

# Bernstein-Sato polynomials and analytic non-equivalence of plane curve singularities

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## Abstract

We compute Bernstein-Sato polynomials of some pairs of topologically equivalent plane curve singularities. Some pairs have the same Tjurina number but distinct Bernstein-Sato polynomials, which implies that they are not analytically equivalent.

Key words: plane curve, analytic equivalence, Bernstein-Sato polynomial, Puiseux characteristic, Tjurina number

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## 1 Introduction

The purpose of this paper is to present some examples of pairs of plane curve singularities with the same topological type and the same Tjurina number that have different Bernstein-Sato polynomials, which implies that the pair are not analytically equivalent.

In general, let  $f$  and  $g$  be complex analytic functions defined on a neighborhood of the origin  $0$  in  $\mathbb{C}^n$ . Then  $f$  and  $g$ , or more precisely, the germs  $f = 0$  and  $g = 0$  of complex hypersurfaces, are said to be analytically equivalent at the origin if there exist a germ of biholomorphic map  $\varphi$  at  $0$  and a germ of holomorphic function  $u$  at  $0$  such that  $\varphi(0) = 0$ ,  $u(0) \neq 0$ , and  $g = u(f \circ \varphi)$ .

On the other hand,  $f$  and  $g$  are said to be topologically equivalent at the origin if there exists a homeomorphism  $\varphi$  of a neighborhood  $U$  onto  $V$  such that  $\varphi(0) = 0$  and  $\varphi(\{x \in U \mid f(x) = 0\}) = \{x \in V \mid g(x) = 0\}$ .

Let  $\mathbb{C}\{x\}$  be the ring of convergent power series in the variables  $x = (x_1, \dots, x_n)$ . We denote by  $J_f$  the ideal of  $\mathbb{C}\{x\}$  generated by  $f$  and its derivatives  $\partial f / \partial x_i$  ( $i = 1, \dots, n$ ). The quotient algebra  $T(f) := \mathbb{C}\{x\} / J_f$  is called the Tjurina algebra and its dimension as the vector space over  $\mathbb{C}$  is called the Tjurina number of  $f$ , which we denote by  $\tau(f)$ .

Mather and Yau [8] proved under the condition  $f$  and  $g$  have isolated singularity at 0 that  $f$  and  $g$  are analytically equivalent if and only if  $T(f)$  and  $T(g)$  are isomorphic as  $\mathbb{C}$ -algebras. In particular, the equality  $\tau(f) = \tau(g)$  is a necessary condition for  $f$  and  $g$  to be analytically equivalent. See [5], [11] for classification of analytic equivalence of curves defined by some specific polynomials including cubics, and e.g. [3] for related problems for curves. In any case, not much seems to be known about classification of analytic equivalence classes of germs of plane curves.

The local Bernstein-Sato polynomial, which is also called the local  $b$ -function, is also invariant under analytic equivalence. Let us denote by  $\mathcal{D}$  the ring of linear differential operators with coefficients in  $\mathbb{C}\{x\}$ . Then the local  $b$ -function  $b_f(s)$  of  $f$  at the origin 0 is the nonzero polynomial  $b(s)$  of the least degree in an indeterminate  $s$  that satisfies

$$b(s)f^s \in \mathcal{D}[s]f^{s+1},$$

that is, there exists a polynomial  $P(s)$  in  $s$  with coefficients in  $\mathcal{D}$  such that  $b(s)f^s = P(s)f^{s+1}$  holds. Here  $f^s$  is regarded as a formal function, on which  $\mathcal{D}[s]$  acts naturally. The above  $b_f(s)$  is uniquely determined up to nonzero constant multiple, hence is unique if we impose that  $b_f(s)$  be monic.

Kashiwara [6] proved that there always exists such  $b_f(s)$  and its roots are negative rational numbers. Yano [13] calculated the local Bernstein-Sato polynomials of a variety of examples including curves. An algorithm for computing  $b_f(s)$  of an arbitrary polynomial  $f$  was given by the present author [10]. See also [9] for improvements.

On the other hand, complete classification of topological equivalence is well-known for germs of plane curves. In what follows, we restrict our attention to germs of holomorphic functions in two variables which are irreducible in the unique factorization domain  $\mathbb{C}\{x, y\}$  in two variables  $x, y$ .

By a locally holomorphic change of the coordinates, the germ of the curve  $f = 0$  is parametrized by a Puiseux expansion

$$x = t^p, \quad y = t^q + c_{q+1}t^{q+1} + c_{q+2}t^{q+2} + \dots \quad (1.1)$$

in the complex parameter  $t$  with  $|t|$  sufficiently small, where  $p, q$  are positive integers and  $c_{q+1}, c_{q+2}, \dots$  are complex numbers. We set  $c_q = 1$ . We may also assume  $p < q$  and that the greatest common divisor of the elements of

the set  $\{p, q\} \cup \{j \in \mathbb{Z} \mid j > q, c_j \neq 0\}$  is one. Then the Puiseux characteristic of (1.1) is defined as follows (see [12]): First set

$$r_1 = \min\{r \in \mathbb{Z} \mid r \geq q, c_r \neq 0, p \nmid r\}, \quad e_1 = \gcd(p, r_1),$$

where gcd means the greatest common divisor. If  $e_1 > 1$ , then set

$$r_2 = \min\{r \in \mathbb{Z} \mid r \geq q, c_r \neq 0, e_1 \nmid r\}, \quad e_2 = \gcd(e_1, r_2).$$

Define  $r_i$  and  $e_i$  recursively in the same way. Then we have  $e_m = 1$  for some  $m$  and terminate this procedure. The sequence  $(p; r_1, \dots, r_m)$  is called the Puiseux characteristic of (1.1). In particular, if  $p$  and  $q$  are relatively prime, then the Puiseux characteristic is simply  $(p; q)$ .

It is a classical result attributed to Burau and Zariski dating back to the 1930s that the Puiseux characteristic is in one-to-one correspondence to each topological equivalence class of the germs of plane curves (see e.g., [12]). In particular, if  $p$  and  $q$  are relatively prime, then the curve germ defined by (1.1) is topologically equivalent to that of  $y^p - x^q = 0$ . It should be noted that the Alexander polynomial of the knot defined by (1.1), which can be thought of as a topological counterpart of the Bernstein-Sato polynomial, plays an essential role in the proof of the classical theorem above.

Yano [14] made a conjecture about the generic Bernstein-Sato polynomial of a plane curve germ in terms of its Puiseux characteristic. There are many works related to his conjecture; see e.g., [4],[1],[2]. However, complete theoretical description of the behavior of the Bernstein-Sato polynomials of the curves with the same Puiseux characteristic seems to be unknown.

## 2 Examples

We give examples of topologically equivalent germs of plane curves some of which have the same Tjurina number but different Bernstein-Sato polynomials.

In what follows  $p$  and  $q$  are relatively prime positive integers with  $p < q$  and  $f_0(x, y) = y^p - x^q$ . Let  $\mathbb{N}$  be the set of non-negative integers and set

$$G(p, q) = \{i \in \mathbb{N} \mid i > q\} \setminus (\mathbb{N}p + \mathbb{N}q).$$

Then it is easy to see that by a holomorphic change of local coordinates, the parametrization (1.1) can be transformed to a simple form

$$x = t^p, \quad y = t^q + \sum_{r \in G(p, q)} c_r t^r.$$

Among such parametrizations, we pick up the one

$$x = t^p, \quad y = t^q + t^r \quad (2.2)$$

for each  $r \in G(p, q)$ . Note that if  $c_r \neq 0$ , then this is analytically equivalent to  $x = t^p, y = t^q + c_r t^r$ .

For each  $r \in G(p, q)$ , let  $f_r(x, y)$  be the polynomial whose germ at 0 is the defining function of the plane curve germ parametrized by (2.2). See 2.3 of [12] for a method of computing  $f_r$ , other than the elimination method based on an appropriate Gröbner basis. We denote by  $\tau(f_r)$  the Tjurina number of  $f_r$  and by  $b_r(s)$  the local Bernstein polynomial of  $f_r$  at 0. Note that  $f_0$  and  $f_r$  with  $r \in G(p, q)$  are all topologically equivalent. Note also that there is an explicit formula (see 6.4 of [7]) for the Bernstein-Sato polynomials of quasi-homogeneous polynomials with isolated singularity, which applies to the binomial  $x^p - y^q$ .

The following examples were computed by using the library file “nn.ndbf.rr” of Risa/Asir developed by Nishiyama and Noro [9].

**Example 1** We set  $p = 4, q = 9$ . Then we have  $f_0 = y^4 - x^9$  and  $G(4, 9) = \{10, 11, 14, 15, 19, 23\}$ . Corresponding polynomials are

$$\begin{aligned} f_{10} &= y^4 - 2x^5y^2 - 4x^7y - x^9 + x^{10}, \\ f_{11} &= y^4 - 4x^5y^2 - x^9 + 2x^{10} - x^{11}, \\ f_{14} &= y^4 - 2x^7y^2 - 4x^8y - x^9 + x^{14}, \\ f_{15} &= y^4 - 4x^6y^2 - x^9 + 2x^{12} - x^{15}, \\ f_{19} &= y^4 - 4x^7y^2 - x^9 + 2x^{14} - x^{19}, \\ f_{23} &= y^4 - 4x^8y^2 - x^9 + 2x^{16} - x^{23}. \end{aligned}$$

The Tjurina numbers of  $f_r$  are as follows:

$f$	$f_0$	$f_{10}$	$f_{11}$	$f_{14}$	$f_{15}$	$f_{19}$	$f_{23}$
$\tau(f)$	24	21	21	23	22	23	24

The local Bernstein-Sato polynomial  $b_0(s) = b_{f_0}(s)$  of  $f_0$  at 0 is

$$\begin{aligned} b_0(s) &= (36s + 13)(36s + 17)(36s + 25)(36s + 29)(36s + 31)(36s + 35) \\ &\quad \times (36s + 37)(36s + 41)(36s + 43)(36s + 47)(36s + 55)(36s + 59) \\ &\quad \times (18s + 11)(18s + 13)(18s + 17)(18s + 19)(18s + 23)(18s + 25) \\ &\quad \times (12s + 7)(12s + 11)(12s + 13)(12s + 17)(6s + 5)(6s + 7)(s + 1). \end{aligned}$$

The Bernstein-Sato polynomials of  $f_r$  with  $r \in G(4, 9)$  are as follows:

$$\begin{aligned}
b_{10}(s) &= b_0(s) \frac{(36s+19)(36s+23)(18s+7)(12s+5)}{(36s+55)(36s+59)(18s+25)(12s+17)}, \\
b_{11}(s) &= b_0(s) \frac{(36s+19)(36s+23)(12s+5)}{(36s+55)(36s+59)(12s+17)}, \\
b_{14}(s) &= b_0(s) \frac{36s+23}{36s+59}, \\
b_{15}(s) &= b_0(s) \frac{(36s+19)(36s+23)}{(36s+55)(36s+59)}, \\
b_{19}(s) &= b_{14}(s), \\
b_{23}(s) &= b_0(s).
\end{aligned}$$

Thus  $f_{10}$  and  $f_{11}$  have the same Tjurina number 21 but have different Bernstein-Sato polynomials. We do not know if each of the pairs  $(f_0, f_{23})$  and  $(f_{14}, f_{19})$  is one with analytically equivalent germs.

**Example 2**  $p = 5, q = 6, f_0 = y^5 - x^6, G(5, 6) = \{7, 8, 9, 13, 14, 19\}$ . The corresponding polynomials are

$$\begin{aligned}
f_7 &= y^5 - 5x^4y^2 - 5x^5y - x^6 - x^7, \\
f_8 &= y^5 - 5x^4y^2 - 5x^6y - x^6 - x^8, \\
f_9 &= y^5 - 5x^3y^3 + 5x^6y - x^6 - x^9, \\
f_{13} &= y^5 - 5x^5y^2 - 5x^9y - x^6 - x^{13}, \\
f_{14} &= y^5 - 5x^4y^3 + 5x^8y - x^6 - x^{14}, \\
f_{19} &= y^5 - 5x^5y^3 + 5x^{10} - x^6 - x^{19}.
\end{aligned}$$

The Tjurina numbers are as follows:

$f$	$f_0$	$f_7$	$f_8$	$f_9$	$f_{13}$	$f_{14}$	$f_{19}$
$\tau(f)$	20	18	18	18	20	19	20

The Bernstein-Sato polynomial of  $f_0$  is

$$\begin{aligned}
b_0(s) &= (30s+11)(30s+17)(30s+23)(30s+29)(30s+31)(30s+37) \\
&\quad \times (30s+43)(30s+49)(15s+8)(15s+11)(15s+13)(15s+14) \\
&\quad \times (15s+16)(15s+17)(15s+19)(15s+22)(10s+7)(10s+9) \\
&\quad \times (10s+11)(10s+13)(s+1).
\end{aligned}$$

Those of  $f_r$  are as follows:

$$b_7(s) = b_0(s) \frac{(30s + 13)(30s + 19)(15s + 7)}{(30s + 43)(30s + 49)(15s + 22)},$$

$$b_8(s) = b_0(s) \frac{(30s + 13)(30s + 19)}{(30s + 43)(30s + 49)},$$

$$b_9(s) = b_0(s) \frac{(30s + 19)(15s + 7)}{(30s + 49)(15s + 22)},$$

$$b_{13}(s) = b_0(s),$$

$$b_{14}(s) = b_0(s) \frac{30s + 19}{30s + 49},$$

$$b_{19}(s) = b_0(s).$$

Thus  $f_7, f_8, f_9$  have the same Tjurina number 18 but distinct Bernstein-Sato polynomials.

**Example 3**  $p = 5, q = 7, f_0 = y^5 - x^7, G(5, 7) = \{8, 9, 11, 13, 16, 18, 23\}$ .  
The corresponding polynomials are

$$\begin{aligned} f_8 &= y^5 - 5x^3y^3 + 5x^6y - x^7 - x^8, \\ f_9 &= y^5 - 5x^5y^2 - 5x^6y - x^7 - x^9, \\ f_{11} &= y^5 - 5x^5y^2 - 5x^8y - x^7 - x^{11}, \\ f_{13} &= y^5 - 5x^4y^3 + 5x^8y - x^7 - x^{13}, \\ f_{16} &= y^5 - 5x^6y^2 - 5x^{11} - x^7 - x^{16}, \\ f_{18} &= y^5 - 5x^5y^3 + 5x^{10}y - x^7 - x^{18}, \\ f_{23} &= y^5 - 5x^6y^3 + 5x^{12}y - x^7 - x^{23}. \end{aligned}$$

The Tjurina numbers are as follows:

$f$	$f_0$	$f_8$	$f_9$	$f_{11}$	$f_{13}$	$f_{16}$	$f_{18}$	$f_{23}$
$\tau(f)$	24	21	22	22	22	24	23	24

The Bernstein-Sato polynomial of  $f_0$  is

$$\begin{aligned} b_0(s) &= (35s + 12)(35s + 17)(35s + 19)(35s + 22)(35s + 24)(35s + 26) \\ &\quad \times (35s + 27)(35s + 29)(35s + 31)(35s + 32)(35s + 33)(35s + 34) \\ &\quad \times (35s + 36)(35s + 37)(35s + 38)(35s + 39)(35s + 41)(35s + 43) \\ &\quad \times (35s + 44)(35s + 46)(35s + 48)(35s + 51)(35s + 53)(35s + 58)(s + 1). \end{aligned}$$

Those of other  $f_r$  are as follows:

$$\begin{aligned}
b_8(s) &= \frac{(35s + 13)(35s + 18)(35s + 23)}{(35s + 48)(35s + 53)(35s + 58)}, \\
b_9(s) &= \frac{(35s + 16)(35s + 18)(35s + 23)}{(35s + 51)(35s + 53)(35s + 58)}, \\
b_{11}(s) &= \frac{(35s + 16)(35s + 23)}{(35s + 51)(35s + 58)}, \\
b_{13}(s) &= \frac{(35s + 18)(35s + 23)}{(35s + 53)(35s + 58)}, \\
b_{16}(s) &= b_0(s), \\
b_{18}(s) &= \frac{35s + 23}{35s + 58}, \\
b_{23}(s) &= b_0(s).
\end{aligned}$$

Thus  $f_9$ ,  $f_{11}$ , and  $f_{13}$  have the same Tjurina number 22 but distinct Bernstein-Sato polynomials.

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