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AN ALGORITHM OF COMPUTING b-FUNCTIONS

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To Professor Hikosaburo Komatsu on the occasion of his sixtieth birthday

1. Introduction. Let $f(x) \in K[x] = K[x_1, \dots, x_n]$ be a polynomial of n variables with coefficients in a field K of characteristic zero. Let us denote by

$$A_n(K) := K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle, \qquad \hat{\mathscr{D}}_n(K) := K[[x_1, \dots, x_n]]\langle \partial_1, \dots, \partial_n \rangle$$

the rings of differential operators with polynomial and formal power series coefficients, respectively, with $\partial_i = \partial/\partial x_i$ and $\partial = (\partial_1, \dots, \partial_n)$. ($A_n(K)$ is called the Weyl algebra over K.)

Let s be a parameter. Then the (local) b-function (or the Bernstein-Sato polynomial) $b_f(s)$ associated with $f(x)$ is the monic polynomial of the least degree $b(s) \in K[s]$ satisfying

$$P(s, x, \partial) f(x)^{s+1} = b(s) f(x)^s \tag{1.1}$$

with some $P(s, x, \partial) \in \hat{\mathscr{D}}_n(K)[s]$. The monic polynomial of the least degree $b(s) \in K[s]$ satisfying (1.1) with some $P(s, x, \partial) \in A_n(K)[s]$ is denoted by $\tilde{b}_f(s)$. The existence of $\tilde{b}_f(s)$ was proved by I. N. Bernstein [Be1], [Be2], which implies the existence of $b_f(s)$. Note that $b_f(s)$ divides $\tilde{b}_f(s)$, but $b_f(s)$ and $\tilde{b}_f(s)$ are not necessarily identical. More generally, the existence of $b_f(s)$ for $f(x) \in K[[x]]$ was proved by J. E. Björk [Bj].

In this paper, we present an algorithm for, given $f(x) \in K[x]$, computing $b_f(s)$ and finding a $P(s, x, \partial) \in \hat{\mathscr{D}}_n(K)$ that satisfies (1.1) with $b(s) = b_f(s)$. More precisely, our algorithm finds a $Q(s, x, \partial) \in A_n(K)[s]$ and an $a(x) \in K[x]$ with $a(0) \neq 0$ such that $P(s, x, \partial) = (1/a(x)) Q(s, x, \partial)$ satisfies (1.1) with $b(s) = b_f(s)$. Computing $\tilde{b}_f(s)$ and an associated $P \in A_n(K)[s]$ is slightly easier.

An algorithm of computing $b_f(s)$ was first given by M. Sato et al. [SKKO] when $f(x)$ is a relative invariant of a prehomogeneous vector space. J. Briançon et al. [BGMM] and Ph. Maisonobe [Mai] gave an algorithm of computing $b_f(s)$ for $f(x)$ with isolated singularity. Also note that T. Yano [Y] worked out many interesting examples of b-functions systematically.

Our method consists in computing the (generalized) b-function for a section of a holonomic system (or more generally, a specializable D-module) via Gröbner basis computation in the Weyl algebra. In general, let M be a finitely generated

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left $A_{n+1}(K)$-module. We write $t = x_{n+1}$, $\partial_t = \partial_{n+1}$, and $\partial = (\partial_1, \dots, \partial_n)$. Put $\mathscr{M} = \hat{\mathscr{D}}_{n+1}(K) \otimes_{A_{n+1}(K)} M$. For $u \in M$, the (local) b-function of u is the monic polynomial $b_u(s) \in K[s]$ of the least degree, if any, satisfying

$$(b(t\partial_t) + tP(t, x, t\partial_t, \partial))(1 \otimes u) = 0 \quad \text{in } \mathscr{M}$$

with some $P(t, x, t\partial_t, \partial) \in \hat{\mathscr{D}}_{n+1}(K)$ (cf. [KK], [L]). We give an algorithm which computes $b_u(s)$ and P, or determines that there is none, by using a kind of Gröbner basis for left ideals of the Weyl algebra related to a filtration introduced by M. Kashiwara [K3]. Such Gröbner bases were used by Oaku [O2], [O3]. Especially we use the FW-Gröbner basis introduced in [O2]. The computation of the Bernstein-Sato polynomial $b_f(s)$ is reduced to finding the b-function $b_u(s)$ for $u = \delta(t - f(x))$, the delta function supported by $t - f(x) = 0$, as was observed by B. Malgrange [M1] (cf. also [M2], [K1], [K2]).

Our algorithm is strict (at least if K is algebraic over $\mathbf{Q}$) in the sense that, given a finite set of data, it returns an answer as a finite set of data, or else determines that there is no answer, in a finite number of steps using a finite amount of memory in the computation. For example, a system Kan of N. Takayama [T3] is available for actual execution of our algorithm.

We could also write down an algorithmic procedure for computing $b_f(s)$ even if $f(x) \in K[[x]]$ by using the FD-Gröbner basis of [O3] instead of the FW-Gröbner basis. However, this would not yield an algorithm in the strict sense as above in general.

2. Gröbner bases for ideals of the Weyl algebra. In this section we recall briefly the theory and algorithm of Gröbner bases for ideals of $A_n(K)$. We fix a (total) order $\prec$ of $\mathbf{N}^{2n}$ with $\mathbf{N} := \{0, 1, 2, \dots\}$ that satisfies the following conditions:

(A-1) $\alpha \succ \beta$ implies $\alpha + \gamma \succ \beta + \gamma$ for any $\alpha, \beta, \gamma \in \mathbf{N}^{2n}$;

(A-2) $\alpha \succeq 0$ for any $\alpha \in \mathbf{N}^{2n}$.

Under the condition (A-1), the condition (A-2) is equivalent to the order $\prec$ being a well-order (cf. [CLO]).

Let us write $A_n = A_n(K)$ with a field K of characteristic zero. An element P of A_n is written uniquely as a finite sum

$$P = \sum_{\alpha, \beta \in \mathbf{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta$$

with $a_{\alpha\beta} \in K$, where we use the notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then we define the *leading exponent* $\operatorname{lexp}(P)$ and the *leading coefficient* $\operatorname{lcoef}(P)$ of P by

$$\operatorname{lexp}(P) := \max{}_{\prec}\{(\alpha, \beta) \in \mathbf{N}^{2n} \mid a_{\alpha\beta} \neq 0\},$$

$$\operatorname{lcoef}(P) := a_{\alpha\beta} \quad \text{with } (\alpha, \beta) = \operatorname{lexp}(P),$$

where $\max_{\prec}$ denotes taking the maximum element with respect to the order $\prec$ (we assume $P \neq 0$). Let I be a left ideal of A_n. Then the set $E(I)$ of leading exponents of I is defined by

$$E(I) := \{\operatorname{lexp}(P) \mid P \in I \setminus \{0\}\} \subset \mathbf{N}^{2n}.$$

Definition 2.1 (Gröbner basis). A finite subset $\mathbf{G}$ of a left ideal I of A_n is called a *Gröbner basis* of I (with respect to the order $\prec$) if

$$E(I) = \bigcup_{P \in \mathbf{G}} (\operatorname{lexp}(P) + \mathbf{N}^{2n})$$

holds. (Then $\mathbf{G}$ generates I.)

The algorithm of computing a Gröbner basis (i.e., the Buchberger algorithm [Bu]) consists of computing division (or reduction) and S-polynomials (or operators).

LEMMA 2.2 (Division). *For $P, P_1, \dots, P_k \in A_n$, there exist $Q_1, \dots, Q_k, R \in A_n$ such that*

$$P = \sum_{i=1}^{k} Q_i P_i + R, \qquad \operatorname{lexp}(R) \notin \bigcup_{i=1}^{k} (\operatorname{lexp}(P_i) + \mathbf{N}^{2n})$$

and that, for each i, $\operatorname{lexp}(Q_i P_i) \preceq \operatorname{lexp}(P)$ if $Q_i \neq 0$. Moreover, there is an algorithm (the division, or the reduction algorithm) of computing $Q_1, \dots, Q_k, R$.

In general, for vectors $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ in $\mathbf{N}^m$, we put

$$\alpha \vee \beta := (\max\{\alpha_1, \beta_1\}, \dots, \max\{\alpha_m, \beta_m\}).$$

Definition 2.3 (S-operator). For $P, Q \in A_n$, put $\operatorname{lexp}(P) = (\alpha, \beta)$ and $\operatorname{lexp}(Q) = (\alpha', \beta')$. Then the *S-operator* of P and Q is defined by

$$\operatorname{sp}(P, Q) := \operatorname{lcoef}(Q) x^{\beta \vee \beta' - \beta} \partial^{\alpha \vee \alpha' - \alpha} P - \operatorname{lcoef}(P) x^{\beta \vee \beta' - \beta'} \partial^{\alpha \vee \alpha' - \alpha'} Q.$$

THEOREM 2.4 ([G], [C], [T1], [KW]). *Let $\mathbf{G} = \{P_1, \dots, P_k\}$ be a finite subset of A_n which generates a left ideal I of A_n. Then the following two conditions are equivalent:*

(1) *$\mathbf{G}$ is a Gröbner basis of I;*

(2) *for any $i, j \in \{1, \dots, s\}$ with $i < j$, there exist $Q_{ij1}, \dots, Q_{ijk} \in A_n$ so that*

$$\operatorname{sp}(P_i, P_j) = \sum_{\ell=1}^{k} Q_{ij\ell} P_\ell$$

and that $Q_{ij\ell} = 0$ or $\operatorname{lexp}(Q_{ij\ell} P_\ell) \prec \operatorname{lexp}(P_i) \vee \operatorname{lexp}(P_j)$ for each ℓ.

Condition (2) of this theorem provides the Buchberger algorithm of computing a Gröbner basis from a given set of generators.

PROPOSITION 2.5. *Let $\{P_1, \dots, P_k\}$ be a Gröbner basis of a left ideal I of A_n. Then for $P \in A_n$, P belongs to I if and only if there exist $Q_1, \dots, Q_k \in A_n$ such that $P = \sum_{i=1}^k Q_i P_i$ and that $\mathrm{lexp}(Q_i P_i) \preceq \mathrm{lexp}(P)$ if $Q_i \neq 0$ for each i.*

The following theorem can be proved in the same way as its counterpart in the polynomial ring (see [CLO], [BW], [E]).

THEOREM 2.6. *Let $\mathbf{G} = \{P_1, \dots, P_k\}$ be a Gröbner basis of a left ideal I of A_n. Take $Q_{ij\ell}$ satisfying condition (2) of Theorem 2.4. Put $\mathrm{lexp}(P_i) = (\alpha^{(i)}, \beta^{(i)})$ and*

$$S_{ji} := \mathrm{lcoef}(P_i) x^{\beta^{(i)} \vee \beta^{(j)} - \beta^{(i)}} \partial^{\alpha^{(i)} \vee \alpha^{(j)} - \alpha^{(i)}},$$

$$\vec{V}_{ij} := (0, \dots, \overset{(i)}{S_{ji}}, \dots, -\overset{(j)}{S_{ij}}, \dots, 0) - (Q_{ij1}, \dots, Q_{ijk}) \in (A_n)^k.$$

Then the syzygy module

$$S(P_1, \dots, P_k) := \left\{ (Q_1, \dots, Q_k) \in (A_n)^k \;\middle|\; \sum_{j=1}^k Q_j P_j = 0 \right\}$$

is generated by $\{\vec{V}_{ij} \mid 1 \leqslant i < j \leqslant k\}$.

3. Gröbner basis and homogenization with respect to a filtration. Fixing a field K of characteristic zero, we put

$$A_{n+1} = A_{n+1}(K) := K[t, x_1, \dots, x_n]\langle \partial_t, \partial_1, \dots, \partial_n \rangle,$$

$$\hat{\mathscr{D}}_{n+1} = \hat{\mathscr{D}}_{n+1}(K) := K[[t, x_1, \dots, x_n]]\langle \partial_t, \partial_1, \dots, \partial_n \rangle$$

with $\partial_t := \partial/\partial t$ and $\partial_i = \partial/\partial x_i$. Put $Y = \{(t, x) \in K^{n+1} | t = 0\}$. We use a filtration ($V$-filtration) with respect to Y introduced by Kashiwara [K3] for the study of vanishing cycle sheaves (cf. also [L], [LS]). An element P of A_{n+1} (or of $\hat{\mathscr{D}}_{n+1}$) is written in the form

$$P = \sum_{\mu, \nu \geqslant 0, \alpha, \beta \in \mathbf{N}^n} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta. \tag{3.1}$$

For each integer m, define K-subspaces of A_{n+1} and of $\hat{\mathscr{D}}_{n+1}$, respectively, by

$$F_m(A_{n+1}) := \left\{ P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in A_{n+1} \;\middle|\; a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m \right\},$$

$$F_m(\hat{\mathscr{D}}_{n+1}) := \left\{ P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in \hat{\mathscr{D}}_{n+1} \;\middle|\; a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m \right\}.$$

For a nonzero element P of $\hat{\mathscr{D}}_{n+1}$, we define the F-order $\mathrm{ord}_F(P)$ of P as the minimum integer m that satisfies $P \in F_m(\hat{\mathscr{D}}_{n+1})$. When the F-order of P in the form (3.1) is m, we put

$$\hat{\sigma}(P) = \hat{\sigma}_m(P) := \sum_{\nu - \mu = m} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta$$

and call it the *formal symbol* of P along Y (cf. [LS]). We have $\mathrm{ord}_F(PQ) = \mathrm{ord}_F(P) + \mathrm{ord}_F(Q)$ and $\hat{\sigma}(PQ) = \hat{\sigma}(P)\hat{\sigma}(Q)$ for $P, Q \in \hat{\mathscr{D}}_{n+1}$.

Now let $\prec_F$ be an order on $\mathbf{N}^{2n+2}$ which satisfies (A-1) (with n replaced by $n+1$) and

(A-3) if $\nu - \mu > \nu' - \mu'$, then $(\mu, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta')$;

(A-4) $(\mu, \mu, \alpha, \beta) \succeq_F (0, 0, 0, 0)$ for any $\mu \in \mathbf{N}$ and $\alpha, \beta \in \mathbf{N}^n$.

The condition (A-3) implies that $\prec_F$ is not a well-order. However, the definitions in the preceding section apply to this order $\prec_F$. The only difficulty is that Lemma 2.2 does not hold in general. Let us denote by $\mathrm{lexp}_F(P) \in \mathbf{N}^{2n+2}$, $\mathrm{lcoef}_F(P) \in K$ the leading exponent and the leading coefficient of $P \in A_{n+1} \setminus \{0\}$ with respect to $\prec_F$, respectively. The set of leading exponents $E_F(I) \subset \mathbf{N}^{2n+2}$ is defined in the same way.

Definition 3.1 (FW-Gröbner basis). A finite set $\mathbf{G}$ of generators of a left ideal I of A_{n+1} is called an *FW-Gröbner basis* of I if we have

$$E(I) = \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}(P) + \mathbf{N}^{2n+2}).$$

Since we do not have division algorithm, the Buchberger algorithm does not work directly. To bypass this difficulty to obtain an algorithm of computing FW-Gröbner bases, we use the homogenization technique.

Definition 3.2. For $i, j, \mu, \nu, \mu', \nu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, an order $\prec_H$ on $\mathbf{N}^{2n+3}$ is defined by

$$\begin{aligned}(i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta') \quad &\text{if and only if } (i > j) \quad \text{or} \\ &(i = j \text{ and } (\mu + \ell, \nu, \alpha, \beta) \succ_F (\mu' + \ell', \nu', \alpha', \beta')) \\ &\text{or } (i = j,\ \nu = \nu',\ \alpha = \alpha',\ \beta = \beta',\ \mu > \mu')\end{aligned}$$

with $\ell, \ell' \in \mathbf{N}$ such that $\nu - \mu - \ell = \nu' - \mu' - \ell'$. This definition is independent of the choice of ℓ, ℓ' in view of the condition (A-1).

LEMMA 3.3. (1) *$\prec_H$ is a well-order.*

(2) *If $\nu - \mu - i = \nu' - \mu' - j$, then $(i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta')$ if and only if $(\mu, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta')$.*

Proof. (1) It suffices to show $(i, \mu, \nu, \alpha, \beta) \succeq_H (0, 0, 0, 0, 0)$ for any $i, \mu, \nu \in \mathbf{N}$

and $\alpha, \beta \in \mathbf{N}^n$. Since this is obvious when $i > 0$, let us assume $i = 0$. Take $\ell, \ell' \in \mathbf{N}$ so that $\nu - \mu - \ell = -\ell'$. Then we have

$$(\mu + \ell - \ell', \nu, \alpha, \beta) = (\nu, \nu, \alpha, \beta) \succeq_F (0, 0, 0, 0)$$

by (A-4). This implies $(\mu + \ell, \nu, \alpha, \beta) \succeq_F (\ell', 0, 0, 0)$ in view of (A-1). Hence we have $(0, \mu, \nu, \alpha, \beta) \succeq_H (0, 0, 0, 0, 0)$ by definition.

(2) Assume $\nu - \mu - i = \nu' - \mu' - j$. Then we have $\nu - \mu > \nu' - \mu'$ if and only if $i > j$. If $i = j$, we can take $\ell = \ell' = 0$ in Definition 3.2. Assertion (2) follows from these facts combined with (A-3) and the definition of $\prec_H$. This completes the proof. □

For a nonzero element $P = P(x_0)$ of $A_{n+1}[x_0]$, let us denote by $\operatorname{lexp}_H(P) \in \mathbf{N}^{2n+3}$ and $\operatorname{lcoef}_H(P) \in K$ the leading exponent and the leading coefficient of P with respect to $\prec_H$, respectively. The set $E_H(I)$ of leading exponents of a left ideal I of $A_{n+1}[x_0]$ is also defined.

Definition 3.4 (F-homogeneity). An element P of $A_{n+1}[x_0]$ of the form

$$P = \sum_{i,\mu,\nu,\alpha,\beta} a_{i,\mu,\nu,\alpha,\beta} x_0{}^i t^\mu x^\alpha \partial_t^\nu \partial^\beta$$

is said to be *F-homogeneous* of order m if $a_{i,\mu,\nu,\alpha,\beta} = 0$ whenever $\nu - \mu - i \neq m$.

Definition 3.5 (F-homogenization). For an element P of A_{n+1} of the form

$$P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta,$$

put $m = \min\{\nu - \mu \mid a_{\mu,\nu,\alpha,\beta} \neq 0 \text{ for some } \alpha, \beta \in \mathbf{N}^n\}$. Then the *F-homogenization* $P^h \in A_{n+1}[x_0]$ of P is defined by

$$P^h = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} x_0^{\nu-\mu-m} t^\mu x^\alpha \partial_t^\nu \partial^\beta.$$

P^h is F-homogeneous of order m.

The following two lemmas follow from the Leibniz rule for the product of differential operators.

LEMMA 3.6. *If $P, Q \in A_{n+1}[x_0]$ are both F-homogeneous, then so is PQ.*

LEMMA 3.7. *For $P, Q \in A_{n+1}$, we have $(PQ)^h = P^h Q^h$.*

The following lemma is an immediate consequence of the definition.

LEMMA 3.8. *For $P_1, \ldots, P_k \in A_{n+1}$, put $P = P_1 + \ldots + P_k$. Then there exist $\ell, \ell_1, \ldots, \ell_k \in \mathbf{N}$ so that*

$$x_0^\ell P^h = x_0^{\ell_1} (P_1)^h + \cdots + x_0^{\ell_k} (P_k)^h.$$

Let us define $\varpi\colon \mathbf{N}^{2n+3} \to \mathbf{N}^{2n+2}$ by $\varpi(i, \mu, \nu, \alpha, \beta) = (\mu, \nu, \alpha, \beta)$.

LEMMA 3.9. (1) *If $P(x_0) \in A_{n+1}[x_0]$ is F-homogeneous, then we have* $\operatorname{lexp}_F(P(1)) = \varpi(\operatorname{lexp}_H(P(x_0)))$.

(2) *For any A_{n+1}, we have* $\operatorname{lexp}_F(P) = \varpi(\operatorname{lexp}_H(P^h))$.

Proof. Part (1) follows from (2) of Lemma 3.3, and (2) follows from (1) since $P^h(1) = P$. □

PROPOSITION 3.10. *Let I be a left ideal of $A_{n+1}[x_0]$ generated by F-homogeneous operators. Then there exists an H-Gröbner basis (i.e., a Gröbner basis with respect to $\prec_H$) of I consisting of F-homogeneous operators. Moreover, such an H-Gröbner basis can be computed by the Buchberger algorithm.*

Proof. Since $\prec_H$ is a well-order, the arguments in Section 2 apply. Hence we have only to verify that taking the S-operator and computing division both preserve the F-homogeneity. This follows from Lemma 3.6. □

PROPOSITION 3.11. *Let I be a left ideal of A_{n+1} generated by $P_1, \ldots, P_d \in A_{n+1}$. Let us denote by I^h the left ideal of $A_{n+1}[x_0]$ generated by $(P_1)^h, \ldots, (P_d)^h$. (Here I^h is not defined uniquely by I.) Let $\mathbf{G} = \{Q_1(x_0), \ldots, Q_k(x_0)\}$ be an H-Gröbner basis of I^h consisting of F-homogeneous operators. Then for $P \in A_{n+1}$, the following two conditions are equivalent:*

(1) *$P \in I$;*

(2) *there exist $U_1, \ldots, U_k \in A_{n+1}$ such that $P = \sum_{j=1}^k U_j Q_j(1)$, and that for each $j = 1, \ldots, k$, $\operatorname{lexp}_F(U_j Q_j(1)) \preceq_F \operatorname{lexp}_F(P)$ if $U_j \neq 0$.*

Proof. Assume $P \in I$. Then there exist $V_1, \ldots, V_d \in A_{n+1}$ such that $P = V_1 P_1 + \cdots + V_d P_d$. Then by Lemmas 3.7 and 3.8, there exist $\ell, \ell_1, \ldots, \ell_d \in \mathbf{N}$ such that

$$x_0^\ell P^h(x_0) = x_0^{\ell_1} (V_1)^h(x_0) (P_1)^h(x_0) + \cdots + x_0^{\ell_d} (V_d)^h(x_0) (P_d)^h(x_0) \in I^h.$$

Hence by Proposition 2.5, there exist F-homogeneous $U_1(x_0), \ldots, U_k(x_0) \in A_{n+1}[x_0]$ such that

$$x_0^\ell P^h(x_0) = U_1(x_0) Q_1(x_0) + \cdots + U_k(x_0) Q_k(x_0)$$

and that $\operatorname{lexp}_H(U_j(x_0) Q_j(x_0)) \preceq_H \operatorname{lexp}_H(x_0^\ell P^h(x_0))$ for each j with $U_j(x_0) \neq 0$. Setting $x_0 = 1$, we see that (2) holds in view of Lemmas 3.3 and 3.9.

In order to prove the inverse implication, it suffices to show that each $Q_j(1)$

belongs to I. Since $Q_j(x_0) \in I^h$, there exist $V_1(x_0), \ldots, V_d(x_0) \in A_{n+1}[x_0]$ such that

$$Q_j(x_0) = V_1(x_0)(P_1)^h(x_0) + \cdots + V_d(x_0)(P_d)^h(x_0).$$

Hence we get $Q_j(1) \in I$ setting $x_0 = 1$. This completes the proof. □

THEOREM 3.12. $\mathbf{G}(1) := \{Q_1(1), \ldots, Q_k(1)\}$ *is an FW-Gröbner basis of* I *under the same assumptions as in Proposition 3.11.*

Proof. In view of Proposition 3.11, $\mathbf{G}(1)$ generates I. Let P be an arbitrary element of I. Then it suffices to show that

$$\operatorname{lexp}_F(P) \in \bigcup_{j=1}^{k} (\operatorname{lexp}_F(Q_j(1)) + \mathbf{N}^{2n+2}).$$

Since $P \in I$, we can take U_j satisfying the condition (2) of Proposition 3.11. Then we have

$$\operatorname{lexp}_F(P) = \operatorname{lexp}_F(U_jQ_j(1)) = \operatorname{lexp}_F(U_j) + \operatorname{lexp}_F(Q_j(1)) \in \operatorname{lexp}_F(Q_j(1)) + \mathbf{N}^{2n+2}$$

for some j. This completes the proof. □

THEOREM 3.13. *Under the same assumptions as in Proposition 3.11, there exist, for any* $i, j \in \{1, \ldots, k\}$ *with* $i < j$, *F-homogeneous* $U_{ij1}(x_0), \ldots, U_{ijk}(x_0) \in A_{n+1}[x_0]$ *so that*

$$\operatorname{sp}(Q_i(x_0), Q_j(x_0)) = S_{ji}(x_0)Q_i(x_0) - S_{ij}(x_0)Q_j(x_0) = \sum_{\ell=1}^{k} U_{ijk}Q_\ell(x_0), \tag{3.2}$$

where $S_{ji}(x_0)$ *is defined in the same way as* S_{ji} *in Theorem 2.6, and, for each* ℓ, *we have* $\operatorname{lexp}_H(U_{ij\ell}Q_\ell(x_0)) \prec \operatorname{lexp}_F(Q_i(x_0)) \vee \operatorname{lexp}_F(Q_j(x_0))$, *if* $U_{ij\ell}(x_0) \neq 0$. *Put*

$$\vec{V}_{ij}(x_0) = (0, \ldots, \overset{(i)}{S_{ji}(x_0)}, \ldots, \overset{(j)}{-S_{ij}(x_0)}, \ldots, 0) - (U_{ij1}(x_0), \ldots, U_{ijk}(x_0)) \in (A_{n+1}[x_0])^k.$$

Then the syzygy module

$$S(Q_1(1), \ldots, Q_k(1)) := \left\{ (U_1, \ldots, U_k) \in (A_{n+1})^k \,\middle|\, \sum_{\ell=1}^{k} U_\ell Q_\ell(1) = 0 \right\}$$

is generated by $\{\vec{V}_{ij}(1) \mid 1 \leqslant i < j \leqslant k\}$.

Proof. The first assertion follows from Theorem 2.4. Suppose

$$(U_1, \ldots, U_k) \in S(Q_1(1), \ldots, Q_k(1)).$$

Then in view of Lemmas 3.7 and 3.8, there exist $\ell_1, \ldots, \ell_k \in \mathbf{N}$ such that

$$x_0^{\ell_1}(U_1)^h(x_0)Q_1(x_0) + \cdots + x_0^{\ell_k}(U_k)^h(x_0)Q_k(x_0) = 0.$$

Hence by applying Theorem 2.6 to the order $\prec_H$, there are $W_{ij}(x_0) \in A_{n+1}[x_0]$ such that

$$(x_0^{\ell_1}(U_1)^h(x_0), \ldots, x_0^{\ell_k}(U_k)^h(x_0)) = \sum_{i<j} W_{ij}(x_0)\vec{V}_{ij}(x_0).$$

This completes the proof by setting $x_0 = 1$. □

Definition 3.14. Let P be a nonzero element of A_{n+1} (resp., $\hat{\mathscr{D}}_{n+1}$) of F-order m. Then we define $\psi(P)(s) \in A_n[s]$ (resp., $\hat{\mathscr{D}}_n[s]$) by

$$\hat{\sigma}_0(t^m P) = \psi(P)(t\partial_t) \quad \text{if} \quad m \geqslant 0,$$

$$\hat{\sigma}_0(\partial_t^{-m} P) = \psi(P)(t\partial_t) \quad \text{if} \quad m < 0.$$

The cause of our use of FW-Gröbner basis lies in the following theorem.

THEOREM 3.15. *We use the same notation as in Proposition 3.11. Let* $\psi(I)$ *be the left ideal of* $A_n[s]$ *generated by the set* $\{\psi(P)(s) \mid P \in I \cap (F_0(A_{n+1}) \setminus F_{-1}(A_{n+1}))\}$. *Then* $\psi(I)$ *is generated by* $\psi(Q_1(1)), \ldots, \psi(Q_k(1))$.

Proof. By definition it is easy to see that $\psi(Q_j(1)) \in \psi(I)$ for $j = 1, \ldots, k$. Suppose $P \in I \cap (F_0(A_{n+1}) \setminus F_{-1}(A_{n+1}))$. Let $U_1, \ldots, U_k \in A_{n+1}$ be as in Proposition 3.11. Let $Q_j(1)$ be of F-order m_j. Then the F-order of U_j is not greater than $-m_j$. Hence we can take $U_j' \in F_0(A_{n+1})$ such that $\hat{\sigma}_{-m_j}(U_j) = U_j'S_j$, where $S_j = t^{m_j}$ if $m_j \geqslant 0$ and $S_j = \partial_t^{-m_j}$ if $m_j < 0$. Then we have

$$\psi(P)(t\partial_t) = \sum_{j=1}^{k} \hat{\sigma}_0(U_jQ_j(1)) = \sum_{j=1}^{k} \hat{\sigma}_{-m_j}(U_j)\hat{\sigma}_{m_j}(Q_j(1)) = \sum_{j=1}^{k} U_j'\psi(Q_j(1))(t\partial_t).$$

This completes the proof. □

What is more crucial in the application of FW-Gröbner bases to the D-module theory is that this theorem can be localized as follows.

THEOREM 3.16. *In the same notation as Theorem 3.15, let* $\mathscr{I} = \hat{\mathscr{D}}_{n+1}I$ *be the left ideal of* $\hat{\mathscr{D}}_{n+1}$ *generated by* I. *Let* $\psi(\mathscr{I})$ *be the left ideal of* $\hat{\mathscr{D}}_n[s]$ *generated by the set* $\{\psi(P)(s) \mid P \in \mathscr{I} \cap (F_0(\hat{\mathscr{D}}_{n+1}) \setminus F_{-1}(\hat{\mathscr{D}}_{n+1}))\}$. *Then* $\psi(\mathscr{I})$ *is generated by* $\psi(Q_1(1)), \ldots, \psi(Q_k(1))$.

Proof. We use the same notation as in Theorem 3.13. In general, for a nonzero element $P = P(x_0)$ of $A_{n+1}[x_0]$, let us denote by $\sigma_H(P)(x_0)$ the highestdegree part of P with respect to x_0. Then we have $\hat{\sigma}(P(1)) = \sigma_H(P)(1)$. By taking the maximum degree parts with respect to x_0 in both sides of the equation (3.2) of Theorem 3.13, we know that $\{\sigma_H(Q_1)(x_0), \dots, \sigma_H(Q_k)(x_0)\}$ is an H-Gröbner basis of a left ideal of $A_{n+1}[x_0]$.

Let $Q_j(1)$ be of F-order m_j and set $m_{ij} = \nu - \mu$ with $(\mu, \nu, \alpha, \beta) = \mathrm{lexp}_F(Q_i(1)) \vee \mathrm{lexp}_F(Q_j(1))$. Put

$$\vec{V}_{ij}^0(x_0) := (0, \dots, \overset{(i)}{\sigma_H(S_{ji})}(x_0), \dots, \overset{(j)}{-\sigma_H(S_{ij})}(x_0), \dots, 0) - (U_{ij1}^0(x_0), \dots, U_{ijk}^0(x_0)).$$

Here we put $U_{ij\ell}^0(x_0) := \sigma_H(U_{ij\ell})(x_0)$ if $\mathrm{ord}_F(U_{ij\ell}(1)Q_j(1)) = m_{ij} - m_\ell$, and otherwise $U_{ij\ell}^0(x_0) = 0$. Hence by applying Theorem 3.13 to $\sigma_H(Q_j)(x_0)$, we know that the syzygy module $S(\hat{\sigma}(Q_1(1)), \dots, \hat{\sigma}(Q_k(1)))$ is generated by $\mathbf{V}^0 := \{\vec{V}_{ij}^0(1) | i < j\}$. By virtue of the flatness of $\hat{\mathscr{D}}_{n+1}$ over A_{n+1}, this implies that the syzygy module

$$\hat{S} := \left\{ (U_1, \dots, U_k) \in (\hat{\mathscr{D}}_{n+1})^k \,\middle|\, \sum_{j=1}^k U_j \hat{\sigma}(Q_j(1)) = 0 \right\}$$

is also generated by $\mathbf{V}^0$ over $\hat{\mathscr{D}}_{n+1}$.

Now suppose $P \in \mathscr{I} \cap (F_0(\hat{\mathscr{D}}_{n+1}) \backslash F_{-1}(\hat{\mathscr{D}}_{n+1}))$. Then there exist $U_1, \dots, U_k \in \hat{\mathscr{D}}_{n+1}$ such that

$$P = U_1 Q_1(1) + \cdots + U_k Q_k(1).$$

Put $m := \max\{\mathrm{ord}_F(U_j Q_j(1)) | j = 1, \dots, k\}$. Our aim is to show that we can choose $U_1, \dots, U_k$ so that $m = \mathrm{ord}_F(P)$. For this purpose, assume $m > \mathrm{ord}_F(P)$. Put $U_j^0 = \hat{\sigma}_{m-m_j}(U_j)$. Then we have $(U_1^0, \dots, U_k^0) \in \hat{S}$. Hence there exist F-homogeneous $W_{ij} \in \hat{\mathscr{D}}_{n+1}$ such that

$$(U_1^0, \dots, U_k^0) = \sum_{i<j} W_{ij} \vec{V}_{ij}^0(1).$$

Define $U_1', \dots, U_k' \in \hat{\mathscr{D}}_{n+1}$ by

$$(U_1', \dots, U_k') = \sum_{i<j} W_{ij} \vec{V}_{ij}(1).$$

Then we get

$$P = (U_1 - U_1')Q_1(1) + \cdots + (U_k - U_k')Q_k(1).$$

We also have $\mathrm{ord}_F((U_j - U_j')Q_j(1)) < m$ since $\hat{\sigma}_{m-m_j}(U_j) = \hat{\sigma}_{m-m_j}(U_j')$. By using this argument repeatedly, we can choose $U_1, \dots, U_k \in \hat{\mathscr{D}}_{n+1}$ such that $P = \sum_{j=1}^k U_j Q_j(1)$ and that $\mathrm{ord}_F(U_j Q_j(1)) \leqslant \mathrm{ord}_F(P)$ for any j. Hence we know, by the same argument as in the proof of Theorem 3.15, that $\psi(\mathscr{I})$ is generated by $\psi(Q_1(1)), \dots, \psi(Q_k(1))$. This completes the proof. □

Remark 3.17. Theorems 3.15 and 3.16 hold with $\{Q_1(1), \dots, Q_k(1)\}$ replaced by an arbitrary FW-Gröbner basis of I. To prove this, we have only to develop a theory of FW-Gröbner basis by using the "truncated division" with respect to the filtration as in [O3].

4. Computation of the b-function of a D-module. We retain the notation in the preceding section. Let M be a finitely generated left A_{n+1}-module and u a nonzero element of M. In the sequel, we assume that a system of the equations for u is given explicitly; i.e., we assume that a finite set of generators of a left ideal I of A_{n+1} is given so that $A_{n+1}u \simeq A_{n+1}/I$.

More generally, if a presentation of M and a representation of u are known, i.e., if generators of a left A_{n+1}-submodule N of $(A_{n+1})^r$ is given so that $M \simeq (A_{n+1})^r/N$, and also given is an element $\vec{U} \in (A_{n+1})^r$ such that u corresponds to the modulo class of $\vec{U}$ by the above isomorphism, then there is an algorithm to find generators of the above I by computing syzygies by means of (a generalization of) Theorem 2.6.

Put $\mathscr{M} := \hat{\mathscr{D}}_{n+1} \otimes_{A_{n+1}} M$ and $\mathscr{I} := \hat{\mathscr{D}}_{n+1} I$. Then we have $\hat{\mathscr{D}}_{n+1}(1 \otimes u) \simeq \hat{\mathscr{D}}_{n+1}/\mathscr{I}$. The *b-function* $b_u(s)$ of u is the monic polynomial, if any, $b(s) \in K[s]$ of the least degree satisfying

$$(b(t\partial_t) + P)(1 \otimes u) = 0 \quad \text{in } \mathscr{M} \tag{4.1}$$

with some $P \in F_{-1}(\hat{\mathscr{D}}_{n+1})$. Note that (4.1) is equivalent to $b(t\partial_t) + P \in \mathscr{I}$.

The existence of the b-function in this sense was proved by Kashiwara-Kawai [KK] and Laurent [L] when $K = \mathbf{C}$ and $\mathscr{M}$ is holonomic. Our purpose here is to present an algorithm to determine whether there exists such nonzero $b(s)$, and if it does, to find $b_u(s)$ and an associated P.

Now let M, u, I be as above and let $\mathbf{G} = \{Q_1(x_0), \dots, Q_k(x_0)\}$ be an H-Gröbner basis of I^h consisting of F-homogeneous elements as in the preceding section. Let $\psi(I)$ and $\psi(\mathscr{I})$ be left ideals of $A_{n+1}[s]$ and of $\hat{\mathscr{D}}_{n+1}[s]$, respectively, defined in Theorems 3.15 and 3.16. Then $\psi(I)$ and $\psi(\mathscr{I})$ are both generated by $\psi(\mathbf{G}(1)) := \{\psi(Q_1(1)), \dots, \psi(Q_k(1))\}$.

Let $\prec$ be an order on $\mathbf{N}^{2n+1}$ satisfying (A-1), (A-2) (with $2n+2$ replaced by $2n+1$), and:

(A-5) If $|\beta| > |\beta'|$, then $(\mu, \alpha, \beta) \succ (\mu', \alpha', \beta')$ for any $\mu, \mu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$.

In particular, if the order $\prec_F$ satisfies, in addition to (A-1), (A-3), (A-4),

(A-6) If $\nu - \mu = \nu' - \mu'$ and $|\beta| > |\beta'|$, then $(\mu, \nu, \alpha, \beta) \succ (\mu', \nu', \alpha', \beta')$ for any $\mu, \nu, \mu', \nu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$,

then the order $\prec_{F'}$ on $\mathbf{N}^{2n+1}$ defined by

$$(\mu, \alpha, \beta) \succ_{F'} (\mu', \alpha', \beta') \Leftrightarrow (\mu, \mu, \alpha, \beta) \succ_F (\mu', \mu', \alpha', \beta')$$

satisfies (A-1), (A-2), (A-5).

For an element P of $A_n[s]$ (resp., $\hat{\mathscr{D}}_n[s]$), its order ord(P) is defined to be the usual order with respect to ∂, and the principal symbol $\sigma(P) \in K[x, \xi, s]$ (resp., $K[[x]][\xi, s]$) is defined also in the standard way with $\xi = (\xi_1, \dots, \xi_n)$.

THEOREM 4.1. *Let $\sigma(\psi(I))$ and $\sigma(\psi(\mathscr{I}))$ be ideals of $K[x, \xi, s]$ and of $K[[x]][\xi, s]$ generated by $\{\sigma(P) | P \in \psi(I)\}$ and by $\{\sigma(P) | P \in \psi(\mathscr{I})\}$, respectively. Let $\mathbf{H}$ be a Gröbner basis of $\psi(I)$ with respect to an order $\prec$ on $\mathbf{N}^{2n+1}$ satisfying (A-1), (A-2), (A-5). Then $\sigma(\psi(I))$ and $\sigma(\psi(\mathscr{I}))$ are both generated by $\sigma(\mathbf{H}) := \{\sigma(P) | P \in \mathbf{H}\}$. In particular, if the order $\prec_F$ satisfies (A-6) in addition to (A-1), (A-3), (A-4), then we can take $\sigma(\psi(\mathbf{G}(1))) := \{\sigma(\psi(Q_1(1))), \dots, \sigma(\psi(Q_k(1)))\}$ as $\sigma(\mathbf{H})$.*

Proof. The first assertion can be proved in the same way as the theorem of [O1]; one has only to modify its proof by adding a parameter s. The last statement amounts to saying that $\psi(\mathbf{G}(1))$ is a Gröbner basis with respect to $\prec$. In fact, (A-6) and Proposition 3.11 guarantee that this is the case. □

COROLLARY 4.2. *In the same notation as in Theorem 4.1, put $J := \psi(I) \cap K[x, s]$ and $\mathscr{J} := \psi(\mathscr{I}) \cap K[[x]][s]$. Then J and $\mathscr{J}$ are both generated by $\sigma(\mathbf{H}) \cap K[x, s]$ as ideals of $K[x, s]$ and of $K[[x]][s]$, respectively.*

COROLLARY 4.3. *The b-function of u is the monic generator of $\mathscr{J} \cap K[s]$ if it is not the zero ideal. If $\mathscr{J} \cap K[s] = 0$, then the b-function does not exist.*

At this stage, the problem has become one in commutative algebra. Hence the arguments below should be more or less standard. To be general, let J be an ideal of $K[x, s]$ whose generators are given explicitly and put $\mathscr{J} = K[[x]][s]J$. Our purpose is to compute $\mathscr{J} \cap K[s]$. ($J \cap K[s]$ is computed easily through the elimination by Gröbner basis.)

The following lemma is a consequence of the faithful flatness of $K[[x]]$ over $K[x]_0$.

LEMMA 4.4. *Let $K[x]_0$ be the localization of $K[x]$ with respect to the maximal ideal generated by $x_1, \dots, x_n$. Put $\mathscr{J}' := K[x]_0[s]J$. Then we have $\mathscr{J} \cap K[s] = \mathscr{J}' \cap K[s]$.*

Thus we can compute $\mathscr{J} \cap K[s]$ by using the Gröbner basis computation in the polynomial ring and factorization in $\overline{K}[s]$ in the following steps. (We denote by $\overline{K}$ the algebraic closure of K.)

Algorithm 4.5. Input: a set of generators $f_1(x, s), \dots, f_k(x, s)$ of J.

(1) Determine whether there exists, and find, if any, some $g(x, s) \in J$ such that its leading coefficient with respect to s does not vanish at $x = 0$. This can be done, e.g., as follows. (One can use, instead of the homogenization below, Mora's tangent cone algorithm [Mo].)

(a) Let $(f_i)^h(x_0, x, s)$ be the homogenization of $f_i(x, s)$ with respect to x; i.e., $(f_i)^h$ is homogeneous with respect to x_0 and x, and $(f_i)^h(1, x, s) = f_i(x, s)$.

(b) Let $>$ be an order on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}^n \ni (i, \mu, \alpha)$ with (i, μ, α) corresponding to $x_0{}^i s^\mu x^\alpha$. Assume $>$ satisfies (A-1), (A-2), and $(i, \mu, \alpha) > (j, \nu, \beta)$ if $\mu > \nu$ or ($\mu = \nu$ and $i > j$).

(c) Let $\{g_1(x_0, x, s), \dots, g_r(x_0, x, s)\}$ be a Gröbner basis of the ideal generated by $(f_1)^h, \dots, (f_k)^h$ with respect to $>$.

(d) Let $g(x, s)$ be one of $g_i(1, x, s)$ with the property above; if there is no such $g(x, s)$, then quit (there is no $b(s)$).

(2) Compute the monic generator $f_0(s)$ of the ideal $J(0)$ of $K[s]$ that is generated by $f_1(0, s), \dots, f_k(0, s)$ by Gröbner basis or GCD computation; if $f_0(s) = 1$, then put $b(s) := 1$ and go to (6).

(3) Compute the factorization $f_0(s) = (s - s_1)^{\mu_1} \cdots (s - s_m)^{\mu_m}$ in $\overline{K}[s]$.

(4) Put $\bar{J} := \overline{K}[x, s]J$. For each $i = 1, \dots, m$, determine the least integer $\ell_i = \ell \geqslant 0$ satisfying $h(x, s)(s - s_i)^\ell \in J$ with some $h(x, s) \in \overline{K}[x, s]$ such that $h(0, s_i) \neq 0$, or else determine that there is no such ℓ. This can be done by computing ideal quotient and saturation via Gröbner bases as follows (cf. [BW], [CLO], [E]).

For $i := 1$ to m:

(a) Compute a set of generators G_i of the saturation $\bar{J} : (s - s_i)^\infty$ by means of Gröbner basis.

(b) Determine whether there is some $h(x, s) \in G_i$ such that $h(0, s_i) \neq 0$; if there is no such h, then put $\ell_i := \infty$ and quit (there is no $b(s)$).

(c) By computing the ideal quotient $\bar{J} : (s - s_i)^\ell$ for $\ell = \mu_i, \mu_i + 1, \dots$ repeatedly, determine the least $\ell \geqslant \mu_i$ such that $\bar{J} : (s - s_i)^\ell$ contains an element which does not vanish at $(x, s) = (0, s_i)$. Denote this ℓ by ℓ_i.

(5) Put $b(s) := (s - s_1)^{\ell_1} \cdots (s - s_m)^{\ell_m}$; we have $b(s) \in K[s]$.

(6) Compute $J_0 := (J : b(s)) \cap K[x]$ by Gröbner bases and find an element $a(x) \in J_0$ such that $a(0) \neq 0$; such $a(x)$ exists.

(7) Find by division $q_1(x, s), \dots, q_k(x, s) \in K[x, s]$ such that

$$a(x)b(s) = q_1(x, s)f_1(x, s) + \cdots + q_k(x, s)f_k(x, s).$$

Output: $b(s), a(x), q_1(x, s), \dots, q_k(x, s)$; then $b(s)$ is the monic generator of $\mathscr{J} \cap K[s]$.

THEOREM 4.6. *The above algorithm is correct.*

Proof. Let us first show the correctness of step (1). This is a parameter version of standard basis computation by homogenization (cf. [La]). Let $\mathrm{lexp}(f(x_0))$ be the leading exponent of $f(x_0) \in K[x_0, x, s]$ with respect to $>$. Then the condition for g stated in step (1) is equivalent to $\mathrm{lexp}(g^h) = (i, \mu, 0)$ with some $i, \mu \in \mathbf{N}$. Thus g can be chosen from, if any, $g_1(1, x, s), \dots, g_r(1, x, s)$. If such g does not exist, we have $\mathscr{J} \cap K[s] = 0$ since a nonzero element of $\mathscr{J} \cap K[s]$ would satisfy the desired property.

Hence in the sequel, we assume that $g(x, s)$ of step (1) exists. Suppose $\ell_i < \infty$ for any i and put $Q := \bar{J} : b(s)$. Put

$$\mathbf{V}(Q) := \{(x, s) \in \overline{K}^{n+1} | f(x, s) = 0 \text{ for any } f \in Q\}.$$

Then we have

$$\mathbf{V}(Q) \cap (\{0\} \times \bar{K}) \subset \mathbf{V}(J) \cap (\{0\} \times \bar{K})$$

$$= \{(0,s) | s \in \bar{K},\ f_1(0,s) = \cdots = f_k(0,s) = 0\}$$

$$= \{(0,s_1), \ldots, (0,s_m)\}. \tag{4.2}$$

Moreover, we have $h_i(x,s)(s-s_i)^{\ell_i} \in \bar{J}$, and hence $h_i(x,s) \in Q$, with some $h_i(x,s) \in \bar{K}[x,s]$ such that $h_i(0,s_i) \neq 0$. This implies, together with (4.2), that $\mathbf{V}(Q) \cap (\{0\} \times \bar{K}) = \varnothing$. In addition, there exists $g(x,s) \in J \subset Q$ whose leading coefficient with respect to s does not vanish at $x = 0$. Hence in view of, e.g., the extension theorem of [CLO, p. 162], there exists some $a(x) \in Q \cap \bar{K}[x]$ with $a(0) \neq 0$. This implies $a(x)b(s) \in \bar{J}$. In particular, $b(s)$ belongs to $\bar{\mathscr{J}}' \cap \bar{K}[s]$ with $\bar{\mathscr{J}}' := \bar{K}[x]_0[s]J$.

Let us see that $b(s)$ generates $\bar{\mathscr{J}}' \cap \bar{K}[s]$. If $f(s)$ belongs to $\bar{\mathscr{J}}' \cap \bar{K}[s]$, then $h(x)f(s) \in \bar{J}$ with some $h(x) \in \bar{K}[x]$ with $h(0) \neq 0$. Hence $(s-s_i)^{\ell_i}$ divides $f(s)$ by the definition of ℓ_i. It follows $b(s)$ divides $f(s)$. It also follows that $\mathscr{J}' \cap K[s] = \{0\}$ if $\ell_i = \infty$ for some i.

We have to show $b(x) \in K[s]$. Put $L := K(s_1, \ldots, s_m) \subset \bar{K}$, and let $1, \omega_1, \ldots, \omega_\nu$ be a basis of L over K, and define $\pi(c) := c_0$ for $c = c_0 + c_1\omega_1 + \cdots + c_\nu\omega_\nu \in L$ with $c_0, c_1, \ldots, c_\nu \in K$. We extend π to the mapping of $L[[x]][s]$ to $K[[x]][s]$. Note that $b(s) \in L[s] \cap L[x]_0[s]J$, since the Gröbner basis computation does not require field extension. Hence we have

$$b(s) = q_1(x,s)f_1(x,s) + \cdots + q_k(x,s)f_k(x,s)$$

with some $q_i(x,s) \in L[[x]][s]$. It follows

$$\pi(b(s)) = \pi(q_1(x,s))f_1(x,s) + \cdots + \pi(q_k(x,s))f_k(x,s) \in \mathscr{J},$$

and hence $\pi(b(s)) \in \mathscr{J}'$ by Lemma 4.4. Thus $b(s)$ divides $\pi(b(s))$ in $L[s]$. Since both are monic and of the same degree, we have $b(s) = \pi(b(s)) \in K[s]$. Thus we can take $a(x) \in J : b(s) \subset K[x]$ such that $a(0) \neq 0$.

Finally, the inequality $\ell_i \geqslant \mu_i$ follows from $b(s) \in J(0) = (f_0(s))$. This completes the proof. □

Now let us turn to the computation of P associated with $b_u(s)$. By detecting the FW-Gröbner basis computation and the Gröbner basis computation corresponding to Theorem 4.1, we obtain $P_i \in I$ such that $\psi(P_i) = f_i(x,s)$ in the above notation. Let the F-order of P_i be m_i. By using outputs of Algorithm 4.5, put

$$P := b(t\partial_t) - \frac{1}{a(x)} \sum_{i=1}^{m} q_i(x, t\partial_t) S_i P_i,$$

where $S_i := t^{m_i}$ if $m_i \geqslant 0$ and $S_i = \partial_t^{-m_i}$ if $m_i < 0$. Then we have $P \in F_{-1}(\hat{\mathscr{D}}_{n+1})$ and $(b(t\partial_t) - P)(1 \otimes u) = 0$. How to find a "good" P, e.g., of minimum order with respect to ∂, remains unsolved.

5. Computation of the Bernstein-Sato polynomial. We retain the notation in the preceding sections. Let $f(x) \in K[x]$ be a polynomial with $f(0) = 0$. (One can suppose $f(x) \in K[[x]]$ with $f(0) = 0$ in the argument below except for the algorithm.) The following argument is due to Malgrange [M1]. Put $\mathscr{L} = K[[x]][f^{-1}, s]f^s$, where we regard f^s as a free generator. Then $\mathscr{L}$ has a structure of left $\hat{\mathscr{D}}_n[s]$-module defined by

$$\partial_i(g(s)f^{-m}f^s) = \left(\frac{\partial g}{\partial x_i}(s)f^{-m} + (s-m)g(s)\frac{\partial f}{\partial x_i}f^{-m-1}\right)f^s \qquad (i = 1, \ldots, n)$$

for $g(s) \in K[[x]][s]$ and $m \in \mathbf{N}$. Moreover, $\mathscr{L}$ has also a structure of left $\hat{\mathscr{D}}_{n+1}$-module defined by

$$t(g(s)f^s) = g(s+1)f^{s+1}, \qquad \partial_t(g(s)f^s) = -sg(s-1)f^{s-1}$$

for $g(s) \in K[[x]][f^{-1}, s]$. We can make an element $a(t) \in K[[t]]$ operate on $g(s)f^s$ since $f(0) = 0$. It is easy to see that

$$-\partial_t t(g(s)f^s) = sg(s)f^s \quad \text{for any } g(s) \in K[[x]][f^{-1}, s], \tag{5.1}$$

$$(t - f(x))f^s = 0, \tag{5.2}$$

$$\left(\partial_i + \frac{\partial f}{\partial x_i}(x)\partial_t\right)f^s = 0 \quad (i = 1, \ldots, n). \tag{5.3}$$

Put $\mathscr{N} := \hat{\mathscr{D}}_n[s]f^s$ and $\mathscr{M} := \hat{\mathscr{D}}_{n+1}f^s$. Then we have inclusions $\mathscr{N} \subset \mathscr{M} \subset \mathscr{L}$ in view of (5.1). The following lemma is the same as Lemma 4.1 of [M1].

Lemma 5.1. *Put*

$$I := A_{n+1}(t - f(x)) + \sum_{i=1}^{n} A_{n+1}\left(\partial_i + \frac{\partial f}{\partial x_i}\partial_t\right).$$

Then the left ideal $\mathscr{I} = \hat{\mathscr{D}}_{n+1}I$ of $\hat{\mathscr{D}}_{n+1}$ is maximal.

Proposition 5.2. *$\mathscr{M}$ is isomorphic to $\hat{\mathscr{D}}_{n+1}/\mathscr{I}$.*

Proof. Put $\mathscr{I}' := \{P \in \hat{\mathscr{D}}_{n+1} | Pf^s = 0\}$. Then we have $\mathscr{M} \simeq \hat{\mathscr{D}}_{n+1}/\mathscr{I}'$. Since $\mathscr{M} \ni f^s \neq 0$, $\mathscr{I}'$ is a proper ideal. Equations (5.2) and (5.3) imply $\mathscr{I} \subset \mathscr{I}'$. Since $\mathscr{I}$ is maximal, we must have $\mathscr{I}' = \mathscr{I}$. This completes the proof. □

COROLLARY 5.3. *For $P(s) \in \hat{\mathscr{D}}_n[s]$, we have $P(s)f^s = 0$ in $\mathscr{N}$ if and only if $P(-\partial_t t) \in \mathscr{I}$.*

Proof. In view of (5.1), we have $P(s)f^s = P(-\partial_t t)f^s$. Hence the assertion follows from Proposition 5.2. □

The (local) b-function (Bernstein-Sato polynomial) $b_f(s)$ of $f(x)$ is the monic polynomial of the least degree $b(s) \in K[s]$ satisfying

$$P(s,x,\partial)f^{s+1} = b(s)f^s \qquad \text{in } \mathscr{N} \tag{5.4}$$

with some $P(s) \in \hat{\mathscr{D}}_n[s]$. The monic polynomial of the least degree $b(s)$ satisfying (5.4) with some $P(s) \in A_n[s]$ is denoted by $\tilde{b}_f(s)$. Such $\tilde{b}_f(s)$, and hence $b_f(s)$ also exist (cf. [Be1], [Be2], [Bj], [K1]). By definition $b_f(s)$ divides $\tilde{b}_f(s)$.

In view of Corollary 5.3, equation (5.4) is equivalent to

$$b(-\partial_t t) - P(-\partial_t t, x, \partial)f \in \mathscr{I}.$$

Since $t - f \in \mathscr{I}$, this is also equivalent to

$$b(-\partial_t t) - P(-\partial_t t, x, \partial)t \in \mathscr{I},$$

and we have $P(-\partial_t t, x, \partial)t \in F_{-1}(\hat{\mathscr{D}}_{n+1})$. On the other hand, suppose $b(s) \in K[s]$ and $Q \in F_{-1}(\hat{\mathscr{D}}_{n+1})$ satisfy

$$(b(t\partial_t) - Q)f^s = 0 \qquad \text{in } \mathscr{M}.$$

Expanding Q in the form

$$Q = \sum_{m=1}^{\infty} Q_m(x, t\partial_t, \partial)t^m,$$

put

$$\rho(Q) := \sum_{m=1}^{\infty} Q_m(x, -s-1, \partial)f^{m-1}.$$

Note that $\rho(Q)$ is well defined as an element of $\hat{\mathscr{D}}_n[s]$ since $f(0) = 0$. Then we get, in view of Corollary 5.3,

$$(b(-s-1) - \rho(Q)f)f^s = 0 \quad \text{in } \mathscr{N}.$$

In conclusion, the computation of $b_f(s)$ and an associated $P(s) \in \hat{\mathscr{D}}_n[s]$ of (5.4) can be done as follows.

Algorithm 5.4. Input: $f(x) \in K[x]$.

(1) Letting I be the left ideal of A_{n+1} generated by $t-f$ and $\partial_i + (\partial f/\partial x_i)\partial_t$ ($i = 1, \dots, n$), compute an FW-Gröbner basis **G** of I via F-homogenization;

(2) Compute a Gröbner basis **H** of the left ideal generated by $\psi(\mathbf{G}) := \{\psi(P) | P \in \mathbf{G}\}$ with respect to an order satisfying (A-1), (A-2), (A-5). In the course of this computation, find $P_1, \dots, P_m \in \mathbf{G}$ such that $\mathbf{H} \cap K[x,s] = \{\psi(P_1), \dots, \psi(P_m)\}$.

(3) Compute the outputs $b(s) \in K[s], a(x) \in K[x], q_1(x,s), \dots, q_s(x,s) \in K[x,s]$ of Algorithm 4.5 with $\mathbf{H} \cap K[x,s]$ as inputs.

(4) Put $Q := a(x)b(t\partial_t) - \sum_{i=1}^m q_i(x, t\partial_t)S_iP_i \in F_{-1}(A_{n+1})$, where $S_i := t^{m_i}$ if $m_i := \mathrm{ord}_F(P_i) \geqslant 0$ and $S_i := \partial_t^{-m_i}$ otherwise.

Output: $b_f(s) := b(-s-1)$ and $P(s) := (1/a(x))\rho(Q)$.

Remark 5.5. (1) Step (1) of Algorithm 4.5, which is called in the above algorithm, can be skipped since the existence of $\tilde{b}_f(s)$ is assured by Bernstein [Be2].

(2) If K is a subfield of **C**, the fact that the roots of $b_f(s)$ are rational (see Kashiwara [K1]) makes steps (3) and (4) of Algorithm 4.5 considerably easier since there is no need of field extension.

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