A naive approach to microlocal analysis of Feynman amplitudes

Toshinori Oaku
Department of Mathematics, Tokyo Woman's Christian University

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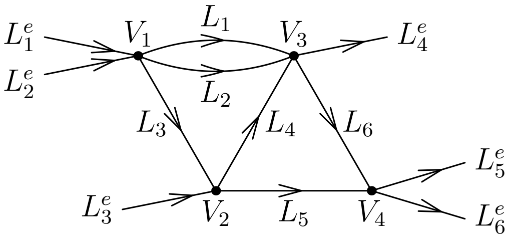
Feynman diagrams and Feynman integrals

Let G be a connected Feynman graph (diagram), i.e., G consists of

- vertices $V_1, \cdots, V_{n'}$,
- oriented line segments $L_1, \ldots, L_N$ called internal lines,
- oriented half-lines $L_1^e, \ldots, L_n^e$ called external lines.

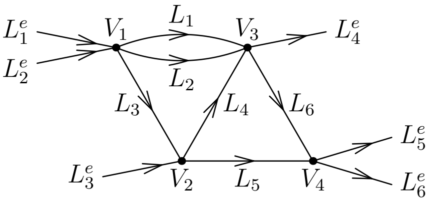
The end-points of each internal line L_l are two distinct vertices, and each external line has only one end-point, which coincides with one of the vertices.



• We associate ν-dimensional vector $\mathbf{p}_r$ to each external line L_r^e $(1 \leq r \leq n')$,
and ν-dimensional vector $\mathbf{k}_l$ and a real number (mass) $m_l \geq 0$ to each internal line L_l $(1 \leq l \leq N)$.

• For a vertex V_j and an internal or external line L_l, the incidence number $[j : l]$ is defined as follows:

$$[j : l] = 1 \text{ if } L_l \text{ ends at } V_j,$$
$$[j : l] = -1 \text{ if } L_l \text{ starts from } V_j,$$
$$[j : l] = 0 \text{ otherwise.}$$



The Feynman integral associated with G is defined to be

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n'} \delta\Big(\sum_{r=1}^{n} [j:r]\mathbf{p}_r + \sum_{l=1}^{N} [j:l]\mathbf{k}_l\Big)}{\prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l.$$

Here δ denotes the ν-dimensional delta function,

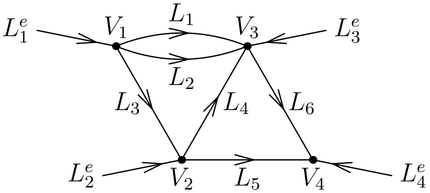
$$\mathbf{k}_l^2 := k_{l0}^2 - k_{l1}^2 - \cdots - k_{l\nu}^2,$$

$d^\nu \mathbf{k}_l$ is the ν-dimensional volume element,
and $(\cdots + \sqrt{-1}\,0)$ means the limit $(\cdots + \sqrt{-1}\,\varepsilon)$ as $\varepsilon \to +0$.

The integrand is well-defined as a generalized function at least if all $m_l > 0$.

In what follows, we assume that G is external, i.e., for each vertex V_j, there exists a unique external line, which we may assume to be L_j^e, that ends at V_j and that no external line starts from V_j. Then $n = n'$ holds and the Feynman integral is

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n} \delta\Big(\mathbf{p}_j + \sum_{l=1}^{N} [j:l]\mathbf{k}_l\Big)}{\prod_{l=1}^{N}(\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l$$



Rewriting the Feynman integral

The delta factors of the integrand of the Feynman integral correspond to the linear equations (momentum preservation)

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \le j \le n)$$

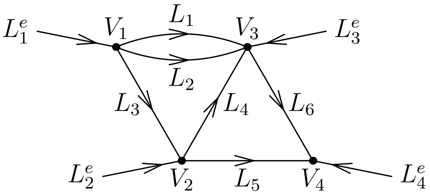
for indeterminates p_j and k_l which correspond to the vectors $\mathbf{p}_j$ and $\mathbf{k}_l$. These equations define an N-dimensional linear subspace of $\mathbb{R}^{n+N}$, which is contained in the hyperplane $p_1 + \cdots + p_n = 0$ since $\sum_{j=1}^{n} [j : l] = 0$.

Lemma

Let A be the $n \times N$ matrix whose (j, l)-element is $[j : l]$. Then the rank of A is $n - 1$.

One can prove this lemma by induction on n.
For the example below, the matrix A is given by

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



In view of the lemma above, we can choose a set of indices

$$J = \{l_1, \dots, l_{N-n+1}\} \subset \{1 \dots, N\}$$

and integers a_{lr} and b_{lj} so that

$$k_l = \sum_{r=1}^{n-1} a_{lr} p_r + \sum_{j=1}^{N-n+1} b_{lj} k_{l_j} = \psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}})$$

$$(l \in J^c := \{1, \dots, N\} \setminus J).$$

More precisely, the system

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \le j \le n)$$

of linear equations is equivalent to

$$\sum_{j=1}^{n} p_j = 0, \quad k_l - \psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}}) = 0 \quad (l \in J^c).$$

In particular, the matrix (a_{lr}) is non-singular.

Then the Feynman integral is written in the form

$$
\begin{aligned}
F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = &\int_{\mathbb{R}^{N\nu}} \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \\
&\times \prod_{l \in J^c} \delta(\mathbf{k}_l - \psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})) \\
&\times \prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \prod_{l=1}^{N} d\mathbf{k}_l \\
= &\,\delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})
\end{aligned}
$$

with the amplitude function

$$
\begin{aligned}
\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) = &\int_{\mathbb{R}^{(N-n+1)\nu}} \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \\
&\times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \prod_{l \in J} d\mathbf{k}_l.
\end{aligned}
$$

Under the condition that $m_l > 0$, the integrand

$$\Psi(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}) = \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\, 0)^{-1} \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}\, 0)^{-1}$$

is well-defined as a hyperfunction on $\mathbb{R}^{\nu N}$ defined by the boundary value of the holomorphic function

$$\Phi(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}) = \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2)^{-1} \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1}$$

on $\mathbb{R}^{\nu N} + \sqrt{-1}\Gamma 0$ with the convex cone Γ generated by vectors $d\mathbf{k}_l^2$ $(l \in J)$ and $d\psi_l^2$ $(l \in J^c)$, which are lineary independent over $\mathbb{R}$.

The annihilator of the integrand as a hyperfunction

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in the variables $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$, $\mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}$. In view of the injectivity of the boundary value map, the annihilator of the hyperfunction Ψ coincides with that of the rational function Φ:

$$\mathrm{Ann}_{D_{\nu N}}\Psi = \mathrm{Ann}_{D_{\nu N}}\Phi$$

and hence it is computable.

Integrals of microfunctions

We use the notation $x = (x', x'')$ with $x' = (x_1, \ldots, x_{n-d})$ and $x'' = (x_{n-d+1}, \ldots, x_n)$ for the coordinate of the base space $\mathbb{R}^n$, and $\xi = (\xi', \xi'')$ for the contangential coordinate. Let $\pi : \mathbb{R}^n \ni x \longmapsto x' \in \mathbb{R}^{n-d}$ be the natural projection. Let

$$\varpi : \mathbb{R}^n \underset{\mathbb{R}^m}{\times} \sqrt{-1}T^*\mathbb{R}^{n-d} \ni (x, \sqrt{-1}\langle \xi', dx' \rangle) \longmapsto (x', \sqrt{-1}\langle \xi', dx' \rangle) \in \sqrt{-1}T^*\mathbb{R}^{n-d}$$

be the natural projection induced by π and

$$\rho : \mathbb{R}^n \underset{\mathbb{R}^m}{\times} \sqrt{-1}T^*\mathbb{R}^{n-d} \ni (x, \sqrt{-1}\langle \xi', dx' \rangle) \longmapsto (x, \sqrt{-1}\langle \xi', dx' \rangle) \in \sqrt{-1}T^*\mathbb{R}^{n}$$

be the natural inclusion. Let $\mathcal{C}_{\mathbb{R}^n}$ and $\mathcal{C}_{\mathbb{R}^{n-d}}$ be the sheaves of microfunctions on $\sqrt{-1}T^*\mathbb{R}^n$ and on $\sqrt{-1}T^*\mathbb{R}^{n-d}$ respectively.

Then the integration along the fibers of $\pi : \mathbb{R}^n \ni x \longmapsto x' \in \mathbb{R}^{n-d}$ is defined as a sheaf homomorphism

$$\pi_* \,:\, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n} \longrightarrow \mathcal{C}_{\mathbb{R}^{n-d}}$$

according to Sato-Kawai-Kashiwara (1973). In particular, for any open set W of $\sqrt{-1}\,T^*\mathbb{R}^{n-d}$, there exists a homomorphism

$$\pi_* \,:\, \Gamma\big(W,\, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n}\big) \ni u(x) \longmapsto \int_{\mathbb{R}^d} u(x)\, dx'' \in \Gamma\big(W,\, \mathcal{C}_{\mathbb{R}^{n-d}}\big).$$

Moreover, it is a homomorphism of left D_{n-d}-modules, where D_{n-d} is the ring of differential operators with polynomial coefficients in x'.

Lemma

Let W be an open set of $\sqrt{-1}\,T^*\mathbb{R}^{n-d}$ and u be an element of $\Gamma\big(W,\, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n}\big)$. Then the integral $\int_{\mathbb{R}^d} \partial_{x_j} u(x)\, dx'' \in \Gamma(W, \mathcal{C}_{\mathbb{R}^{n-d}})$ vanishes for any $n-d+1 \leq j \leq n$.

Proof: I adopt a concrete definition in terms of defining functions following Kashiwara-Kawai-Kimura and A. Kaneko. Let $x_* = (x_0', \sqrt{-1}\langle \xi_0', dx' \rangle)$ be a point of W. We may assume that W is a sufficiently small neighborhood of p'. We may assume that u is the spectrum of the hyperfunction defined as the boundary value of a holomorphic function $F(z)$ on $(U \times \mathbb{R}^d) + \sqrt{-1}V0$ where U is an open neighborhood of x_0' in $\mathbb{R}^{n-d}$ and V is an open convex cone of $\mathbb{R}^n$ such that

$$V^\circ := \{\eta \in \mathbb{R}^n \mid \langle y, \eta \rangle \geq 0 \ (\forall y \in V)\} \subset \{(\eta', \eta'') \mid \langle x_0', \eta' \rangle > 0\}.$$

By the assumption that u belong to $\Gamma(W, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n})$, there exists $R > 0$ such that $F(z)$ continues analytically to $U \times (\mathbb{R}^d \setminus (-R, R)^d)$.

Then (e.g.) $\int_{\mathbb{R}^d} \partial_{x_n} u(x)\, dx''$ is the spectrum of the boundary value $G(x' + \sqrt{-1}V'0)$ of

$$
\begin{aligned}
G(z') &= \int_{[-R,R]^d} \partial_{x_n} F(z', x'')\, dx'' \\
&= \int_{[-R,R]^{d-1}} F(z', x_{n-d+1}, \dots, x_{n-1}, R)\, dx_{n-d+1} \cdots dx_{n-1} \\
&\quad - \int_{[-R,R]^{d-1}} F(z', x_{n-d+1}, \dots, x_{n-1}, -R)\, dx_{n-d+1} \cdots dx_{n-1}
\end{aligned}
$$

with $V' = V \cap (\mathbb{R}^{n-d} \times \{0\})$. (Note that $V' \neq \emptyset$ by the assumption.) Hence $G(z')$ is real analytic on U. This completes the proof.

Now let D_n and D_{n-d} be the rings of differential operators with polynomial coefficients in $x = (x_1, \ldots, x_n)$ and in $x' = (x_1, \ldots, x_{n-d})$ respectively. Then the following is an immediate consequence of the preceding lemma:

Proposition

Let u be an element of $\Gamma(W, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n})$ and let I be a left ideal of D_n such that $Pu = 0$ for any $P \in I$. Let Q be an element of

$$(\partial_{x_{n-d+1}} D_n + \cdots + \partial_{x_n} D_n + I) \cap D_{n-d}.$$

Then Q annihilates $\int_{\mathbb{R}^d} u(x)\, dx''$ as microfunction on W. More generally, the integration induces a linear map

$$\mathrm{Hom}_{D_n}(M, \Gamma(W, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^n})) \longrightarrow \mathrm{Hom}_{D_{n-d}}(M', \Gamma(W, \mathcal{C}_{\mathbb{R}^{n-d}}))$$

with $M' = M/(\partial_{x_{n-d+1}} M + \cdots + \partial_{x_n} M)$.

There is an algorithm to compute the 'integration module' M' if a presentation of M is given.

Feynman amplitudes as microfunctions

As was pointed out by Sato-Kawai-Kashiwara in the 1970's, the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ associated with an external diagram G with positive masses is well-defined as a microfunction on the set

$$\sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\bigl(\Lambda(G) \setminus \Lambda_+(G)\bigr)$$

and its support (analytic wave-front set) is contained in $\varpi(\Lambda_+(G))$. These sets are called Landau-Nakanishi varieties and defined as follows:

We set

$$\Lambda(G) = \{(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}; \sqrt{-1}(\langle \mathbf{u}_1, d\mathbf{p}_1\rangle + \cdots + \langle \mathbf{u}_{n-1}, d\mathbf{p}_{n-1}\rangle) \in \mathbb{R}^{\nu N} \underset{\mathbb{R}^{\nu(n-1)}}{\times} \sqrt{-1}T^*\mathbb{R}^{\nu(n-1)}$$

$$\mid \exists \alpha_l \geq 0 \ (1 \leq l \leq N) \text{ such that}$$

$$\alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \leq j \leq N - n + 1), \tag{1}$$

$$\alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \tag{2}$$

$$\alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj}\psi_l = 0 \ (1 \leq j \leq N - n + 1), \tag{3}$$

$$\mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr}\psi_l \ (1 \leq r \leq n - 1)\} \tag{4}$$

with

$$\psi_l = \sum_{r=1}^{n-1} a_{lr}\mathbf{p}_r + \sum_{j=1}^{N-n+1} b_{lj}\mathbf{k}_{l_j},$$

$$\langle \mathbf{u}, d\mathbf{p}\rangle = u_1 dp_1 - u_2 dp_2 - \cdots - u_\nu dp_\nu,$$

$$\begin{aligned}\Lambda_+(G) = \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}; \sqrt{-1}(\langle \mathbf{u}_1, d\mathbf{p}_1) \rangle + \cdots \\ + \langle \mathbf{u}_{n-1}, d\mathbf{p}_{n-1} \rangle) \in \mathbb{R}^{\nu N} \underset{\mathbb{R}^{\nu(n-1)}}{\times} \sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)}\end{aligned}$$

$$\begin{aligned}&\mid \exists \alpha_l > 0 \ (1 \le l \le N) \text{ such that} \\ &\alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \le j \le N - n + 1), \\ &\alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\ &\alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \le j \le N - n + 1), \\ &\mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \le r \le n - 1)\}.\end{aligned}$$

ϖ is the projection

$$\begin{aligned}\varpi \ : \ \mathbb{R}^{\nu N} \underset{\mathbb{R}^{\nu(n-1)}}{\times} \sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \longmapsto T^*\mathbb{R}^{\nu(n-1)} \\ = \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}; \sqrt{-1}\,(\langle \mathbf{u}_1, d\mathbf{p}_1 \rangle + \cdots + \langle \mathbf{u}_{n-1}, d\mathbf{p}_{n-1}))\}.\end{aligned}$$

Proof: Set $W = \sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi(\Lambda(G) \setminus \Lambda_+(G))$. Let

$$\rho : \mathbb{R}^{\nu N} \underset{\mathbb{R}^{\nu(n-1)}}{\times} \sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \longrightarrow \sqrt{-1}T^*\mathbb{R}^{\nu N}$$

be the natural inclusion. Let $\mathrm{S.S.}\,\Psi$ be the singular spectrum of the integrand Ψ of the Feynman amplitude as a hyperfunction. Then it is easy to see that its singular spectrum $\mathrm{S.S.}\,\Psi$ satisfies

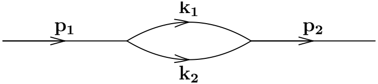
$$\rho^{-1}(\mathrm{S.S.}\,\Psi) \subset \Lambda(G).$$

If $\forall \alpha_l > 0$ and $\mathbf{u}_r$, $\mathbf{p}_r$ $(1 \le r \le n-1)$ are given, then $\alpha_l \psi_l$ and $\alpha_{l_j}\mathbf{k}_{l_j}$ are uniquely determined by (4) and (3) since the matrix (a_{lr}) is non-singular; then α_l are determined by (1) and (2). Thus $\mathbf{k}_{l_j}$ (and ψ_l) are uniqueley determined. This implies that

$$\mathrm{sp}\,\Psi \in \Gamma(W,\, \varpi_! \rho^{-1}\mathcal{C}_{\mathbb{R}^{\nu N}}).$$

Hence $\tilde{F}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = \int_{\mathbb{R}^{\nu(N-n+1)}} \mathrm{sp}\,\Psi \, \prod_{l \in J} d\mathbf{k}_l$ is well-defined as a microfunction on W.

For example, for the graph G below



$$\Lambda(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0,$$
$$\alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2(\mathbf{p}_1 - \mathbf{k}_1), \quad \exists \alpha_1, \alpha_2 \geq 0\}$$
$$\Lambda_+(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0,$$
$$\alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2(\mathbf{p}_1 - \mathbf{k}_1), \quad \exists \alpha_1, \alpha_2 > 0\},$$

from which, we can confirm that

$$\varpi(\Lambda(G) \setminus \Lambda_+(G)) = \{(\mathbf{p}_1, \sqrt{-1}\,\langle \mathbf{u}_1, d\mathbf{p}_1 \rangle) \mid \mathbf{u}_1 = \mathbf{0}\},$$
$$\varpi(\Lambda_+(G)) = \{(\mathbf{p}_1, \sqrt{-1}\,\langle \mathbf{u}_1, d\mathbf{p}_1 \rangle) \mid \mathbf{p}_1^2 - (m_1 + m_2)^2 = 0,\ \mathbf{u} = \alpha \mathbf{p}_1,\ \alpha > 0\}.$$

This implies, in particular, that the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as an element of $\mathcal{B}(\mathbb{R}^\nu)/\mathcal{A}(\mathbb{R}^\nu)$.

Local cohomology

In general, let $f_1, \ldots, f_d$ be polynomials in the variables $x = (x_1, \ldots, x_n)$ with complex coefficients such that the variety

$$Y = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_d(x) = 0\}$$

is d-codimensional, i.e., $f_1, \cdots, f_d$ are of complete intersection. Then the (algebraic) d-th local cohomology group associated with $f_1, \ldots, f_d$ is defined to be the quotient space

$$H^d_{[Y]}(\mathbb{C}[x]) := \mathbb{C}[x, f^{-1}] / \sum_{k=1}^{d} \mathbb{C}[x, (f/f_k)^{-1}]$$

with $f = f_1 \cdots f_d$. It consists of the cohomology classes $[g/f^\nu]$ with $\nu = 1, 2, 3, \ldots$ and $g \in \mathbb{C}[x]$.

• $H^d_{[Y]}(\mathbb{C}[x])$ has a natural structure of left D_n-module and is holonomic as such.

• The simplest example of the local cohomology group is

$$H^1_{[\{0\}]}(\mathbb{C}[x]) = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$$

with $x = x_1$ (one variable), which is spanned by the classes $[x^{-k}]$ with $k = 1, 2, 3, \dots$ as a $\mathbb{C}$-vector space. As a left D_1-module, it is generated only by $[x^{-1}]$ since $\partial_x^k [x^{-1}] = (-1)^k k! [x^{-k-1}]$.

• There are algorithms (U. Walther, Oaku-Takayama) to compute the local cohomology group as a D-module, in particular, the annihilator (the holonomic system) for each cohomology class.

Local cohomology and integrands of Feynman amplitudes

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-1}}$. We regard the integrand

$$\Psi = \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}\,0)^{-1}$$

of the Feynman amplitude as a hyperfunction. Let Φ be the corresponding rational function

$$\Phi = \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2)^{-1} \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1}.$$

Set

$$Y := \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}) \in \mathbb{C}^{\nu N} \mid \mathbf{k}_l^2 - m_l^2 = 0 \quad (l \in J), \quad \psi_l^2 - m_l^2 = 0 \quad (l \in J^c)\}$$

and

$$B_G := H^N_{[Y]}(\mathbb{C}[\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}]).$$

We denote by $[\Phi]$ the modulo class of Φ in B_G.

Propositon

Let $P \in D_{\nu N}$ be an element of $\mathrm{Ann}_{D_{\nu N}}[\Phi]$. Then $P(\mathrm{sp}\,\Psi) = 0$ holds as an element of $\Gamma(W, \varpi_! \rho^{-1} \mathcal{C}_{\mathbb{R}^{\nu N}})$ with $W = \sqrt{-1}\, T^* \mathbb{R}^{\nu(n-1)} \setminus \varpi(\Lambda(G) \setminus \Lambda_+(G))$.

Note that $\mathrm{Ann}_{D_{\nu N}}[\Phi]$ is strictly larger than $\mathrm{Ann}_{D_{\nu N}} \Phi$.

Proof: By the definition, $P\Phi$ is written in the form

$$P\Phi = \sum_{j=1}^{n-1} \prod_{l\in J, l\neq j} a_j(\mathbf{k}_l^2 - m_l^2)^{-d_{jl}} \prod_{l\in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-d_{jl}}$$
$$+ \sum_{j\in J^c} \prod_{l\in J} b_j(\mathbf{k}_l^2 - m_l^2)^{-e_{jl}} \prod_{l\in J^c, l\neq j} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-e_{jl}}$$

with polynomials a_j, b_j and nonnegative integers d_{jl}, e_{jl}. It follows that

$$\rho^{-1}(\mathrm{S.S.}\, P\Psi) \subset \Lambda(G) \setminus \Lambda_+(G) \subset \varpi^{-1}(\varpi(\Lambda(G) \setminus \Lambda_+(G)))$$
$$= \varpi^{-1}(\sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \setminus W).$$

This implies that $P(\mathrm{sp}\, \Psi)$ vanishes as an element of $\Gamma(W, \varpi_! \rho^{-1}\mathcal{C}_{\mathbb{R}^{\nu N}})$. □
Hence we get

Theorem

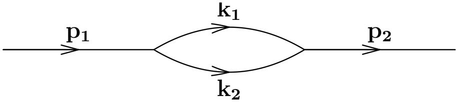
The Feynman amplitude $\tilde{F}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ is a solution of the integration module of the local cohomology B_G as a microfunction on W.

Examples

We present the computation of the integration module of the local cohomology associated with the integrand of the Feynman amplitude for some simple Feynman diagrams.

Example 1

Let us study the Feynman diagram G below:



Then the Feynman integral is written in the form

$$\begin{aligned} F_G(\mathbf{p}_1, \mathbf{p}_2) &= \int_{\mathbb{R}^4} \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) \\ &\quad \times (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2 \\ &= \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1) \end{aligned}$$

with the amplitude

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1.$$

In view of the invariance under Lorentz transformations, let us set $\mathbf{p}_1 = (x, 0, \ldots, 0)$.

In case $\nu = 2$, the integration ideal of the annihilator of the local cohomology class

$$[(k_{10}^2 - k_{11}^2 - m_1^2)^{-1}((x - k_{10})^2 - k_{11}^2 - m_2^2)^{-1}]$$

is generated by

$$(x-m_1-m_2)(x-m_1+m_2)(x+m_1-m_2)(x+m_1+m_2)\partial_x+2x(x^2-m_1^2-m_2^2).$$

The solutions (the kernel) of this operator are constant multiples of

$$(x-m_1+m_2)^{-1/2}(x+m_1-m_2)^{-1/2}(x+m_1+m_2)^{-1/2}(x-m_1-m_2)^{-1/2}.$$

In case of $\nu = 4$, by using the 3-dimensional polar coordinates, we have

$$\begin{aligned}\tilde{F}_G(\mathbf{p}_1) &= \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\,d\mathbf{k}_1 \\ &= 2\pi \int_{\mathbb{R}^2} (k_{10}^2 - r^2 - m_1^2 + \sqrt{-1}\,0)^{-1} \\ &\qquad \times ((x - k_{10})^2 - r^2 - m_2^2 + \sqrt{-1}\,0)^{-1} r_+^2\,dk_{10}dr.\end{aligned}$$

The integration ideal of the annihilator of the cohomology class

$$[r^2(k_{10}^2 - r^2 - m_1^2)^{-1}((x - k_{10})^2 - r^2 - m_2^2)^{-1}]$$

is generated by

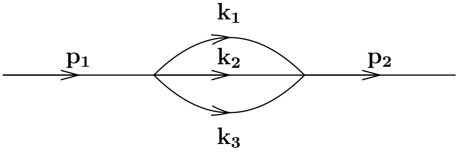
$$\begin{aligned}x(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x \\ - 2((m_1^2 + m_2^2)x^2 - m_1^4 + 2m_2^2 m_1^2 - m_2^4).\end{aligned}$$

The solutions (the kernel) of this operator are constant multiples of

$$x^{-2}(x - m_1 + m_2)^{1/2}(x + m_1 - m_2)^{1/2}(x + m_1 + m_2)^{1/2}(x - m_1 - m_2)^{1/2}.$$

Example 2

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)$$

with

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^4} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}\,0)^{-1} \times ((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2.$$

We work in the 2-dimensional space-time ($\nu = 2$) and compute holonomic systems for $\tilde{F}_G((x, 0))$ by assigning some special values to m_1, m_2, m_3 since the computation for general m_1, m_2, m_3 (as parameters) is intractable.

First let us set $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ so that $(-m_1 + m_2 + m_3)^2$, $(m_1 - m_2 + m_3)^2$, $(m_1 + m_2 - m_3)^2$ are distinct. Then $\tilde{F}_G((x, 0))$ is annihilated by the differential operator

$$
\begin{aligned}
&30x(x-1)(x+1)(x-3)(x+3)(x-5)(x+5)(x-7)(x+7)\underline{\partial_x^3} \\
&\quad + (-2x^{12} + 191x^{10} - 5340x^8 + 35954x^6 + 273082x^4 \\
&\qquad\qquad - 2071305x^2 + 661500)\underline{\partial_x^2} \\
&\quad + (-10x^{11} + 675x^9 - 12108x^7 + 15454x^5 + 936462x^3 \\
&\qquad\qquad - 2692665x)\underline{\partial_x} \\
&\quad - 8x^{10} + 372x^8 - 3300x^6 - 36028x^4 + 457932x^2 - 356760.
\end{aligned}
$$

The singular points $x = 0, \pm 1, \pm 3, \pm 5, \pm 7$ are all regular and the indicial equations are all $s^2(s-1)$.

Next set $m_1 = m_2 = m_3 = 1$. Then $\tilde{F}_G((x,0))$ is annihilated by

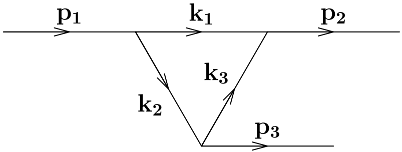
$$x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2.

See Adams-Bogner-Weinzierl (2015) for complete computation with arbitrary m_1, m_2, m_3 by a different (and more efficient) method.

Example 3

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2) = \int_{\mathbb{R}^\nu} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1} \times ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1.$$

We work in the 2-dimensional space-time ($\nu = 2$). In view of the invariance under Lorentz transformation, we set $\mathbf{p}_1 = (x, 0)$, and $\mathbf{p}_2 = (y, z)$.

First let us set $m_1 = m_2 = m_3 = 1$. We compute the integration ideal I of the annihilator of the integrand regarded as a local cohomology class. Since I is too complicated (with tens of generators), we will present only the characteristic variety of $M := D_3/I$.

The characteristic variety of $M := D_3/I$ is given by

$$\mathrm{Char}(M) = T^*_{\{f=0\}}\mathbb{C}^3 \cup T^*_{\{x=0\}}\mathbb{C}^3 \cup T^*_{\{x=f_0=0\}}\mathbb{C}^3 \cup T^*_{\{x=y^2-z^2-4=0\}}\mathbb{C}^3 \cup T^*_{\{x=y+z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3$$

with

$$\begin{aligned} f(x,y,z) &= (y-z)(y+z)x^2 - 2(y-z)(y+z)yx \\ &\qquad + (y-z)^2(y+z)^2 + 4z^2, \\ f_0(y,z) &= f(0,y,z), \end{aligned}$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$.

Only the first component $T^*_{\{f=0\}}\mathbb{C}^3$ should be physically significant. Especially, it roughly means that the Feynman amplitude is analytic outside the surface $f = 0$, which has rather complicated singularities.

The decomposition of $\mathrm{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir for prime and primary decomposition of polynomial ideals developed by M. Noro.
He also computed a primary decomposition of the symbol ideal of I, which enabled us to compute the multiplicity of each component of $\mathrm{Char}(M)$.
Thus the characteristic cycle, i.e., the characteristic variety with multiplicity of each component, of M is

$$T^*_{\{f=0\}}\mathbb{C}^3 + 2\,T^*_{\{x=0\}}\mathbb{C}^3 + T^*_{\{x=f_0=0\}}\mathbb{C}^3 + T^*_{\{x=yz-4=0\}}\mathbb{C}^3 + T^*_{\{x=y+z=0\}}\mathbb{C}^3 + T^*_{\{x=y-z=0\}}\mathbb{C}^3 + 2\,T^*_{\{x=y=z=0\}}\mathbb{C}^3.$$

Singularities of the surface $f = 0$

Let us investigate the singularities of the complex surface

$$Z = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\},$$
$$f = (y - z)(y + z)x^2 - 2(y - z)(y + z)yx + (y - z)^2(y + z)^2 + 4z^2.$$

Following N. Honda and T. Kawai, we rewrite f as

$$f = yzx^2 - yz(y + z)x + y^2z^2 + (y - z)^2$$

by change of coordinates $(y + z, y - z) \to (y, z)$.
Then the singular locus (the set of the singular points) of Z is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$.
The projection $Z \ni (x, y, z) \mapsto (y, z)$ defines a doube covering on $\{(x, y) \mid xy \neq 0\}$ branched along the union of curves $y - z = 0$ and $yz - 4 = 0$.

The stratification of Z with respect to the (local) b-function $b_{f,p}(s)$ of f at a point p is

strata	$b_{f,p}(s)$
$\{(0,0,0)\}$	$(s+1)^3(2s+3)$
$\{(2,0,0),(-2,0,0),(2,2,2),(-2,-2,-2)\}$	$(s+1)^2(2s+3)$
$\{x=y=z\} \cup \{y=z=0\}$ $\setminus \{(0,0,0),(\pm 2,0,0),\pm(2,2,2)\}$	$(s+1)^2$
$\{f=0\} \setminus (\{x=y=z\} \cup \{y=z=0\})$	$s+1$

In comparison, that of $g := x^2 - y^2 z$ (Whitney umbrella) is

strata	$b_{g,p}(s)$
$\{(0,0,0)\}$	$(s+1)^2(2s+3)$
$\{x=y=0\} \setminus \{(0,0,0)\}$	$(s+1)^2$
$\{g=0\} \setminus \{x=y=0\}$	$s+1$

In case $m_1 = 1$, $m_2 = 2$, $m_3 = 3$

In case $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, we succeeded in computation of the integration ideal I of the integrand as a local cohomology class. The characteristic variety of $M := D_3/I$ is given by

$$\mathrm{Char}(M) = T^*_{\{g=0\}}\mathbb{C}^3 \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3,$$

with

$$g(x,y,z) = (y-z)(y+z)x^4 - 2y(y^2-z^2-8)x^3$$
$$+(y^4+(-2z^2-22)y^2+z^4+26z^2+64)x^2+6y(y^2-z^2-8)x+9(y^2-z^2).$$

The decomposition of $\mathrm{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir for prime and primary decomposition of polynomial ideals developed by M. Noro.

The singularities of the surface $g = 0$

Again by change of coordinates $(y+z, y-z) \to (y,z)$, we rewirte g as

$$g = yzx^4 - (y+z)(yz-8)x^3 + ((z^2+1)y^2 - 24zy + z^2 + 64)x^2 + 3(y+z)(yz-8)x + 9yz.$$

The set of the singular points of the surface $g = 0$ is given by the curve

$$zx^2 - (z^2-8)x - 3z = y - z = 0.$$

The local b-function is $(s+1)^2(2s+3)$ at the 8 points

$$\pm(1,2,2), \quad \pm(1,-4,-4), \quad \pm(3,-2,-2), \quad \pm(3,4,4);$$

$(s+1)^2$ on the curve $zx^2 - (z^2-8)x - 3z = y - z = 0$ other than the 8 points above.

The local b-function of g at the 8 points is the same as that of the Whitney umbrella at the origin.