. Set $k_1 = -1$ if $b(s)$ has no integral root. Then as a left $D_n$-module, $\pi_* M$ is generated by the set

$$\{t^\gamma [u] \mid \gamma \in \mathbb{N}^d, |\gamma| \leq k_1\}$$

In particular, $\pi_* M = 0$ if $k_1 < 0$. 

Proof

Since $\pi_* M$ is the cokernel of $M^d (\partial_{t_1}, \ldots, \partial_{t_d}) M$, we have an exact sequence

$$M^d (\partial_{t_1}, \ldots, \partial_{t_d}) \xrightarrow{} M \xrightarrow{} \pi_* M \xrightarrow{} 0$$

of left $D_n$-modules. First let us show that the induced sequence

$$F_{k_1+1}(M)^d (\partial_{t_1}, \ldots, \partial_{t_d}) \xrightarrow{} F_{k_1}(M) \xrightarrow{} \pi_* M \xrightarrow{} 0$$

is also exact. Let $k > k_1$ and $u_k \in F_k(M)$ with nonzero modulo class $\bar{u}_k \in \text{gr}_k(M)$. We have $b(\theta - k)\bar{u}_k = 0$. There exists $c(s) \in \mathbb{C}[s]$ such that $b(\theta - k) - b(\theta) = \theta c(\theta)$. Then $b(-k)\bar{u}_k = \theta c(\theta)\bar{u}_k$ holds. Since $b(-k) \neq 0$, this implies that there exist $v_1, \ldots, v_d \in F_{k+1}(M)$ such that

$$u_k - (\partial_{t_1} v_1 + \cdots + \partial_{t_d} v_d) \in F_{k-1}(M).$$
Continuing this argument, we conclude that there exist $v_1, \ldots, v_d \in F_{k+1}$ such that

$$u_k - (\partial_{t_1} v_1 + \cdots + \partial_{t_d} v_d) \in F_{k_1}(M).$$

Now assume that $v_1, \ldots, v_d \in F_k(M)$ with $k > k_1 + 1$. Let $\bar{v}_j$ be the modulo class of $v_j$ in $\text{gr}_k(M)$. Then we have

$$\partial_{t_1} \bar{v}_1 + \cdots + \partial_{t_d} \bar{v}_d = 0 \in \text{gr}_{k_1}(M).$$

We want to show that $\bar{v}_j = 0$ for any $j$. Since this is rather technical, let us show only in case $d = 1$. Since $0 = b(\partial_1 t_1 - k)\bar{v}_1 = 0$ and $\partial_1 t_1 - k = t_1 \partial_1 - k + 1$, we get $b(-k + 1)\bar{v}_1 = c(t_1 \partial_{t_1}) \partial_t \bar{v}_1 = 0$ with some $c(s) \in \mathbb{C}[s]$. Since $b(-k + 1) \neq 0$ by the assumption, $\bar{v}_1 = 0$. 
Now let
\[(D_{n+d})^r \xrightarrow{\psi} D_{n+d} \xrightarrow{\varphi} M \rightarrow 0\]
be a presentation of $M$, where
\[
\varphi(P) = P u \quad (\forall P \in D_{n+d}),
\]
\[
\psi((Q_1, \ldots, Q_r)) = Q_1 P_1 + \cdots + Q_r P_r \quad (\forall Q_1, \ldots, Q_r \in D_{n+d}).
\]
Here we assume that $P_1, \ldots, P_r$ are a $w$-involutive basis of $I = \text{Ann}_{D_{n+d}} u$ with $\text{ord}_w(P_i) = m_i$. This implies that the sequence
\[
\bigoplus_{i=1}^r F_{k-m_i}(D_{n+d}) \xrightarrow{\psi} F_k(D_{n+d}) \xrightarrow{\varphi} F_k(M) \rightarrow 0
\]
is exact. Set $F_k[\mathbf{m}][(D_\pi)^r] := \bigoplus_{i=1}^r F_{k-m_i}(D_\pi)$ with $\mathbf{m} = (m_1, \ldots, m_r)$, and so on.
Then $\psi$ induces homomorphisms

$$\overline{\psi} : (D_\pi)^r \longrightarrow D_\pi,$$

$$\overline{\psi} : F_k[m]( (D_\pi)^r ) := \bigoplus_{i=1}^{r} F_{k-m_i}(D_\pi) \longrightarrow F_k(D_\pi),$$

where $\{ F_k(D_\pi) \}$ denotes the filtration induced by $\{ F_k^w(D_{n+d}) \}$. 
We have a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
F_{k_1+1} \left[ m \right] \left( (D_{n+d})^r \right)^d & \longrightarrow & F_{k_1} \left[ m \right] \left( (D_{n+d})^r \right) & \longrightarrow & F_{k_1} \left[ m \right] \left( (D_{\pi})^r \right) & \longrightarrow & 0 \\
\downarrow (\psi, \ldots, \psi) & & \downarrow \psi & & \downarrow \overline{\psi} & & \\
F_{k_1+1} (D_{n+d})^d & \longrightarrow & F_{k_1} (D_{n+d}) & \longrightarrow & F_{k_1} (D_{\pi}) & \longrightarrow & 0 \\
\downarrow (\phi, \ldots, \phi) & & \downarrow \phi & & \downarrow \pi_* M & & \\
F_{k_1+1} (M)^d & \longrightarrow & F_{k_1} (M) & \longrightarrow & \pi_* M & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \\
0 & & 0 & & 0 & &
\end{array}
\]
where the upper leftmost morphisms send

$$
\begin{pmatrix}
Q_{11} & \cdots & Q_{1r} \\
\vdots & & \vdots \\
Q_{d1} & \cdots & Q_{dr}
\end{pmatrix}
\in F_{k_1+1}[m]((D_{n+d})^r)^d
$$

to

$$
\begin{pmatrix}
Q_{11} & \cdots & Q_{1r} \\
\vdots & & \vdots \\
Q_{d1} & \cdots & Q_{dr}
\end{pmatrix}
\begin{pmatrix}
P_1 \\
\vdots \\
P_r
\end{pmatrix}
\in F_{k_1+1}(D_{n+d})^d,
$$

respectively.
In the commutative diagram, the three horizontal sequences and the two vertical sequences except the rightmost one are exact. This implies that the rightmost vertical sequence is also exact; i.e.,

$$\pi_* M = \operatorname{coker} (\overline{\psi} : F_{k_1}[m]((D_\pi)^r) \longrightarrow F_{k_1}(D_\pi)).$$

Note that

$$F_{k_1}(D_\pi) = \bigoplus_{|\gamma| \leq k_1} t^\gamma D_n, \quad F_{k_1}[m]((D_\pi)^r) = \bigoplus_{i=1}^r \bigoplus_{|\gamma| \leq k_1 - m_i} t^\gamma D_n$$

as left $D_n$-modules. Hence $\psi$ is a homomorphism of free left $D_n$-modules of finite rank, $\operatorname{coker} \psi$ can be explicitly computed by linear algebra over $D_n$. This gives the relations among the generators $\{t^\gamma[u] \mid |\gamma| \leq k_1\}$ of $\pi_* M$. By elimination, we can obtain $\operatorname{Ann}_{D_n}[u]$ so that $D_n[u] \cong D_n/\operatorname{Ann}_{D_n}[u]$ is a left $D_n$-submodule of $\pi_* M$. 
Example (again)

Set $n = d = 1$ and write $x = x_1$, $t = t_1$. Consider

$$M := D_2/(D_2P_1 + D_2P_2) \quad \text{with} \quad P_1 = t - \partial_x, \quad P_2 = t(t-1)(\partial_t - x).$$

Set $u := \overline{1} \in M$ and $F_k(M) := F_k^w(D_n)u$.

Let $\overline{u}$ be the residue class of $u$ in $\text{gr}_0(M)$. Since $0 = \partial_t P_1 \overline{u} = \partial_t t \overline{u}$, the $b$-function of $M$ w.r.t. $w$ is $b(s) = s$. So $k_1 = 0$.

A Gröbner basis of $I := D_2P_1 + D_2P_2$ with respect to a monomial order adapted to $(1, 0, -1, 0)$ is $\{P_1, P_2, P_3\}$ with

$$P_3 = x\partial_x^2 - x\partial_x + 2\partial_x - 1 - \partial_t \partial_x^2 + \partial_t \partial_x.$$

The $w$-order of $P_1, P_2, P_3$ are $1, 2, 0$ respectively.
Hence we have an exact sequence

\[
F_{-2}(D_\pi) \oplus F_{-1}(D_\pi) \oplus F_0(D_\pi) \xrightarrow{\overline{\psi}} F_0(D_\pi) \twoheadrightarrow \pi_* M \twoheadrightarrow 0,
\]

where \(\overline{\psi}\) is induced from the column vector \(^t(P_1 \ P_2 \ P_3)\). Since \(D_\pi \cong D_1[t]\), we have \(F_k(D_\pi) = 0\) for \(k < 0\) and \(F_0(D_\pi) = D_1\). Since operators of the form \(\partial_t P\) with \(P \in D_2\) vanishes in \(D_\pi\),

\[
\overline{\psi}((0 \ 0 \ Q)) = QP_3 = Q(x, \partial_x)(x\partial_x^2 - x\partial_x + 2\partial_x - 1)
\]

holds for any \(Q = Q(x, \partial_x) \in D_1\). This implies

\[
\pi_* M = D_1/D_1(x\partial_x^2 - x\partial_x + 2\partial_x - 1)
\]