Algorithms for $D$-modules, integration, and generalized functions

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Plan of the course

1st lecture **Introduction**: Aim and an example

Chapter 1: Basics of $D$-modules

2nd lecture **Chapter 2**: Gröbner bases in the ring of differential operators

**Chapter 3**: Distributions as generalized functions

3rd lecture **Chapter 4**: $D$-module theoretic integration algorithm

**Chapter 5**: Integration over the domain defined by polynomial inequalities
**Introduction**

A *D*-module is a system of linear (partial or ordinary) differential equations with polynomial (or analytic) coefficients.

There is a special class of *D*-modules which are called **holonomic**, the solution spaces of which are finite dimensional vector spaces.

A **holonomic function** is a differentiable or generalized function which is a solution of a holonomic system. For example, $\exp(f) = e^f$ is a holonomic function for any polynomial $f = f(x_1, \ldots, x_n)$.

**Aim:** To find a holonomic system which the integral of a holonomic function satisfies. The integration is to be performed over a domain defined by polynomial inequalities.
Example

Let us consider the integral

\[ v(x) = \int_0^1 e^{xy} \, dy = \frac{e^x - 1}{x} \]

and assume that we did not know the answer. The integrand \( u(x, y) := e^{xy} \) satisfies a holonomic system

\[ (\partial_x - y)u(x, y) = (\partial_y - x)u(x, y) = 0 \]

with \( \partial_x = \partial / \partial x \) and \( \partial_y = \partial / \partial y \).

In order to apply integration algorithms for holonomic functions, we have to get rid of the boundary \( y = 0, 1 \).
The Heaviside function and the delta function

The Heaviside function $Y(t)$ is defined by $Y(t) = 1$ for $t > 0$ and $Y(t) = 0$ for $t < 0$. (Never mind the value at $t = 0$.)

$Y(t)$ is discontinuous at $t = 0$ and its derivative $Y'(t)$ as a generalized function coincides with Dirac’s delta function $\delta(t)$.

$\delta(t)$ vanishes outside of $t = 0$ and $t\delta(t) = 0$ holds everywhere in $\mathbb{R}$. 

\[ Y(t) \]

\[ \delta(t) \]
Olivier Heaviside (1850–1925)

a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell’s field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis.

an excerpt from Wikipedia
Example (continued)

We rewrite the integral as

\[ v(x) = \int_{-\infty}^{\infty} e^{xy} Y(y) Y(1 - y) \, dy. \]

The new integrand \( \tilde{u}(x, y) := e^{xy} Y(y) Y(1 - y) \) satisfies a holonomic system

\[ y(y - 1)(\partial_y - x)\tilde{u}(x, y) = (\partial_x - y)\tilde{u}(x, y) = 0. \tag{1} \]

In fact one has

\[ y(y - 1)(\partial_y - x)(e^{xy} Y(y) Y(1 - y)) = y(y - 1)e^{xy}(\delta(y) - \delta(y - 1)) = 0. \]
The integration algorithm applied to (1) outputs an answer

\[(x \partial_x^2 - (x - 2) \partial_x - 1) v(x) = 0.\]

By rewriting the differential operator on the left-hand side as

\[x \partial_x^2 - (x - 2) \partial_x - 1 = \partial_x(\partial_x - 1)x,\]

we know that \(v(x) = x^{-1}(C_1 e^x + C_2)\) holds for \(x \neq 0\) with some constants \(C_1, C_2\). We get \(C_1 = 1\) and \(C_2 = -1\) from the fact that \(v(x)\) is continuous at \(x = 0\) with \(v(0) = 1\). Thus we get

\[v(x) = \frac{e^x - 1}{x}.\]
General integrals

For a holonomic function $u(x_1, \ldots, x_n)$, consider the integral

$$v(x_1, \ldots, x_{n-d}) = \int_{D(x_1, \ldots, x_{n-d})} u(x_1, \ldots, x_n) \, dx_{n-d+1} \cdots dx_n,$$

where

$$D(x_1, \ldots, x_{n-d}) := \{(x_{n-d+1}, \ldots, x_n) \in \mathbb{R}^d \mid f_j(x_1, \ldots, x_n) \geq 0 \quad (1 \leq j \leq m)\}$$

with polynomials $f_1, \ldots, f_m$ with real coefficients. We rewrite it as

$$v(x_1, \ldots, x_{n-d}) = \int_{\mathbb{R}^d} u(x_1, \ldots, x_n) Y(f_1) \cdots Y(f_m) \, dx_{n-d+1} \cdots dx_n$$
and apply the $D$-module theoretic integration algorithm. Since generalized functions are involved, we cannot use differential operators with rational function coefficients.

For example, $x\delta(x) = 0$ DOES NOT imply

$$\delta(x) = \frac{1}{x}(x\delta(x)) = \frac{1}{x}0 = 0.$$ 

Hence we must work exclusively with differential operators with polynomial coefficients, that is, in the framework of (algebraic) $D$-modules.

We also need an algorithm to compute a holonomic system for the product $uY(f_1)\cdots Y(f_m)$ as a generalized function.

(The end of Introduction)
1. Basics of $D$-modules

References of Chapter 1

1.1 The ring of differential operators

Let $K$ be a field of characteristic zero. In this course, we assume mostly that $K$ is $\mathbb{C}$, the field of complex numbers.

Let $K[x] := K[x_1, \ldots, x_n]$ be the ring of polynomials in indeterminates $x = (x_1, \ldots, x_n)$ with coefficients in $K$.

A derivation $\theta : K[x] \to K[x]$ is a $K$-linear map that satisfies

$$\theta(fg) = \theta(f)g + f\theta(g) \quad (\forall f, g \in K[x]).$$

The set $\text{Der}_K K[x]$ of the derivations constitutes a $K[x]$-module.
For \( i = 1, \ldots, n \), define a derivation \( \partial_i \) by the partial derivative

\[
\partial_i = \partial_{x_i} : K[x] \ni f \mapsto \frac{\partial f}{\partial x_i} \in K[x].
\]

Then \( \partial_1, \ldots, \partial_n \) are a \( K[x] \)-basis of \( \text{Der}_K K[x] \).

In fact, if \( \theta \in \text{Der}_K K[x] \), then it is easy to see that

\[
\theta = \theta(x_1) \partial_1 + \cdots + \theta(x_n) \partial_n.
\]

Let \( \text{End}_K K[x] \) be the \( K \)-algebra consisting of the \( K \)-linear endomorphisms of \( K[x] \).

The ring \( D_n \) is defined as the \( K \)-subalgebra of \( \text{End}_K K[x] \) that is generated by \( K[x] \) and \( \text{Der}_K K[x] \), or equivalently, by \( x_1, \ldots, x_n \) and \( \partial_1, \ldots, \partial_n \).
An element \( a = a(x) \) of \( K[x] \) is regarded as an element of \( D_n \) as the multiplication \( f \mapsto af \) for \( f \in K[x] \). With this identification, \( D_n \) contains \( K[x] \) as a subring.

\( D_n \) is called the \textit{ring of differential operators} in the variables \( x = (x_1, \ldots, x_n) \) with polynomial coefficients, or, more simply, the \textit{n-th Weyl algebra} over \( K \).

\( D_n \) is a non-commutative \( K \)-algebra. In fact, for \( a \in K[x] \) regarded as an element of \( D_n \), the product in \( D_n \) satisfies

\[
\partial_i a = a \partial_i + \partial_i(a) = a \partial_i + \frac{\partial a}{\partial x_i} \quad (i = 1, \ldots, n).
\]
For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) with \( \mathbb{N} = \{0, 1, 2, \ldots \} \), we use the notation \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \). Then an element \( P \) of \( D_n \) is uniquely written in a finite sum

\[
P = P(x, \partial) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} x^\alpha \partial^\beta \quad (a_{\alpha, \beta} \in K),
\]

which is called the normal form of \( P \).

Introducing commutative indeterminates \( \xi = (\xi_1, \ldots, \xi_n) \) corresponding to \( \partial \), we associate with this \( P \) a polynomial

\[
P(x, \xi) := \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} x^\alpha \xi^\beta \in K[\mathbb{x, \xi}] = K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]
\]

and call it the total symbol of \( P \). Note that \( P \) must be in the normal form when \( \xi \) is substituted for \( \partial \).
By this correspondence, $D_n$ is isomorphic to $K[x, \xi]$ as a $K$-vector space but not as a ring, of course.

The product $R = PQ$ in $D_n$ can be effectively computed by using the Leibniz formula

$$R(x, \xi) = \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} \left( \frac{\partial}{\partial \xi} \right)^\nu P(x, \xi) \cdot \left( \frac{\partial}{\partial x} \right)^\nu Q(x, \xi)$$

where $\nu! = \nu_1! \cdots \nu_n!$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$. 
Example

Set $n = 1$ and write $x = x_1$ and $\partial = \partial_1$. Consider the product $R := \partial^m x^m$ with a non-negative integer $m$. Since the total symbol of $\partial^m$ and $x^m$ are $\xi^m$ and $x^m$ respectively, the Leibniz formula gives the total symbol $R(x, \xi)$ as

$$R(x, \xi) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( \frac{\partial}{\partial \xi} \right)^\nu \xi^m \cdot \left( \frac{\partial}{\partial x} \right)^\nu x^m$$

$$= \sum_{\nu=0}^{m} \frac{1}{\nu!} \left\{ m(m-1) \cdots (m-\nu+1) \right\}^2 \xi^{m-\nu} x^{m-\nu}.$$

This implies

$$\partial^m x^m = \sum_{\nu=0}^{m} \frac{1}{\nu!} \left\{ m(m-1) \cdots (m-\nu+1) \right\}^2 x^{m-\nu} \partial^{m-\nu}. $$
Exercise 1.1. Prove the Leibniz formula.

Exercise 1.2. Set \( n = 1 \) and \( x = x_1, \partial = \partial_1 \). For a positive integer \( m \), prove
\[
x^m \partial^m = x \partial (x \partial - 1) \cdots (x \partial - m + 1).
\]
The $D$-module formalism

Given $P_1, \ldots, P_r \in D_n$, let us consider a system of linear (partial or ordinary) differential equations

$$P_1 u = \cdots = P_r u = 0 \quad (3)$$

for an unknown function $u$. Let $I := D_n P_1 + \cdots + D_n P_r$ be the left ideal of $D_n$ generated by $P_1, \ldots, P_r$. Then (3) is equivalent to

$$P u = 0 \quad (\forall P \in I).$$

Here we suppose that the unknown function $u$ belongs to some ‘function space’ $\mathcal{F}$ which is a left $D_n$-module.
For $\mathcal{F}$ to be a left $D_n$-module, it is necessary that any function $f$ belonging to $\mathcal{F}$ be infinitely differentiable and multiplication $af$ by an arbitrary polynomial $a \in K[x]$ make sense.

Here are examples of ‘function spaces’:

**Example** By the definition, $K[x]$ has a natural structure of left $D_n$-module since $D_n$ is a subalgebra of $\text{End}_K K[x]$. So $K[x]$ has two structures: a subring of $D_n$ and a left $D_n$-module. Hence for $f \in K[x]$ and $P \in D_n$, $Pf$ has two meanings:

- $Pf$ as the product in $D_n$ with $f$ regarded as an element of the subring $K[x]$ of $D_n$.
- $Pf$ as the action of $P$ on the element $f$ of the left $D_n$-module $K[x]$. In other words, we regard $f$ as a function subject to derivations.
This might cause some confusion.

In “Gröbner Deformations of Hypergeometric Differential Equations” by Saito, Sturmfels, Takayama (Springer, 2000), the action of $P$ on an element $f$ of a left $D_n$-module is denoted conspicuously by $P \bullet f$ for distinction, which is new, I guess, both to analysts and algebraists.

I would like to write, only if needed, $Pf = P(f)$ to clarify the action of $P$ on $f$, and $Pf = P \cdot f$ to emphasize that it be the product in $D_n$, following the traditional notation in $D$-module theory.
Example  The field $K(x) = K(x_1, \ldots, x_n)$ of rational functions has a natural structure of left $D_n$-module. For a point $p = (p_1, \ldots, p_n)$ of the affine space $K^n$, the set of regular functions at $p$, i.e. the elements of $K(x)$ whose denominators do not vanish at $p$ also has a natural structure of $D_n$-modules. More generally, for a multiplicative subset $S$ of $K[x]$, the localization $K[x][S^{-1}]$ is also a left $D_n$-module.

Example  Set $K = \mathbb{C}$. Let $C^\infty(U)$ be the set of the complex-valued $C^\infty$ functions on an open set $U$ of the $n$-dimensional real Euclidean space $\mathbb{R}^n$. Then each $\partial_i$ acts on $C^\infty(U)$ as differentiation. This makes $C^\infty(U)$ a left $D_n$-module. Let $C_0^\infty(U)$ be the set of $C^\infty$ functions on $U$ with compact support. More precisely, $f \in C^\infty(U)$ belongs to $C_0^\infty(U)$ if and only if there is a compact subset $K$ of $U$ such that $f(x) = 0$ for any $x \in U \setminus K$. Then $C_0^\infty(U)$ is a left $D_n$-submodule of $C^\infty(U)$. 

Other examples of such $\mathcal{F}$ are the set $\tilde{\mathcal{O}}(U)$ of possibly multi-valued analytic functions on an open subset $U$ of $\mathbb{C}^n$, the set $\mathcal{D}'(U)$ of the Schwartz distributions on an open subset $U$ of $\mathbb{R}^n$, and the set $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, which shall be introduced later, as well as the set $\mathcal{B}(U)$ of the hyperfunctions (of Mikio Sato) on an open subset $U$ of $\mathbb{R}^n$.

Now for a left ideal $I$ of $D_n$, consider the residue module $M := D_n/I$, which is a left $D_n$-module generated by the residue class $\bar{1}$ of $1 \in K[x] \subset D_n$. Fix a left $D_n$-module $\mathcal{F}$ as your favorite function space. A map $\varphi : M \to \mathcal{F}$ is $D_n$-linear, or a $D_n$-homomorphism, if

$$
\varphi(u + v) = \varphi(u) + \varphi(v), \quad \varphi(Pu) = P\varphi(u) \quad (\forall u, v \in M, \forall P \in D_n).
$$
Let $\text{Hom}_{D_n}(M, F)$ be the set of the $D_n$-homomorphisms of $M$ to $F$, which is a $K$-vector space.

Since $M$ is generated by $\overline{1}$ as left $D_n$-module, $\varphi \in \text{Hom}_{D_n}(M, F)$ is uniquely determined by $\varphi(\overline{1})$.

On the other hand, for $\varphi$ to be well-defined, it is necessary and sufficient that $\varphi(\overline{1})$ be annihilated by $I$, i.e., $P\varphi(\overline{1}) = 0$ for any $P \in I$.

In conclusion, we have an isomorphism as $K$-vector space

$$\text{Hom}_{D_n}(M, F) \ni \varphi \sim \varphi(\overline{1}) \in \{ f \in F \mid Pf = 0 \quad (\forall P \in I) \}.$$ 

We started with a left ideal $I$ of $D_n$ generated by given $P_1, \ldots, P_n \in D_n$ and considered a left $D_n$-module $M = D_n/I$. Note that

$$I = \text{Ann}_{D_n} \overline{1} = \{ P \in D_n \mid P\overline{1} = 0 \in M \}.$$
We can argue in the reverse order:
Let $M$ be a finitely generated left $D_n$-module and let $u_1, \ldots, u_m \in M$ be generators of $M$, i.e. assume that for any $u \in M$, there exist $P_1, \ldots, P_m \in D_n$ such that $u = P_1u_1 + \cdots + P_mu_m$. Set

$$N := \{(P_1, \ldots, P_m) \in (D_n)^m \mid P_1u_1 + \cdots + P_mu_m = 0\},$$

which is a left $D_n$-submodule of $M$.

Since $D_n$ is left (and right) Noetherian ring (this can be proved by using a Gröbner basis in $D_n$), $N$ is also finitely generated over $D_n$.

Hence there exist

$$Q_i = (Q_{i1}, \ldots, Q_{im}) \in (D_n)^m \quad (i = 1, \ldots, r)$$

which generate $N$ as left $D_n$-module.
Then we have an exact sequence of left $D_n$-modules

$$(D_n)^r \xrightarrow{\psi} (D_n)^m \xrightarrow{\varphi} M \rightarrow 0,$$  \hspace{1cm} (4)

which is called a presentation of $M$. Here $\varphi$ and $\psi$ are $D_n$-homomorphisms defined by, for $P_i \in D_n$,

$$\varphi((P_1, \ldots, P_m)) = P_1 u_1 + \cdots + P_m u_m,$$

$$\psi((P_1, \ldots, P_r)) = (P_1 \quad \cdots \quad P_r) \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & \ddots & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix}$$

and $N = \ker \varphi = \text{im} \psi$ holds.
From (4) we get an exact sequence

$$0 \longrightarrow \text{Hom}_{D_n}(M, \mathcal{F}) \xrightarrow{\varphi^*} \text{Hom}_{D_n}((D_n)^m, \mathcal{F}) \xrightarrow{\psi^*} \text{Hom}_{D_n}((D_n)^r, \mathcal{F}).$$

Since $\text{Hom}_{D_n}((D_n)^m, \mathcal{F})$ is isomorphic to $\mathcal{F}^m$, this yields

$$0 \longrightarrow \text{Hom}_{D_n}(M, \mathcal{F}) \xrightarrow{\varphi^*} \mathcal{F}^m \xrightarrow{\psi^*} \mathcal{F}^r.$$

Regarding the elements of $\mathcal{F}^m$ as column vectors, we have, for $h \in \text{Hom}_{D_n}(M, \mathcal{F})$ and $f_1, \ldots, f_m \in \mathcal{F}$,

$$\varphi^*(h) = \begin{pmatrix} h(u_1) \\ \vdots \\ h(u_m) \end{pmatrix}, \quad \psi^* \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & \ddots & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$
Hence $\text{Hom}_{D_n}(M, \mathcal{F})$ is isomorphic to the solution space of the system of linear differential equations

$$\sum_{j=1}^{m} Q_{ij} f_j = 0 \quad (i = 1, \ldots, r)$$

for unknown functions $f_1, \ldots, f_m \in \mathcal{F}$. Note that the generators $u_1, \ldots, u_m$ of $M$ also satisfy the same equations in $M$. 
Example Let us consider $K[x]$ as a left $D_n$-module. Since $D_n$ contains $K[x]$ as a subring, $K[x]$ is generated by 1 as a left $D_n$-module. For $P \in D_n$, there exist $Q_1, \ldots, Q_n \in D_n$ and $r(x) \in K[x]$ such that

$$P = Q_1 \partial_1 + \cdots + Q_n \partial_n + r(x).$$

Then $P(1) = r(x)$ vanishes if and only if $r(x) = 0$. This implies $K[x] \cong D_n/(D_n \partial_1 + \cdots + D_n \partial_n)$ and a presentation of $K[x]$ is given by

$$(D_n)^n \xrightarrow{t(\partial_1, \ldots, \partial_n)} D_n \xrightarrow{\varphi} K[x] \rightarrow 0$$

with $\varphi(P) = P(1)$. We have

$$\text{Hom}_{D_n}(K[x], \mathcal{F}) \cong \{f \in \mathcal{F} \mid \partial_1 f = \cdots = \partial_n f = 0\} = K$$

for $\mathcal{F} = K[x], K(x), K[x]_\rho$, or for $\mathcal{F} = \mathcal{C}^\infty(U)$ if $K \subset \mathbb{C}$.

Exercise 1.3 Confirm the formulae above for $\varphi^*$ and $\psi^*$. 
1.3. Weight vector and filtration

A weight vector $w$ for $D_n$ is an integer vector

$$w = (w_1, \ldots, w_n; w_{n+1}, \ldots, w_{2n}) \in \mathbb{Z}^{2n}$$

with the conditions $w_i + w_{n+i} \geq 0$ for $i = 1, \ldots, n$, which are necessary in view of the commutation relation $\partial_i x_i = x_i \partial_i + 1$ in $D_n$. For a nonzero differential operator $P$ of the form (2), we define its $w$-order to be

$$\text{ord}_w(P) := \max\{\langle w, (\alpha, \beta) \rangle = w_1 \alpha_1 + \cdots + w_n \alpha_n + w_{n+1} \beta_1 + \cdots + w_{2n} \beta_n \mid a_{\alpha, \beta} \neq 0\}.$$ 

We set $\text{ord}_w(0) := -\infty$. 

A weight vector \( w \) induces the \( w \)-filtration

\[
F^w_k(D_n) := \{ P \in D_n \mid \text{ord}_w(P) \leq k \} \quad (k \in \mathbb{Z})
\]

on the ring \( D_n \). This filtration satisfies the properties:

\[
F^w_k(D_n) \subset F^w_{k+1}(D_n), \quad \bigcup_{k \in \mathbb{Z}} F^w_k(D_n) = D_n,
\]

\[
1 \in D^w_0(D_n), \quad F^w_k(D_n)F^w_l(D_n) \subset F^w_{k+l}(D_n), \quad \bigcap_{k \in \mathbb{Z}} F^w_k(D_n) = \{0\}.
\]
The \textit{w-graded ring} associated with this filtration is defined to be

\[
\text{gr}^w(D_n) := \bigoplus_{k \in \mathbb{Z}} \text{gr}_k^w(D_n), \quad \text{gr}_k^w(D_n) := F_k^w(D_n)/F_{k-1}^w(D_n).
\]

If \(w_i + w_{n+i} > 0\) holds for all \(i = 1, \ldots, n\), then \(\text{gr}^w(D_n)\) is isomorphic to the polynomial ring \(K[x, \xi]\) as a \(K\)-algebra since in the right-hand side of the Leibniz formula, the terms with \(\nu \geq 1\) are of lower \(w\)-order than the term with \(\nu = 0\).
The *Rees algebra* $R^w(D_n)$ associated with the $w$-filtration is defined by

$$R^w(D_n) := \bigoplus_{k \in \mathbb{Z}} F_k^w(D_n) T^k \subset D_n[T]$$

with an indeterminate $T$. We have isomorphisms

$$R^w(D_n)/(T - 1)R^w(D_n) \cong D_n, \quad R^w(D_n)/TR^w(D_n) \cong \text{gr}(D_n)$$

as $K$-algebra. Note that $D_n$, $\text{gr}^w(D_n)$, and $R^w(D_n)$ are left (and right) Noetherian rings. (This can be proved by using Gröbner bases.)
Let $M$ be a left $D_n$-module. A family $\{F_k(M)\}_{k \in \mathbb{Z}}$ of $K$-subspaces $F_k(M)$ of $M$ is called a $w$-filtration if it satisfies

1. $F_k(M) \subset F_{k+1}(M)$ for all $k \in \mathbb{Z}$;
2. $\bigcup_{k \in \mathbb{Z}} F_k(M) = M$;
3. $F_j^w(D_n)F_k(M) \subset F_{j+k}(M)$ for all $j, k \in \mathbb{Z}$;
4. Each $F_k(M)$ is finitely generated as a left $F_0^w(D_n)$-module.

A $w$-filtration $\{F_k(M)\}$ is called good if it also satisfies

5. there exists $k_1 \in \mathbb{Z}$ such that $F_j^w(D_n)F_{k_1}(M) = F_{j+k_1}(M)$ for any $j \leq 0$.
6. there exists $k_2 \in \mathbb{Z}$ such that $F_j^w(D_n)F_{k_2}(M) = F_{j+k_2}(M)$ for any $j \geq 0$. 
A $w$-filtration $\{F_k(M)\}$ is good if and only if the associated Rees module $R(M) := \bigoplus_{k \in \mathbb{Z}} F_k(M) T^k$ is finitely generated as a left $R^w(D_n)$-module.

For a $w$-filtration $\{F_k(M)\}$, let

$$\text{gr}(M) := \bigoplus_{k \in \mathbb{Z}} \text{gr}_k(M), \quad \text{gr}_k(M) := F_k(M)/F_{k-1}(M)$$

be the associated graded module, which is a left $\text{gr}^w(D_n)$-module. If $\{F_k(M)\}$ is good, then $\text{gr}(M)$ is finitely generated over $\text{gr}^w(D_n)$.

**Exercise 1.4** Prove the $K$-algebra isomorphisms (5).

**Exercise 1.5** Show that a $w$-filtration $\{F_k(M)\}$ of $M$ is good if and only if the associated Rees module is finitely generated over $R^w(D_n)$. 
1.4. Holonomic $D$-module and characteristic variety

Following J. Bernstein, let us define the notion of holonomic system by using the weight vector $(1, 1) = (1, \ldots, ; 1, \ldots, 1) \in \mathbb{Z}^{2n}$. Note that

$$\langle (1, 1), (\alpha, \beta) \rangle = |\alpha| + |\beta| = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n.$$ 

Let $M$ be a finitely generated left $D_n$-module and $\{ F_k(M) \}$ be a good $(1, 1)$-filtration. Then each $F_k(M)$ is a finite dimensional $K$-vector space since $F_0^{(1,1)}(D_n) = K$, and $F_k(M) = \{0\}$ for sufficiently small $k$. 

By considering $\text{gr}(M)$ as a graded $K[x, \xi]$-algebra, we see that there exists a (Hilbert) polynomial $p(k)$ in $k$ such that

$$p(k) = \dim_K F_k(M) = \sum_{j \leq k} \dim_K F_j(M)/F_{j-1}(M) \quad (k \gg 0).$$

The degree of $p(k)$ does not depend on the choice of a good $(1, 1)$-filtration $\{F_k(M)\}$ and is called the *dimension* of the module $M$, which we denote by $\dim M$.

Let $d = \dim M$. Then the leading coefficient $c_d$ of $p(k)$ is a positive rational number such that $d!c_d$ is an integer. We call $\text{mult} M \coloneqq d!c_d$ the *multiplicity* of $M$. 
Theorem (Bernstein’s inequality (around 1970))
\[ \dim M \geq n \text{ holds if } M \neq 0. \]

Definition
A finitely generated left $D_n$-module $M$ is called a holonomic system if $\dim M = n$ or $M = 0$.

Example
Since
\[ \dim_K F_k^{(1,1)}(D_n) = \left( \begin{array}{c} 2n + k \\ 2n \end{array} \right) = \frac{1}{(2n)!} k^{2n} + (\text{lower order terms in } k), \]
\[ \dim D_n = 2n. \]
Example  $K[x]$ is holonomic as a left $D_n$-module. In fact, set

$$F_k(K[x]) = \{ f \in K[x] \mid \deg f \leq k \} \quad (k \in \mathbb{Z}).$$

It is easy to see that

$$F_k^{(1,1)}(D_n)F_l(K[x]) = F_{l+k}(K[x])$$

holds for any integers $j, k \geq 0$. Hence $\{F_k(K[x])\}$ is a good $(1, 1)$-filtration. It follows that $\dim K[x] = n$ from

$$\dim_K F_k(K[x]) = \binom{n + k}{n} = \frac{1}{n!}k^n + \text{(lower order terms in $k$)}.$$
Let us recall another characterization of a holonomic system by using the weight vector \( w = (0, 1) = (0, \ldots, 0; 1, \ldots, 1) \). Let \( M \) be a finitely generated left \( D_n \)-module and \( \{ F_k(M) \} \) be a good \((0, 1)\)-filtration of \( M \). Then \( \text{gr}(M) \) is a finitely generated \( K[x, \xi] \)-module. Its support is an algebraic set of \( K^{2n} \) defined by

\[
\text{Supp \, gr}(M) := \{(p, q) \in K^n \times K^n \mid \text{gr}(M)_{(p, q)} := K[x, \xi]_{(p, q)} \otimes_{K[x, \xi]} \text{gr}(M) = 0\},
\]

where \( K[x, \xi]_{(p, q)} \) denotes the localization of \( K[x, \xi] \) at \((p, q)\), i.e., at the maximal ideal corresponding to \((p, q)\). It can be proved that \( \text{Supp \, gr}(M) \) is independent of a good \((0, 1)\)-filtration \( \{ F_k(M) \} \) of \( M \). \( \text{Supp \, gr}(M) \) is called the characteristic variety of \( M \) and denoted by \( \text{Char}(M) \).
**Theorem (Sato-Kashiwara-Kawai (1973), Gabber (1981))**

Let $K$ be an algebraically closed field of characteristic 0 and $M$ be a finitely generated left $D_n$-module. If $M \neq 0$, then the dimension of each irreducible component of $\text{Char}(M)$ is $\geq n$. More precisely, $\text{Char}(M)$ is an involutive subset of the symplectic manifold (the cotangent bundle) $T^*\mathbb{C}^n = \mathbb{C}^{2n}$.

**Theorem (Björk(1979))**

Under the same assumption as the theorem above, $\dim M$ coincides with the maximum dimension of the irreducible components of $\text{Char}(M)$.

Especially, $M$ is holonomic if and only if the dimension of the characteristic variety is $n$ or else $M = 0$. 
Let $P$ be a nonzero differential operator written in the form

$$P = P(x, \partial) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} x^\alpha \partial^\beta \quad (a_{\alpha, \beta} \in K)$$

and set $m := \text{ord}_{(0,1)}(P)$. Then the principal symbol of $P$ is the polynomial defined by

$$\sigma(P)(x, \xi) = \sum_{|\beta|=m} \sum_{\alpha} a_{\alpha, \beta} x^\alpha \xi^\beta.$$

This can be identified with the residue class of $P$ in $\text{gr}^{(0,1)}(D_n) \cong K[x, \xi]$. Note that $\sigma(P)(x, \xi)$ is homogeneous with respect to $\xi$. 

If $M := D_n/I$ with a left ideal $I$ of $D_n$, then we have, by the definition,

$$\text{Char}(M) = \{(x, \xi) \in K^{2n} \mid \sigma(P)(x, \xi) = 0 \text{ for any } P \in I \setminus \{0\}\}.$$

In particular, if $I$ is generated by $P_1, \ldots, P_r$, then we have

$$\text{Char}(M) \subset \{(x, \xi) \in K^{2n} \mid \sigma(P_i)(x, \xi) = 0 \ (\forall i = 1, \ldots, r)\}.$$
Let $\pi : K^{2n} \ni (x, \xi) \mapsto x \in K^n$ be the projection. Then the singular locus of $M$

$$\text{Sing}(M) := \pi(\text{Char}(M) \setminus K^n \times \{0\})$$

is an algebraic set of $K^n$ since $\text{gr}(M)$ is homogeneous with respect to $\xi$. In particular, if $M$ is holonomic, then $\text{Sing}(M)$ is an algebraic set of codimension $\geq 1$, or an empty set, since $\text{Char}(M)$ is homogeneous with respect to $\xi$. 
Example Let $P \in D_1$ with $P \neq 0$; i.e., let $P$ be a linear ordinary differential operator with coefficients in $K[x] = K[x_1]$. Then $P$ is written in the form

$$P = a_m(x) \partial^m + \cdots + a_0(x) \quad (a_i(x) \in K[x], a_m(x) \neq 0).$$

Set $M = D_1/DP_1$. Then the dimension of the algebraic set

$$\text{Char}(M) = \{(x, \xi) \in K^2 \mid \sigma(P)(x, \xi) = a_m(x)\xi^m = 0\}$$

$$= \{(x, \xi) \in K^2 \mid a_m(x) = 0\} \cup \{(x, 0) \mid x \in K\}$$

is one. Hence $M$ is holonomic. $\text{Sing}(M) = \{x \in K \mid a_m(x) = 0\}$. 

Toshinori Oaku (Tokyo Woman's Christian U) Algorithms for $D$-modules, integration, and go
Example  As a left $D_n$-module, $K[x] \cong D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$.
Since $\sigma(\partial_i) = \xi_i$, we have

$$\text{Char}(K[x]) \subset \{(x, \xi) \in K^{2n} \mid \xi = 0\},$$

and it follows that $\text{Sing}(K[x]) = \emptyset$.
Since the dimension of $\text{Char}(K[x])$ is $\geq n$, we have

$$\text{Char}(K[x]) = \{(x, \xi) \in K^{2n} \mid \xi = 0\}.$$

This can be also proved as follows:
If $P \in D_n\partial_1 + \cdots + D_n\partial_n$, then $\sigma(P)(x, \xi)$ belongs to the ideal of $K[x, \xi]$ generated by $\xi_1, \ldots, \xi_n$. 
Exercise 1.5. Let $K = \mathbb{C}$ and let $f(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. Consider a $C^\infty$ function $e^{f(x)}$ on $\mathbb{R}^n$. Set $f_i = \partial_i(f)$ and

$$I := D_n(\partial_1 - f_1) + \cdots + D_n(\partial_n - f_n).$$

1. Show that $I = \text{Ann}_{D_n} e^{f(x)} := \{ P \in D_n \mid Pe^{f(x)} = 0 \}$.
2. Show that $\text{Hom}_{D_n}(M, C^\infty(\mathbb{R}^n)) \cong Ke^{f(x)}$.
3. Show that $M := D_n/I$ is holonomic by using the two definitions respectively.