On various b-functions of specializable D-modules

By

Toshinori Oaku*

Abstract

We recall several notions known under the name of b-function, indicial polynomial, or Bernstein-Sato polynomial, for D-modules. We summarize relations among these notions as well as algorithms for computation.

§ 1. Introduction

Mikio Sato, Masaki Kashiwara, and J. Bernstein introduced the b-function or the Bernstein-Sato polynomial for a polynomial, or more generally, for a germ of holomorphic function. This notion is closely related to the classical notion of the indicial polynomial. So let us first recall the defintion of the indicial polynomial, or the b-function, of a section of a D-module along a submanifold.

Let X be an n-dimensional complex manifold and Y a complex analytic submanifold of X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{I}_Y$ be the defining ideal of Y, which is a sheaf of ideals of $\mathcal{O}_X$. We denote by $\mathcal{D}_X$ the sheaf on X of (rings of) differential operators with holomorphic coefficients.

The V-filtration $\{V_Y^i(\mathcal{D}_X)\}_{i\in\mathbb{Z}}$ of $\mathcal{D}_X$ along Y is defined by

$$V_Y^i(\mathcal{D}_X) := \{P \in \mathcal{D}_X|_Y \mid P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-i} \quad (\forall j \in \mathbb{Z})\}$$

with the convention $\mathcal{I}_Y^j = \mathcal{O}_X|_Y$ for $j < 0$. Let θ be a vector field on a neighborhood of Y in X which induces the identity map on $\mathcal{I}_Y/\mathcal{I}_Y^2$. In a local coordinate $x = (x_1, \dots, x_d, x_{d+1}, \dots, x_n)$ such that $Y = \{x_1 = \cdots = x_d = 0\}$, we may take

$$\theta = x_1\frac{\partial}{\partial x_1} + \cdots + x_d\frac{\partial}{\partial x_d}.$$

2010 Mathematics Subject Classification(s): 14F10, 32C38, 35A27.
Key Words: D-module, b-function, indicial polynomial
Supported by JSPS Grant-in-Aid for Scientific Research (C) 26400123
*Department of Mathematics, Tokyo Woman's Christian University, Suginami-ku, Tokyo, 167-8585, Japan.

Definition 1.1. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module defined on X. Let u be a section of $\mathcal{M}$ defined on a neighborhood of $x_0 \in Y$. The *indicial polynomial* or the *b-function* of u along Y at x_0 is the monic polynomial $b(s)$, if any, in an indeterminate s of the least degree such that

$$b(\theta)u \in V_Y^{-1}(\mathcal{D}_X)_{x_0}u$$

holds, that is, there exists $P \in V_Y^{-1}(\mathcal{D}_X)_{x_0}$ such that

$$(b(\theta) + P)u = 0.$$

If we impose the condition $\operatorname{ord} P \leq \deg b(s)$, then $b(s)$ is called a *regular indicial polynomial* of u along Y at x_0. $\mathcal{M}$ is called *(regular) specializable* along Y if each section u of $\mathcal{M}$ on a neighborhood of an arbitrary $x_0 \in Y$ has a (regular) indicial polynomial along Y at x_0.

If P is an ordinary differential operator and x_0 is its regular singular point, then the indicial polynomial of u such that $Pu = 0$ (i.e., u is the residue class of 1 in $\mathcal{D}_X/\mathcal{D}_X P$) along the point $\{x_0\}$ coincides (basically) with the classical definition of the indicial polynomial at x_0. For example, the indicial (and the regular indicial) polynomial of u such that $(x\partial_x - a)u = 0$ along $\{0\}$ is $s - a$, where we assume $X = \mathbb{C}$ and denote $\partial_x = d/dx$. On the other hand, the b-function of u with $(x^2\partial_x - a)u = 0$ along $\{0\}$ is 1 if $a \neq 0$, and $s(s+1)$ if $a = 0$ (this is the classical indicial polynomial of $\partial_x x^2 \partial_x$). There is no regular indicial polynomial of a section u such that $(x^2\partial_x - a)u = 0$ along $\{0\}$ if $a \neq 0$.

Regular indicial polynomial is not necessarily unique. For example, for u such that

$$x^2\partial_x^2 u = x(\partial_x + \partial_y^2)u = 0$$

in two variables (x, y), the indicial polynomial of u along $x = 0$ is s, while $s(s - c)$ is a regular indicial polynomial of u along $x = 0$ for any c, of the least degree. This follows from $x\partial_x u = -x\partial_y^2 u$ and $(x^2\partial_x^2 + cx\partial_x)u = -cx\partial_y^2 u$. Note that $\mathcal{D}_X u$ is holonomic since its characteristic variety is

$$\{(x, y, \xi dx + \eta dy) \mid x = \eta = 0\} \cup \{(x, y, \xi dx + \eta dy) \mid \xi = \eta = 0\}.$$

Hence we mean by 'the regular indicial polynomial' the set of the regular indicial polynomials of the least degree.

In general, the computation of (regular) indicial polynomial is not trivial. If u satisfies a system of linear (ordinary or partial) differential equation with polynomial coefficients, then an algorithm to detect the existence of and to compute, if any, the b-function of u along a hyperplane was given in [2] by the present author, and consequently in [4] for linear submanifold of arbitrary codimension. A similar algorithm for the regular indicial polynomial along a linear submanifold was introduced in [3].

Example 1.2. The D-module for Appell's F_1 is defined by $P_1 u = P_2 u = 0$ with

$$P_1 = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + (c-(a+b_1+1)x)\partial_x - b_1 y\partial_y - ab_1,$$
$$P_2 = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + (c-(a+b_2+1)y)\partial_y - b_2 x\partial_x - ab_2$$

and parameters a, b_1, b_2, c. Both the indicial and the regular indicial polynomials along the origin $(0,0)$ are $s(s+c-1)$ for arbitrary values of the parameters although $\mathcal{D}_X u$ is holonomic if and only if $c \neq a+1$.

Example 1.3. Let $\mathcal{M}_A(\beta)$ be the A-hypergeometric (GKZ) system for an arbitrary $d \times n$ integer matrix A such that $\operatorname{rank} A = d$ with parameters $\beta = (\beta_1, \dots, \beta_d)$. Then $\mathcal{M}_A(\beta)$ is regular specializable along the origin for any β (see [3]). In particular we have isomorphisms

$$\mathrm{Ext}^k_{(\mathcal{D}_X)_0}(\mathcal{M}_A(\beta), \mathbb{C}\{x\}) \simeq \mathrm{Ext}^k_{(\mathcal{D}_X)_0}(\mathcal{M}_A(\beta), \mathbb{C}[[x]]) \qquad (\forall k \in \mathbb{Z}).$$

§ 2. The b-function with respect to a graph embedding

We follow the method of the definition of Bernstein-Sato polynomial for an arbitrary variety, possibly with multiplicities, by Budur-Mustata-Saito [1]. Let $\mathcal{J}$ be a coherent ideal of $\mathcal{O}_X$. Let $f_1, \dots, f_d$ be a set of local generators of $\mathcal{J}$ on an neighborhood U of $x_0 \in Y$. Consider the associated embedding

$$\iota : U \ni x \longmapsto (x, f_1(x), \dots, f_d(x)) \in U \times \mathbb{C}^d$$

and set $Z = \iota(U)$, which depend on the choice of local generators $f_1, \dots, f_d$ of $\mathcal{J}$.

Let $\mathcal{B}_{Z|U\times\mathbb{C}^d} = \mathcal{H}^d_{[Z]}(\mathcal{O}_{U\times\mathbb{C}^d})$ be the d-th local cohomology group. Suppose that $\mathcal{M}$ is a coherent left $\mathcal{D}_X$-module defined on X and u is a section of $\mathcal{M}$ defined on a neighborhood of x_0. Then $\iota_*(u) = u \otimes \delta(t_1 - f_1)\cdots\delta(t_d - f_d)$ is defined as a section of $\iota_*(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_{Z|U\times\mathbb{C}^d}$.

Theorem 2.1. *Let $b(s)$ be the (regular) indicial polynomial of $\iota_*(u)$ along $U \times \{0\}$. Then $b(s-d)$ does not depend on the choice of local generators $f_1, \dots, f_d$ of $\mathcal{J}$. We call $b(s-d)$ the* (regular) b-function of u *with respect to $\mathcal{J}$.*

If $\mathcal{M} = \mathcal{O}_X$ and $u = 1$, then the b-function in the above sense conicides with $b(-s)$, where $b(s)$ is the Bernstein-Sato polynomial of the variety (with respect to the ideal $\mathcal{J}$) defined by Budur-Mustata-Saito [1].

Proof. Suppose that there exist sections $a_1, \dots, a_d$ of $\mathcal{O}_X$ at x_0 such that $f_{d+1} = a_1 f_1 + \cdots + a_d f_d$. Define an embedding

$$\iota : X \times \mathbb{C}^d \longrightarrow X \times \mathbb{C}^{d+1}$$

by

$$\iota(x, t_1, \dots, t_d) = (x, t_1, \dots, t_d, a_1(x)t_1 + \cdots + a_d(x)t_d).$$

Set $Z = \{(x, t_1, \dots, t_d) \mid t_1 = \cdots = t_d = 0\}$. Then we have

$$\iota(Z) = \{(x, t_1, \dots, t_d, t_{d+1}) \in X \times \mathbb{C}^d \mid t_1 = \cdots = t_d = t_{d+1} = 0\}$$

and

$$\begin{aligned}
&\iota_*(u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - a_1(x)t_1 - \cdots - a_d(x)t_d) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - a_1(x)f_1 - \cdots - a_d(x)f_d) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - f_{d+1}).
\end{aligned}$$

Let $b(s)$ be the b-function of $u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)$ along Z and $\tilde{b}(s)$ be that of $u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - f_{d+1})$ along $\iota(Z)$. Then it is easy to see that $b(s-d) = \tilde{b}(s-d-1)$ holds in view of the lemma below. This completes the proof. □

Lemma 2.2 (Budur-Mustata-Saito). *Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module and u a section of $\mathcal{M}$. Let Y be a non-singular complex hypersurface of X and let $\iota : X \to X \times \mathbb{C}$ be a holomorphic embedding. Let $b(s)$ be the b-function of u along Y at $x_0 \in Y$ and $\tilde{b}(s)$ be that of $u \otimes \delta(t)$ along $\iota(Y)$ at $(x_0, 0)$. Then one has $\tilde{b}(s-1) = b(s)$.*

Proof. We may assume $Y = \{x \in X \mid x_1 = \cdots = x_d = 0\}$ and $\iota(x) = (x, 0)$, and conequently

$$\iota(Y) = \{(x, t) \in X \times \mathbb{C} \mid x_1 = \cdots = x_d = t = 0\}.$$

There exists $Q \in V_Y^{-1}(\mathcal{D}_X)$ such that

$$(b(x_1\partial_1 + \cdots + x_d\partial_d) + Q)u = 0.$$

Then we have

$$(b(x_1\partial_1 + \cdots + x_d\partial_d + \partial_t t) + Q)(u \otimes \delta(t)) = 0$$

and Q belongs to $V_{\iota(Y)}^{-1}(\mathcal{D}_{X\times\mathbb{C}})$. Thus $\tilde{b}(s)$ is a factor of $b(s+1)$.

On the other hand, there exists $Q \in V_{\iota(Y)}^{-1}(\mathcal{D}_{X\times\mathbb{C}})$ such that

$$(\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + t\partial_t) + Q)(u \otimes \delta(t)) = 0.$$

Writing Q in the form

$$Q = \sum_{i,j\geq 0} Q_{ij}(x, \partial)\partial_t^i t^j,$$

we have

$$\begin{aligned} 0 &= (\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + t\partial_t) + Q)(u \otimes \delta(t)) \\ &= \tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + \partial_t t - 1)(u \otimes \delta(t)) + \sum_{i,j\geq 0} Q_{ij}(x,\partial)u \otimes \partial_t^i t^j \delta(t) \\ &= \tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d - 1)u \otimes \delta(t) + \sum_{i\geq 0} Q_{i0}(x,\partial)u \otimes \delta^{(i)}(t). \end{aligned}$$

This implies, in particular,

$$(\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d - 1) + Q_{00})u = 0.$$

Since Q_{00} belongs to $V_Y^{-1}(\mathcal{D}_X)$, we know that $b(s)$ divides $\tilde{b}(s-1)$. In conclusion, we get $b(s) = \tilde{b}(s-1)$. □

Existing algorithms ([4], [3]) for (regular) indicial polynomials along linear submanifolds provide ones for the (regular) b-function with respect to a graph embedding.

Example 2.3. Let $\mathcal{M}_A(\beta) = \mathcal{D}_X u$ be the A-hypergeometric system for $A = \begin{pmatrix} 1\,1\,1 \\ 0\,1\,2 \end{pmatrix}$ with parameters $\beta = (\beta_1, \beta_2)$; i.e.,

$$(x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - \beta_1)u = (x_2\partial_2 + 2x_3\partial_3 - \beta_2)u = (\partial_1\partial_3 - \partial_2^2)u = 0.$$

The singular locus of $\mathcal{M}_A(\beta)$ is

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1x_3(4x_1x_3 - x_2^2) = 0\}.$$

The b-function of u with respect to the ideal $\mathcal{O}_X f$ with $f := 4x_1x_3 - x_2^2$ is $s(s-\beta_1-1/2)$ at any point $p \in \mathbb{C}^3$ such that $f(p) = 0$.

Example 2.4. With $X = \mathbb{C}^3 \ni (x,y,z)$, set $\mathcal{J} = \mathcal{O}_X(x^3 - y^2) + \mathcal{O}_X(x^2 - z)$, which is the defining ideal of a monomial curve $x^3 - y^2 = x^2 - z = 0$.

1. The b-function of $1 \in \mathcal{O}_X$ with respect to $\mathcal{J}$ at 0 is $(s-2)(6s-11)(6s-13)$.
2. The b-function of u such that $\partial_x u = \partial_y u = (z\partial_z - a)u = 0$ with respect to $\mathcal{J}$ at 0 is

   $$b(s,a) := (s-2)(s-a-2)(2s-2a-5)(6s-4a-11) \times (6s-4a-13)(6s-4a-15)$$

   if $a \neq 0, -1, -2$. If $a = 0$, then the b-function of u is $(s-2)(2s-5)(6s-11)(6s-13)$ whereas $b(s,0) = (s-2)^2(2s-5)^2(6s-11)(6s-13)$. If $a = -1$, then the b-function is $(s-1)(s-2)(2s-3)(6s-7)(6s-11)$ whereas $b(s,-1) = (s-1)(s-2)(2s-3)^2(6s-7)(6s-11)$. If $a = -2$, then the b-function is $s(s-2)(2s-1)(6s-5)(6s-7)$ whereas $b(s,-2) = s(s-2)(2s-1)^2(6s-5)(6s-7)$.

§ 3. Comparison between the indicial polynomial and the b-function via graph embedding

Probably, the following theorem is well-known to specialists:

Theorem 3.1. *Let X be a complex manifold and $\mathcal{M}$ be a coherent (sheaf of) left $\mathcal{D}_X$-module on X. Let $f_1, \dots, f_d$ be holomorphic functions on X and set $Y = \{x \in X \mid f_1(x) = \cdots = f_d(x) = 0\}$. Assume that $df_1 \wedge \cdots \wedge df_d \neq 0$ at x_0. Set $Z = \{(x, t_1, \dots, t_d) \in X \times \mathbb{C}^d \mid t_i = f_i(x)\,(i = 1, \dots, d)\}$ and $\mathcal{B}_{Z|X\times\mathbb{C}^d} = \mathcal{H}^d_Z(\mathcal{O}_{X\times\mathbb{C}^d})$. Set $\mathcal{N} := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_{Z|X\times\mathbb{C}^d}$, which has a structure of left $\mathcal{D}_{X\times\mathbb{C}^d}$-module. Then $\mathcal{M}$ is (regular) specializable along Y at x_0 if and only if $\mathcal{N}$ is (regular) specializable along $X \times \{0\}$ at $(x_0, 0)$. Moreover, for any u in the stalk of $\mathcal{M}$ at x_0, the (regular) b-function of u along Y at x_0 coincides with that of $u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)$ along $X \times \{0\}$ at $(x_0, 0)$.*

Proof. We may assume that X is an open set of $\mathbb{C}^n$ containing $x_0 = 0 \in \mathbb{C}^n$, and that $f_i = x_i$ for $i = 1, \dots, d$. We use the notation $x = (x', x'')$ with $x' = (x_1, \dots, x_d)$, $x'' = (x_{d+1}, \dots, x_n)$, and $\partial' = (\partial_1, \dots, \partial_d)$, $\partial'' = (\partial_{d+i}, \dots, \partial_n)$ with $\partial_i = \partial/\partial x_i$. .

Assume that $\mathcal{M}$ is specializable along Y at 0 and let u belong to the stalk of $\mathcal{M}$ at 0. Let $b(s) \in \mathbb{C}[s]$ be the b-function of u along Y at 0. Then there exists a differential operator $Q = Q(x', x'', \partial', \partial'') \in V_Y^{-1}(\mathcal{D}_X)_0$ such that

$$(b(x_1\partial_1 + \cdots + x_d\partial_d) + Q)u = 0.$$

It follows that

$$\begin{aligned}
&(b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d)) + \tilde{Q})(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) \\
&= (b(x_1\partial_1 + \cdots + x_d\partial_d) + Q)u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d) = 0
\end{aligned}$$

with

$$\tilde{Q} = Q(t_1, \dots, t_d, x'', \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'')$$

since

$$\begin{aligned}
&(t_i - x_i)(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = 0, \\
&(\partial_{t_i} + \partial_i)(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = (\partial_i u) \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)
\end{aligned}$$

hold for $i = 1, \dots, d$.

Then $\tilde{Q}$ belongs to $V^{-1}_{X\times\{0\}}(\mathcal{D}_{X\times\mathbb{C}^d})$. Moreover, we have

$$b(t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}) - b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d)) \in V^{-1}_{X\times\{0\}}(\mathcal{D}_{X\times\mathbb{C}^d}).$$

Hence the b-function of $u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)$ along $X \times \{0\}$ at $(0,0)$ is a factor of $b(s)$.

Conversely, let $b(s)$ be the b-function of $u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)$ along $X \times \{0\}$ at $(0,0)$. Then there eixsts a differential operator

$$Q = Q(t_1, \dots, t_d, x', x'', \partial_{t_1}, \dots, \partial_{t_d}, \partial', \partial'') \in V^{-1}_{X \times \{0\}}(\mathcal{D}_{X \times \mathbb{C}^d})$$

such that

$$(b(t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d}) + Q)(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = 0.$$

By using $\partial_{t_i} = (\partial_{t_i} + \partial_i) - \partial_i$, we rewrite the operator as

$$\begin{aligned} P &:= b(t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d}) + Q \\ &= b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d)) \\ &\quad + \sum_{\alpha \in \mathbb{N}^d} Q_\alpha(t_1, \dots, t_d, x, \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'') \partial'^{\alpha} \end{aligned}$$

with $Q_\alpha \in V^{-1}_{X \times \{0\}}(\mathcal{D}_{X \times \mathbb{C}^d})$.

We have

$$[t_1 - x_1, P](u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = 0$$

with

$$[t_1 - x_1, P] = \sum_{\alpha \in \mathbb{N}^d} \alpha_1 Q_\alpha(t_1, \dots, t_d, x, \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'') \partial'^{\alpha - (1,0,\dots,0)}.$$

Now let m be the maximum of α_1 such that $Q_\alpha \neq 0$ with $\alpha = (\alpha_1, \dots, \alpha_d)$. It follows that

$$\begin{aligned} \tilde{P} &:= P - \frac{1}{m} \partial_1 [t_1 - x_1, P] \\ &= b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d)) \\ &\quad + \sum_{\alpha \in \mathbb{N}^d} Q_\alpha(t_1, \dots, t_d, x, \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'') \partial'^{\alpha} \\ &\quad - \sum_{\alpha \in \mathbb{N}^d} \frac{\alpha_1}{m} \partial_1 Q_\alpha(t_1, \dots, t_d, x, \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'') \partial'^{\alpha - (1,0,\dots,0)}, \end{aligned}$$

which is of order at most $m-1$ with repect to ∂_1, also annihilates $u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)$. By induction, we conclude that there exists an operator

$$Q_0(t_1, \dots, t_d, x, \partial_{t_1} + \partial_1, \dots, \partial_{t_d} + \partial_d, \partial'') \in V^{-1}_{X \times \{0\}}(\mathcal{D}_{X \times \mathbb{C}^d})$$

such that

$$(b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d))$$
$$+ Q_0(t_1, \ldots, t_d, x, \partial_{t_1} + \partial_1, \ldots, \partial_{t_d} + \partial_d, \partial''))(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = 0,$$

and consequently

$$\begin{aligned}
&(b(x_1\partial_1 + \cdots + x_d\partial_d) + Q_0(x', x, \partial', \partial''))u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d) \\
&= (b(t_1(\partial_{t_1} + \partial_1) + \cdots + t_d(\partial_{t_d} + \partial_d)) \\
&\qquad + Q_0(t_1, \ldots, t_d, x, \partial_{t_1} + \partial_1, \ldots, \partial_{t_d} + \partial_d, \partial''))(u \otimes \delta(t_1 - x_1) \cdots \delta(t_d - x_d)) = 0.
\end{aligned}$$

This implies that the b-function of u along Y at 0 is a factor of $b(s)$. □

If u satisfies a system of linear differential equations with polynomial coefficients, then in view of the theorem above, existing algorithms [2], [4], [3] for linear submanifolds immediately provide us with algorithms for computing the (regular) indicial polynomial along an arbitrary algebraic subvariety of $\mathbb{C}^n$ at a non-singular point.

Example 3.2. Let $\mathcal{M}_A(\beta) = \mathcal{D}_X u$ be the A-hypergeometric system for $A = \begin{pmatrix} 1\,1\,1 \\ 0\,1\,2 \end{pmatrix}$ with parameters $\beta = (\beta_1, \beta_2)$ as in Example 2.3. Then the indicial polynomial of u along the hypersurface $Y = \{(x_1, x_2, x_3) \mid 4x_1x_3 - x_2^2 = 0\}$ is $s(s-\beta_1-1/2)$ at any non-singular point (i.e., other than the origin) of Y.

References

[1] Budur, N., Mustata, M., Saito, M., Bernstein-Sato polynomials of arbitrary varieties, *Compositio Math.* **142** (2006), 779–798.
[2] Oaku, T., An algorithm of computing b-functions, *Duke Math. J.* **87** (1997), 115–132.
[3] Oaku, T., Regular b-functions of D-modules, *J. Pure and Applied Algebra* **213** (2009), 1545-1557.
[4] Oaku, T., Takayama, N., Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups, *J. Pure Appl. Algebra* **156** (2001), 267–308.