Operational calculus for holonomic distributions
in the framework of $D$-module theory

Dedicated to Professor H. Komatsu and Professor T. Kawai

By

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Abstract

Let $f$ be a real polynomial of $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function of $x$ which satisfies a holonomic system of linear differential equations. We study the distribution $f + \varphi$ with a meromorphic parameter $\lambda$, especially its Laurent expansion and integration, from an algorithmic viewpoint in the framework of $D$-module theory.

§ 1. Introduction

Let $f$ be a non-constant real polynomial in $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function on an open subset $U$ of $\mathbb{R}^n$. Then $\varphi$ can be regarded as a distribution (generalized function in the sense of L. Schwartz) on $U$. We assume that there exists a left ideal $I$ of the ring $D_n$ of differential operators with polynomial coefficients in $x$ which annihilates $\varphi$ on $U_f := \{ x \in U \mid f(x) \neq 0 \}$, i.e., $P \varphi$ vanishes on $U_f$ for any $P \in I$. Moreover, we assume that $M := D_n/I$ is a holonomic $D_n$-module. In this situation, $\varphi$ is called a (locally integrable) holonomic function or a holonomic distribution.

Let us consider the distribution $f^\lambda \varphi$ on $U$ with a holomorphic parameter $\lambda$. This distribution can be analytically extended to a distribution-valued meromorphic function of $\lambda$ on the complex plane $\mathbb{C}$. Such a distribution was systematically studied by Kashiwara and Kawai in [2] with $f$ being, more generally, a real-valued real analytic function. Their investigation was focused on a special case where $M$ has regular singularities but most of the arguments work without this assumption.
The main purpose of this article is to give algorithms to compute
1. A holonomic system for the distribution $f_+^{\lambda_0}\varphi$ with $\lambda_0$ not being a pole of $f_+^{\lambda}\varphi$.
2. A holonomic system for each coefficient of the Laurent series of $f_+^{\lambda}\varphi$ about an arbitrary point.
3. Difference equations for the local zeta function $Z(\lambda) = \int_{\mathbb{R}^n} f_+^{\lambda}\varphi \, dx$.

As was pointed out in [2], an answer to the first problem provides us with an algorithm to compute a holonomic system for the product of two locally $L^2$ holonomic functions. Note that the product does not necessarily satisfies the tensor product of the two holonomic systems for both functions.

In Section 2, we review the theoretical properties of $f_+^{\lambda}\varphi$ mostly following Kashiwara [1] and Kashiwara and Kawai [2] in the analytic category; i.e, under a weaker assumption that $f$ is a real-valued real analytic function and that $\varphi$ satisfies a holonomic system of linear differential equations with analytic coefficients.

In Section 3, we give algorithms to computes holonomic systems considered in Section 2. As a byproduct, we obtain an algorithm to compute difference equations for the local zeta function, which was outlined in [4].

§ 2. Theoretical background

Let $D_{\mathbb{C}^n}$ be the sheaf on $\mathbb{C}^n$ of linear partial differential operators with holomorphic coefficients, which is generated by the derivations $\partial_j = \partial x_j = \partial/\partial x_j \ (j = 1, \ldots, n)$ over the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of rings of holomorphic functions on $\mathbb{C}^n$, with the coordinate system $x = (x_1, \ldots, x_n)$ of $\mathbb{C}^n$.

We denote by $Db$ the sheaf on $\mathbb{R}^n$ of the Schwartz distributions. Assume that $f = f(x)$ is a nonzero real-valued real analytic function defined on an open connected set $U$ of $\mathbb{R}^n$. Let $\varphi$ be a locally integrable function on $U$. Then $f_+^{\lambda}\varphi$ is also locally integrable on $U$ for any $\lambda \in \mathbb{C}$ with $\text{Re} \ \lambda \geq 0$, where $f_+(x) = \max\{f(x), 0\}$.

Let $\mathcal{M}$ be a holonomic $D_{\mathbb{C}^n}$-module defined on an open set $\Omega$ of $\mathbb{C}^n$ such that $U \subset \Omega \cap \mathbb{R}^n$. We also assume that $f$ is holomorphic on $\Omega$. We say that a distribution $\varphi$ is a solution of $\mathcal{M}$ on $U$ if there exist a section $u$ of $\mathcal{M}$ on $U$ and a $D_{\mathbb{C}^n}$-linear homomorphism $\Phi : D_{\mathbb{C}^n}u \to Db$ defined on $U$ such that $\Phi(u) = \varphi$. As a matter of fact, we have only to assume that $\varphi$ is a solution of $\mathcal{M}$ on $U_f := \{x \in U \ | \ f(x) \neq 0\}$ and that $\mathcal{M}$ is holonomic on $\Omega_f := \{x \in \Omega \ | \ f(x) \neq 0\}$.

§ 2.1. Fundamental lemmas

Under the assumptions above, $f_+^{\lambda}\varphi$ is a $Db(U)$-valued holomorphic function of $\lambda$ on the right half-plane

$\mathbb{C}_+ := \{\lambda \in \mathbb{C} \ | \ \text{Re} \ \lambda > 0\}$. 
In other words, let $\mathcal{OD}b$ be the sheaf on $\mathbb{C} \times \mathbb{R}^n \ni (\lambda, x)$ of distributions with a holomorphic parameter $\lambda$. Then $f^\lambda_+ \varphi$ belongs to

$$\mathcal{OD}b(\mathbb{C}_+ \times U) = \left\{ v(\lambda, x) \in \mathcal{D}b(\mathbb{C}_+ \times U) \mid \frac{\partial v}{\partial \lambda} = 0 \right\}.$$  

Let $s$ be an indeterminate corresponding to $\lambda$. The following lemma (Lemma 2.9 of [2]) plays an essential role in the following arguments.

**Lemma 2.1** (Kashiwara-Kawai [2]). Let $\Omega'$ be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega'$ is non-empty and contained in $U$. Assume $P(s) \in \mathcal{D}_{\mathbb{C}^n}(\Omega')[s]$ and $P(\lambda)(f^\lambda_+ \varphi) = 0$ holds in $\mathcal{OD}b(\mathbb{C}_+ \times V_f)$ with $V_f := \{ x \in V \mid f(x) \neq 0 \}$. Then $P(\lambda)(f^\lambda_+ \varphi) = 0$ holds in $\mathcal{OD}b(\mathbb{C}_+ \times V)$.

Let us generalize this lemma slightly. For a positive integer $m$, let us define a section $f^\lambda_+(\log f_+)^m \varphi$ of the sheaf $\mathcal{OD}b$ on $\mathbb{C}_+ \times U$ by

$$\langle f^\lambda_+(\log f_+)^m \varphi, \psi \rangle = \int_{\{ x \in U \mid f(x) > 0 \}} \varphi(x) f(x)^\lambda (\log f(x))^m \varphi(x) \psi(x) \, dx \quad (\forall \psi \in C^\infty_0(U)),$$

where $C^\infty_0(U)$ denotes the space of $C^\infty$ functions on $U$ with compact supports. In fact, $f^\lambda_+(\log f_+)^m \varphi$ is the $m$-th derivative of the distribution $f^\lambda_+ \varphi$ with respect to $\lambda$.

**Lemma 2.2.** Let $\Omega'$ be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega'$ is non-empty and contained in $U$. Let $\varphi_0, \ldots, \varphi_m$ be locally integrable functions on $V$. Assume $P_k(s) \in \mathcal{D}_{\mathbb{C}^n}(\Omega')[s]$ ($k = 0, 1, \ldots, m$) and

$$\sum_{k=0}^m P_k(\lambda)(f^\lambda_+(\log f_+)^k \varphi_k) = 0$$

holds in $\mathcal{OD}b(\mathbb{C}_+ \times V_f)$. Then (2.1) holds in $\mathcal{OD}b(\mathbb{C}_+ \times V)$.

**Proof.** We follow the argument of the proof of Lemma 2.9 in [2]. Let $\phi$ belong to $C^\infty_0(V)$ with $K := \text{supp} \phi$. Let $\chi(t)$ be a $C^\infty$ function of a variable $t$ such that $\chi(t) = 1$ for $|t| \leq 1/2$ and $\chi(t) = 0$ for $|t| \geq 1$. Then we have

$$\left\langle \sum_{k=0}^m P_k(\lambda)(f^\lambda_+(\log f_+)^k \varphi_k), \phi \right\rangle = \left\langle \sum_{k=0}^m P_k(\lambda)(f^\lambda_+(\log f_+)^k \varphi_k), \chi\left(\frac{f}{\tau}\right) \phi \right\rangle = \sum_{k=0}^m \int_V f^\lambda_+(\log f_+)^k \varphi_k^t P_k(\lambda)\left(\chi\left(\frac{f}{\tau}\right) \phi \right) \, dx$$

for any $\tau > 0$, where $^t P_k(\lambda)$ denotes the adjoint operator of $P_k(\lambda)$. Let $m_k$ be the order of $P_k(s)$ and $d_k$ be the degree of $P_k(s)$ in $s$. Then there exist constants $C_k$ such that

$$\sup_{x \in K} \left| ^t P_k(\lambda)\left(\chi\left(\frac{f(x)}{\tau}\right) \phi(x) \right) \right| \leq C_k (1 + |\lambda|)^{d_k} \tau^{-m_k} \quad (0 < \forall \tau < 1).$$


Assume $\text{Re } \lambda > \max\{m_k + 1 \mid 0 \leq k \leq m\}$ and $0 < \tau < 1$. Then we have

$$\left| \int_V f_+^k(\log f_+)^k \varphi_1^i P_k(\lambda) \left( \chi \left( \frac{f}{\tau} \right) \phi \right) dx \right|$$

$$\leq C_k (1 + |\lambda|)^{d_k} \tau^{-m_k} \int_{\{x \in V \mid 0 < f(x) \leq \tau\}} |f_+^k(\log f_+)^k \varphi_1(x) dx|$$

$$\leq k! C_k (1 + |\lambda|)^{d_k} \tau^{\text{Re } \lambda - m_k - 1} \int_{\{x \in V \mid 0 < f(x) \leq \tau\}} |\varphi_1(x) dx|$$

since $|\log t|^k \leq k! t^{-1}$ holds for $0 < t < 1$. This implies

$$\left\langle \sum_{k=0}^m P_k(\lambda)(f_+^k(\log f_+)^k \varphi_k), \phi \right\rangle = \lim_{\tau \to +0} \sum_{k=0}^m \int_V \int_V f_+^k(\log f_+)^k \varphi_k^i P_k(\lambda) \left( \chi \left( \frac{f}{\tau} \right) \phi \right) dx = 0.$$ 

The assertion of the lemma follows from the uniqueness of analytic continuation.

\[\square\]

§ 2.2. Generalized $b$-function and analytic continuation

We assume that there exists on $\Omega$ a sheaf $I$ of coherent left ideals of $D_{\mathbb{C}^n}$ which annihilates $\varphi$ on $U_f = \{x \in U \mid f(x) \neq 0\}$, namely, $P \varphi = 0$ holds on $W \cap U_f$ for any section $P$ of $I$ on an open set $W$ of $\mathbb{C}^n$. We set $M = D_{\mathbb{C}^n}/I$ and denote by $u$ the residue class of $1 \in D_X$ modulo $I$. In the sequel, we assume that $M$ is holonomic on $\Omega_f = \{z \in \Omega \mid f(z) \neq 0\}$, i.e., that $\text{Char}(M) \cap \pi^{-1}(\Omega_f)$ is of dimension $n$, where $\text{Char}(M)$ denotes the characteristic variety of $M$ and $\pi : T^*\mathbb{C}^n \to \mathbb{C}^n$ is the canonical projection.

Let $L = O_{\mathbb{C}^n}[f^{-1}, s] f^s$ be the free $O_{\mathbb{C}^n}[f^{-1}, s]$-module generated by the symbol $f^s$. Then $L$ has a natural structure of left $D_{\mathbb{C}^n}[s]$-module induced by the derivation $\partial_i f^s = s(\partial f/\partial x_i) f^{-1} f^s$. Let us consider the tensor product $L \otimes_{O_{\mathbb{C}^n}} M$ of $O_{\mathbb{C}^n}$-modules, which has a natural structure of left $D_{\mathbb{C}^n}[s]$-module.

**Lemma 2.3.** Let $v$ and $P(s)$ be sections of $M$ and of $D_{\mathbb{C}^n}[s]$ respectively on an open subset of $\Omega$. Then $P(s)(f^s \otimes v) = 0$ holds in $L \otimes_{O_{\mathbb{C}^n}} M$ if and only if $(f^m P(s)f^s) (1 \otimes v) = 0$ holds in $\mathbb{C}[s] \otimes_{\mathbb{C}} M$ for a sufficiently large $m \in \mathbb{N}$.

**Proof.** Set $M[s] = \mathbb{C}[s] \otimes_{\mathbb{C}} M$, which has a natural structure of left module over $\mathbb{C}[s] \otimes_{\mathbb{C}} D_{\mathbb{C}^n} = D_{\mathbb{C}^n}[s]$. Then we have $L \otimes_{O_{\mathbb{C}^n}} M = L \otimes_{O_{\mathbb{C}^n}[s]} M[s]$ as left $D_{\mathbb{C}^n}[s]$-module. Let $v$ be a section of $M[s]$. Since $L$ is isomorphic to $O_{\mathbb{C}^n}[f^{-1}, s]$ as $O_{\mathbb{C}^n}[s]$-module, $f^s \otimes v$ vanishes in $L \otimes_{O_{\mathbb{C}^n}[s]} M[s]$ if and only if $1 \otimes v$ vanishes in $O_{\mathbb{C}^n}[f^{-1}, s] \otimes_{O_{\mathbb{C}^n}[s]} M[s]$. First, let us show that this happens if and only if $f^m v = 0$ in $M[s]$ with some $m \in \mathbb{N}$.

Let $\rho : O_{\mathbb{C}^n}[s, t] \to O_{\mathbb{C}^n}[s, f^{-1}]$ be the homomorphism defined by $\rho(h(s, t)) = h(s, f^{-1})$ for $h(s, t) \in O_{\mathbb{C}^n}[s, t]$. Let $K$ be the kernel of $\rho$. Then we have an exact sequence

$$K \otimes_{O_{\mathbb{C}^n}[s]} M[s] \to O_{\mathbb{C}^n}[s, t] \otimes_{O_{\mathbb{C}^n}[s]} M[s] \xrightarrow{\rho \otimes \text{id}} O_{\mathbb{C}^n}[s, f^{-1}] \otimes_{O_{\mathbb{C}^n}[s]} M[s] \to 0.$$
Hence $1 \otimes v$ vanishes in $\mathcal{O}_C^n[s, f^{-1}] \otimes \mathcal{O}_C^n[s] \mathcal{M}[s]$ if and only if there exists $h(s, t) = \sum_{k=0}^{m} h_k(s) t^k \in \mathcal{K}$ such that $1 \otimes v = h(s, t) \otimes v$ holds in $\mathcal{O}_C^n[s, t] \otimes \mathcal{O}_C^n[s] \mathcal{M}[s]$, which is equivalent to $h_k(s)v = \delta_{hk}(v)$ $(k = 0, 1, \ldots, m)$ since $\mathcal{O}_C^n[s, t]$ is free over $\mathcal{O}_C^n[s]$. On the other hand, $\sum_{k=0}^{m} h_k(s) f^{-k} = \rho(h(s, t)) = 0$ implies

$$0 = f^m h_0(s)v + f^{m-1} h_1(s)v + \cdots + fh_{m-1}(s)v + h_m(s)v = f^m v.$$  

Conversely, if $f^m v = 0$ for some $m \in \mathbb{N}$, then we have $1 \otimes v = f^{-m} \otimes f^m v = 0$ in $\mathcal{O}_C^n[s, f^{-1}] \otimes \mathcal{O}_C^n[s] \mathcal{M}$.

Let $P(s)$ be a section of $\mathcal{D}_C^n[s]$ of order $m$. For $i = 1, \ldots, n$,

$$\partial_i(f^s \otimes v) = f^{s-1} \partial_i f^s v = f^{s-1} \partial_i(f^1 s f^s)v$$

holds in $\mathcal{L} \otimes \mathcal{O}_C^n[s] \mathcal{M}[s]$ with $f_i = \partial f/\partial x_i$. This allows us to show that

$$P(s)(f^s \otimes v) = f^{s-m} \otimes (f^{m-s} P(s)f^s)v$$

holds in $\mathcal{L} \otimes \mathcal{O}_C^n[s] \mathcal{M}[s]$. (Note that $f^{m-s} P(s)f^s$ belongs to $\mathcal{D}_C^n[s]$.) Summing up, we have shown that $P(s)(f^s \otimes v)$ vanishes in $\mathcal{L} \otimes \mathcal{O}_C^n[s] \mathcal{M}[s]$ if and only if $(f^{l-s} P(s)f^s)v$ vanishes in $\mathcal{M}[s]$ for some $l \geq m$.

Lemma 2.3 with $P(s) = 1$ immediately implies

**Proposition 2.4.** Let $\mathcal{M}[f^{-1}] := \mathcal{O}_C^n[f^{-1}] \otimes \mathcal{O}_C^n \mathcal{M}$ be the localization of $\mathcal{M}$ with respect to $f$, which has a natural structure of left $\mathcal{D}_C^n$-module. Then the natural homomorphism $\mathcal{L} \otimes \mathcal{O}_C^n \mathcal{M} \to \mathcal{L} \otimes \mathcal{O}_C^n \mathcal{M}[f^{-1}]$ is an isomorphism.

**Proposition 2.5.** Let $P(s)$ be a section of $\mathcal{D}_C^n[s]$ on an open set $\Omega'$ of $\mathbb{C}^n$ and suppose $P(s)(f^s \otimes u) = 0$ in $\mathcal{L} \otimes \mathcal{O}_C^n \mathcal{M}$. Set $V = U \cap \Omega'$. Then $P(\lambda)(f_+^s \varphi) = 0$ holds in $\mathcal{O} Db(\mathbb{C}_+ \times V)$.

**Proof.** Let $\mathcal{O}_{+\infty} Db$ be the sheaf on $\mathbb{R}^n$ associated with the presheaf

$$W \mapsto \lim_{\longrightarrow} \mathcal{O} Db(\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda > a \} \times W)$$

for every open set $W$ of $\mathbb{R}^n$, where the inductive limit is taken as $a \to \infty$. The $\mathbb{C}$-bilinear sheaf homomorphism

$$\mathcal{L} \times \mathcal{M} \ni (a(s)f^{s-m}, Pu) \mapsto (a(\lambda)f_+^{\lambda-m})P\varphi \in \mathcal{O}_{+\infty} Db$$

with $a(s) \in \mathcal{O}_X[s], \ m \in \mathbb{N}, P \in \mathcal{D}_X$, which is well-defined and $\mathcal{O}_C^n$-balanced on $V_f$ since $f_+^{\lambda-m}$ is real analytic there, induces a $\mathcal{D}_C^n$-linear homomorphism

$$\Psi : \mathcal{L} \otimes \mathcal{O}_C^n \mathcal{M} \to \mathcal{O}_{+\infty} Db$$
on $V_f$ such that $\Psi(a(s)f^{s-m} \otimes Pu) = a(\lambda)f_+^{\lambda-m}P\varphi$. In particular, if $P(s) \in D_{\mathbb{C}^n}[s]$ satisfies $P(s)(f^s \otimes u) = 0$ in $L \otimes_{O_{\mathbb{C}^n}} M$, then $P(\lambda)(f_+^{\lambda} \varphi) = 0$ holds in $O_{+\infty}Db(V_f)$, hence also in $O_{+\infty}Db(V)$ by Lemma 2.1. Since $f_+^{\lambda} \varphi$ belongs to $O Db(\mathbb{C}_+ \times V)$, it follows that $P(f_+^{\lambda} \varphi) = 0$ holds in $ODb(\mathbb{C}_+ \times V)$. This completes the proof.

Kashiwara proved in [1] (Theorem 2.7) that on a neighborhood of each point $p$ of $\Omega$, there exist nonzero $b(s) \in \mathbb{C}[s]$ and $P(s) \in D_{\mathbb{C}^n}[s]$ such that

$$P(s)(f^{s+1} \otimes u) = b(s)f^s \otimes u \text{ in } L \otimes_{O_{\mathbb{C}^n}} M.$$ 

Such $b(s)$ of the smallest degree $b(s) = b_p(s)$ is called the (generalized) $b$-function for $f$ and $u$ at $p$.

Assume $p \in U$. Then by the proposition above,

$$P(\lambda)(f_+^{\lambda+1} \varphi) = b(\lambda)f_+^{\lambda} \varphi$$

holds in $ODb(\mathbb{C}_+ \times V)$ with an open neighborhood $V$ of $p$. It follows that $f_+^{\lambda} \varphi$ is a $Db(V)$-valued meromorphic function of $\lambda$ on $\mathbb{C}$. Let us assume that $U$ is relatively compact in $\Omega$. The poles of $f_+^{\lambda} \varphi$ are contained in

$$\{\lambda - k \mid b_p(\lambda) = 0 \ (\exists p \in U), \ k \in \mathbb{N}\}.$$

**Proposition 2.6** (Lemma 2.10 of [2]). There exists a positive real number $\varepsilon$ such that $f_+^{\lambda} \varphi$ belongs to $ODb(\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > -\varepsilon\} \times U)$.

**Proof.** Let $\lambda_0$ be an arbitrary pole of $f_+^{\lambda} \varphi$. There exists $\psi \in C^\infty_0(U)$ such that $\lambda_0$ is a pole of $Z(\lambda) := (f_+^{\lambda} \varphi, \psi)$. In particular, $|Z(\lambda_0 + t)|$ tends to infinity as $t \to +0$. On the other hand, $Z(\lambda)$ is continuous on $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0\}$. This implies $\text{Re } \lambda_0 < 0$. The conclusion follows since there are at most a finite number of poles of $f_+^{\lambda} \varphi$ in the set $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > -1\}$.

In conclusion, $f_+^{\lambda} \varphi$ is a $Db(U)$-valued meromorphic function on $\mathbb{C}$ whose poles are contained in $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}$.

**§ 2.3. Holonomicity of $f_+^{\lambda} \varphi$ and its applications.**

Let $f$, $\varphi$, $\mathcal{M} = D_{\mathbb{C}^n}/\mathcal{I}$ be as in the preceding subsection. Let $\mathcal{N} = D_{\mathbb{C}^n}[s](f^s \otimes u)$ be the left $D_{\mathbb{C}^n}[s]$-submodule of $L \otimes_{O_{\mathbb{C}^n}} \mathcal{M}$ generated by $f^s \otimes u$. Theorem 2.5 of Kashiwara [1] guarantees that $\mathcal{N}_{\lambda_0} := \mathcal{N}/(s - \lambda_0)\mathcal{N}$ is a holonomic $D_{\mathbb{C}^n}$-module on $\Omega$ for any $\lambda_0 \in \mathbb{C}$.

**Proposition 2.7.** Let $\lambda_0$ be an arbitrary complex number and $f^{\lambda_0} \otimes \varphi$ the residue class of $f^s \otimes u \in \mathcal{N}$ modulo $(s - \lambda_0)\mathcal{N}$.
1. \( N_0 \) is isomorphic to \( \mathcal{M} \) as \( \mathcal{D}_{\mathbb{C}^n} \)-module on \( \Omega_f \).

2. If \( \mathcal{M} \) is \( f \)-saturated, i.e., if \( fv = 0 \) with \( v \in \mathcal{M} \) implies \( v = 0 \), then there is a surjective \( \mathcal{D}_{\mathbb{C}^n} \)-homomorphism \( \Phi : N_0 \to \mathcal{M} \) on \( \Omega \) such that \( \Phi(f^0 \otimes u) = u \).
   Moreover, \( \Phi \) is an isomorphism on \( \Omega_f \).

   \textbf{Proof.} Since \( \mathcal{M}[f^{-1}] = \mathcal{M} \) on \( \Omega_f \), we may assume that \( \mathcal{M} \) is \( f \)-saturated. In view of Lemma 2.3 and the definition of \( N_0 \), \( P \in \mathcal{D}_{\mathbb{C}^n} \) annihilates \( f^0 \otimes u \) if and only if there exist \( Q(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) and an integer \( m \geq \text{ord} Q(s) \) such that \( (f^{m-s}Q(s)f^s)(1 \otimes u) = 0 \) in \( \mathcal{M}[s] \) and \( P = Q(0) \). If there exist such \( Q(s) \) and \( m \), set
   \[
   f^{m-s}Q(s)f^s = Q_0 + Q_1 s + \cdots + Q_l s^l \quad (Q_i \in \mathcal{D}_{\mathbb{C}^n}).
   \]
   Then \( Q_i u = 0 \) holds for any \( i \). In particular, \( Q_0 = f^m P \) annihilates \( u \). This implies \( Pu = 0 \) since \( \mathcal{M} \) is \( f \)-saturated. Hence the homomorphism \( \Phi \) is well-defined.

   Now assume \( p \in \Omega_f \) and \( Pu = 0 \) in the stalk \( \mathcal{M}_p \) of \( \mathcal{M} \) at \( p \). Then \( Q(s) := f^s P f^{-s} \) belongs to \( \mathcal{D}_{\mathbb{C}^n,p}[s] \) and annihilates \( f^s \otimes u \) by Lemma 2.3. Hence \( P = Q(0) \) annihilates \( f^0 \otimes u \). This implies that \( \Phi \) is an isomorphism on \( \Omega_f \). \( \Box \)

\textbf{Theorem 2.8.} If \( \lambda_0 \) is not a pole of \( f_+^\lambda \varphi \), then \( f_+^{\lambda_0} \varphi \) is a solution of \( N_{\lambda_0} \).

\textbf{Proof.} Assume that \( \lambda_0 \in \mathbb{C} \) is not a pole of \( f_+^\lambda \varphi \). Let \( P \) be a section of \( \mathcal{D}_{\mathbb{C}^n} \) which annihilates \( f^{\lambda_0} \otimes u \). Then there exist \( Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) such that
   \[
   P = Q(s) + (s - \lambda_0)R(s), 
   Q(s)(f^s \otimes u) = 0 \text{ in } \mathcal{N}.
   \]
   Proposition 2.5 implies that \( Q(\lambda)(f_+^\lambda \varphi) \) vanishes as section of the sheaf \( \mathcal{O}Db \). In particular, \( P(f_+^{\lambda_0} \varphi) = Q(\lambda_0)(f_+^{\lambda_0} \varphi) = 0 \) holds as distribution. Thus the homomorphism
   \[
   \mathcal{N}_{\lambda_0} = \mathcal{D}_{\mathbb{C}^n}(f^{\lambda_0} \otimes u) \ni P(f^{\lambda_0} \otimes u) \longrightarrow Q(f_+^{\lambda_0} \varphi) \in Db
   \]
   is well-defined and \( \mathcal{D}_{\mathbb{C}^n} \)-linear. Hence \( f_+^{\lambda_0} \varphi \) is a solution of \( \mathcal{N}_{\lambda_0} \). \( \Box \)

The following two theorems are essentially due to Kashiwara and Kawai [2] although they are stated with additional assumptions and stronger results.

\textbf{Theorem 2.9.} \( \varphi \) is a solution of the holonomic \( \mathcal{D}_{\mathbb{C}^n} \)-module \( N_0 \).

\textbf{Proof.} First note that \( \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s](-f)^s \) is isomorphic to \( \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s \) as left \( \mathcal{D}_{\mathbb{C}^n}[s] \)-module since \( \partial_i(-f)^s = sf_i f^{-1}(-f)^s \) holds in \( \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s](-f)^s \) with \( f_i = \partial f/\partial x_i \). Assume that \( P(f^0 \otimes u) = 0 \) holds in \( N_0 = \mathcal{N}/s\mathcal{N} \). Then there exist \( Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) such that
   \[
   P = Q(s) + sR(s), 
   Q(s)(f^s \otimes u) = 0 \text{ in } \mathcal{N}.
   \]
Let $\theta(t)$ be the Heaviside function; i.e., $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t \leq 0$. Then we have $\theta(f) = f^0_+$ and $\theta(-f) = (-f)^0_+$. Theorem 2.8 implies that $P = Q(0)$ annihilates both $\theta(f)\varphi$ and $\theta(-f)\varphi$, and hence also $\varphi = \theta(f)\varphi + \theta(-f)\varphi$. Thus $\varphi$ is a solution of $\mathcal{N}_0$. \hfill $\square$

**Theorem 2.10.** Let $\varphi_1$ and $\varphi_2$ be locally $L^p$ and $L^q$ functions respectively on an open set $U \subset \mathbb{R}^n$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Assume that $\varphi_1$ and $\varphi_2$ are solutions of holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively on $U$. Then for any point $x_0$ of $U$, there exists a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M}$ on a neighborhood of $x_0$ of which the product $\varphi_1\varphi_2$ is a solution.

**Proof.** There exist analytic functions $f_1$ and $f_2$ on a neighborhood $V$ of $x_0$ such that the singular support (the projection of the characteristic variety minus the zero section) of $\mathcal{M}_k$ is contained in $f_k = 0$ for $k = 1, 2$. Set $f(z) = f_1(z)f_2(\overline{z})f_2(z)f_2(\overline{z})$. Then $f(x)$ is a real-valued real analytic function and $\varphi_1$ and $\varphi_2$ are real analytic on $V_f$. Then it is easy to see, in the same way as in the proof of Theorem 2.8, that $\varphi_1\varphi_2$ is a solution of $\mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2$ on $V_f$. To complete the proof, we have only to apply Theorem 2.9 to $\mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2$ and $\varphi_1\varphi_2$ in place of $\mathcal{M}$ and $\varphi$ respectively. \hfill $\square$

### § 2.4. Laurent coefficients of $f_+^\lambda \varphi$

Let $f$, $\varphi$, $\mathcal{M}$ be as in preceding subsections.

**Theorem 2.11.** Let $p$ be a point of $U$. Then each coefficient of the Laurent expansion of $f_+^\lambda \varphi$ about an arbitrary $\lambda_0 \in \mathbb{C}$ is a solution of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module on a common neighborhood of $p$.

**Proof.** Fix $m \in \mathbb{N}$ such that $\text{Re} \lambda_0 + m \geq 0$. By using the functional equation involving the generalized $b$-function, we can find a nonzero $b(s) \in \mathbb{C}[s]$ and a germ $P(s)$ of $\mathcal{D}_{\mathbb{C}^n}[s]$ at $p$ such that

$$b(\lambda)f_+^\lambda \varphi = P(\lambda)(f_+^{\lambda+m} \varphi).$$

Factor $b(s)$ as $b(s) = (s - \lambda_0)^lc(s)$ with $c(s) \in \mathbb{C}[s]$ such that $c(\lambda_0) \neq 0$ and an integer $l \geq 0$. Then we have

$$(\lambda - \lambda_0)^lf_+^\lambda \varphi = \frac{1}{c(\lambda)}P(\lambda)(f_+^{\lambda+m} \varphi).$$

The right-hand side is holomorphic in $\lambda$ on an neighborhood of $\lambda = \lambda_0$. Let

$$f_+^\lambda \varphi = \sum_{k=-l}^{\infty} (\lambda - \lambda_0)^k \varphi_k$$
be the Laurent expansion with \( \varphi_k \in \mathcal{D}b(U) \), which is given by

\[
\varphi_k = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} (\lambda - \lambda_0)^l f_+^\lambda \varphi = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} \left( \frac{1}{c(\lambda)} P(\lambda)(f_+^{\lambda+m}) \varphi \right).
\]

Hence there exist \( Q_{kj} \in \mathcal{D}_{\mathbb{C}^n} \) such that

\[
(2.2) \quad \varphi_k = \sum_{j=0}^{l+k} Q_{kj}(f_+^{\lambda_0+m}(\log f_+)^j \varphi).
\]

First let us show that \( f_+^{\lambda_0+m}(\log f_+)^j \varphi \) with \( 0 \leq j \leq k \) satisfy a holonomic system. Consider the free \( \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]-\)module

\[
\tilde{\mathcal{L}} := \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]f^s \oplus \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]f^s \log f \oplus \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]f^s(\log f)^2 \oplus \cdots,
\]

which has a natural structure of left \( \mathcal{D}_{\mathbb{C}^n}[s]-\)module. Let

\[
\mathcal{N}[k] := \mathcal{D}_{\mathbb{C}^n}[s](f^s \otimes u) + \mathcal{D}_{\mathbb{C}^n}[s]((f^s \log f) \otimes u) + \cdots + \mathcal{D}_{\mathbb{C}^n}[s]((f^s(\log f)^k) \otimes u)
\]

be the left \( \mathcal{D}_{\mathbb{C}^n}[s]-\)submodule of \( \tilde{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{C}^n} \mathcal{M} \) generated by \( (f^s(\log f)^j) \otimes u \) with \( j = 0, 1, \ldots, k \). It is easy to see that \( \mathcal{N}[k]/\mathcal{N}[k-1] \) is isomorphic to \( \mathcal{N} = \mathcal{N}[0] \) as left \( \mathcal{D}_{\mathbb{C}^n}[s]-\)module since

\[
P(s)((f^s(\log f)^k) \otimes u) \equiv (f^{s-m}(\log f)^k) \otimes (f^{m-s}P(s)f^s)u \quad \text{mod } \mathcal{N}[k-1]
\]

holds for any \( P(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) with \( m = \text{ord } P(s) \). Moreover, \( \mathcal{N}_{\lambda_0}[k] := \mathcal{N}[k]/(s-\lambda_0)\mathcal{N}[k] \) is a holonomic \( \mathcal{D}_{\mathbb{C}^n}-\)module since \( \mathcal{N}_{\lambda_0}[k]/\mathcal{N}_{\lambda_0}[k-1] \) is isomorphic to \( \mathcal{N}_{\lambda_0} = \mathcal{N}_{\lambda_0}[0] \), and hence is holonomic as left \( \mathcal{D}_{\mathbb{C}^n}-\)module.

Let \( (f_+^{\lambda_0+m}(\log f)^j) \otimes u \in \mathcal{N}_{\lambda_0+m}[k] \) be the residue class of \( (f^s(\log f)^j) \otimes u \) modulo \( (s-\lambda_0-m)\mathcal{N}[k] \). Suppose \( \sum_{j=0}^{k} P_j((f_+^{\lambda_0+m}(\log f)^j) \otimes u) \) vanishes in \( \mathcal{N}_{\lambda_0+m}[k] \) with \( P_j \) being a section of \( \mathcal{D}_{\mathbb{C}^n} \) on an open neighborhood of a point \( p \) of \( U \). Then there exist \( Q_j(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) such that

\[
\sum_{j=0}^{k} P_j((f^s(\log f)^j) \otimes u) = (s-\lambda_0-m) \sum_{j=0}^{k} Q_j(s)((f^s(\log f)^j) \otimes u)
\]

holds in \( \mathcal{N}[k] \). Then it is easy to see that

\[
(2.3) \quad \sum_{j=0}^{k} P_j(\lambda)(f_+^\lambda(\log f_+)^j \varphi) = (\lambda - \lambda_0 - m) \sum_{j=0}^{k} Q_j(\lambda)(f_+^\lambda(\log f_+)^j \varphi)
\]

holds in \( \mathcal{O}Db(\mathbb{C}_+ \times W_f) \) with an open neighborhood \( W \) of \( p \). Lemma 2.2 and analytic continuation imply that (2.3) holds in \( \mathcal{O}Db(\mathbb{C}_+ \times W) \). By Proposition 2.6, we have in \( \mathcal{D}b(W) \)

\[
\sum_{j=0}^{k} P_j((f_+^{\lambda_0+m}(\log f_+)^j) \varphi) = 0.
\]
In conclusion, with $k$ replaced by $l + k$, there exists a $D_{\mathbb{C}^n}$-homomorphism $\Phi : \mathcal{N}_{\lambda_0 + m}[l + k] \to Db$ such that

$$\Phi((f^{\lambda_0 + m}(\log f)^j) \otimes u) = f^{\lambda_0 + m}_+(\log f_+)^j \varphi \quad (0 \leq j \leq l + k).$$

Set

$$w := \sum_{j=0}^{l+k} Q_{kj}((f^{\lambda_0 + m}(\log f)^j) \otimes u), \quad \mathcal{M}_k := D_{\mathbb{C}^n}w.$$

Then $\mathcal{M}_k$ is a $D_{\mathbb{C}^n}$-submodule of $\mathcal{N}_{\lambda_0 + m}[l + k]$ and hence holonomic. Since $\Phi(w) = \varphi_k$ in view of (2.2), $\varphi_k$ is a solution of $\mathcal{M}_k$. This completes the proof. \(\square\)

§ 3. Algorithms

We give algorithms for computing holonomic systems introduced in the previous section assuming that $f$ is a real polynomial and that $\mathcal{M}$ is algebraic, i.e., defined by differential operators with polynomial coefficients. Let $D_n := \mathbb{C}(x, \partial) = \mathbb{C}(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$ be the ring of differential operators with polynomial coefficients with $\partial_j = \partial/\partial x_j$. The ring $D_n$ is also called the $n$-th Weyl algebra over $\mathbb{C}$.

In the sequel, let $f$ be a non-constant real polynomial of $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function on an open connected set $U$ of $\mathbb{R}^n$. We assume that there exists a left ideal $I$ of $D_n$ which annihilates $\varphi$ on $U_f$, i.e., $P\varphi = 0$ holds on $U_f$ for any $P \in I$, such that $M := D_n/I$ is a holonomic $D_n$-module. We denote by $u$ the residue class of $1 \in D_n$ modulo $I$. Let $L = \mathbb{C}[x, f^{-1}, s]f^s$ be the free $\mathbb{C}[x, f^{-1}, s]$-module generated by $f^s$, which has a natural structure of left $D_n[s]$-module. Let $N := D_n[s](f^s \otimes u)$ be the left $D_n$-submodule of $L \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$.

As was established in the previous section, $f^{\lambda}_+ \varphi$ is a $Db(U)$-valued meromorphic function on $\mathbb{C}$ and is a solution of $N$.

§ 3.1. Mellin transform

Let us assume that $\varphi$ is real analytic on $U_f$ and set

$$\tilde{\varphi}(x, \lambda) := \int_{-\infty}^{\infty} t^{\lambda}_+ \delta(t - f(x)) \varphi(x) \, dt.$$

This is well-defined and coincides with $f^{\lambda}_+ \varphi$ as a distribution on $U_f \times \mathbb{C}_+$. Then we have

$$\int_{-\infty}^{\infty} t^{\lambda}_+ t \delta(t - f(x)) \varphi(x) \, dt = \tilde{\varphi}(x, \lambda + 1),$$

$$\int_{-\infty}^{\infty} t^{\lambda}_+ \partial_t \delta(t - f(x)) \varphi(x) \, dt = -\int_{-\infty}^{\infty} \partial_t(t^{\lambda}_+) \delta(t - f(x)) \varphi(x) \, dt = -\lambda \tilde{\varphi}(x, \lambda - 1).$$
Let $D_{n+1} = D_n(t, \partial_t)$ be the $(n + 1)$-th Weyl algebra with $\partial_t = \partial/\partial t$. Let us consider the ring $D_n(s, E_s, E_s^{-1})$ of difference-differential operators with the shift operator $E_s : s \mapsto s + 1$, where $s$ is an indeterminate corresponding to $\lambda$. In view of the identities above, let us define the ring homomorphism (Mellin transform of operators)

$$\mu : D_{n+1} \longrightarrow D_n(s, E_s, E_s^{-1})$$

by

$$\mu(t) = E_s, \quad \mu(\partial_t) = -sE_s^{-1}, \quad \mu(x_j) = x_j, \quad \mu(\partial_{x_j}) = \partial_{x_j}.$$  

It is easy to see that $\mu$ is well-defined and injective since $[\partial_t, t] = [\mu(\partial_t), \mu(t)] = 1$. Hence we may regard $D_{n+1}$ as a subring of $D_n(s, E_s, E_s^{-1})$. Since $\mu(\partial_t t) = -s$, we can also regard $D_n[s]$ as a subring of $D_{n+1}$. Thus we have inclusions

$$D_n[s] \subset D_{n+1} \subset D_n(s, E_s, E_s^{-1})$$

of rings and $L \otimes_{\mathbb{C}[x]} M$ has a structure of left $D_n(s, E_s, E_s^{-1})$-module compatible with that of left $D_n[s]$-module. Let $\mathcal{F}(U)$ be the $\mathbb{C}$-vector space of the $Db(U)$-valued meromorphic functions on $\mathbb{C}$. Then $\mathcal{F}(U)$ has a natural structure of left $D_n(s, E_s, E_s^{-1})$-module, which is compatible with that of $D_n[s]$-module. In particular, we can regard $\mathcal{F}(U)$ as a left $D_{n+1}$-module.

\section{Computation of $N = D_n[s](f^s \otimes u)$}

The inclusion $D_{n+1}f^s \subset L = \mathbb{C}[x, f^{-1}, s]f^s$ induces a natural $D_{n+1}$-homomorphism

$$D_{n+1}f^s \otimes_{\mathbb{C}[x]} M \xrightarrow{\iota} L \otimes_{\mathbb{C}[x]} M$$

where $N'$ is the left $D_n[s]$-submodule of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$ and $N$ is the left $D_n[s]$-submodule of $L \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$. The homomorphism $\iota$ induces a surjective $D_n[s]$-homomorphism $\iota' : N' \rightarrow N$.

**Proposition 3.1.** The homomorphism $\iota$ is injective if and only if $M$ is $f$-saturated; i.e., the homomorphism $f : M \rightarrow M$ is injective.

**Proof.** First note that $D_{n+1}f^s$ is isomorphic to the first local cohomology group $\mathbb{C}[x, t, (t-f)^{-1}]/\mathbb{C}[x, t]$ of $\mathbb{C}[x, t]$ supported in the non-singular hypersurface $t - f(x) = 0$ since

$$(t - f)f^s = 0, \quad (\partial_{x_i} + f_i \partial_t)f^s = 0 \quad (i = 1, \ldots, n).$$

In particular, $D_{n+1}f^s$ is a free $\mathbb{C}[x]$-module generated by $\partial^j_t f^s$ with $j \geq 0$. Hence an arbitrary element $w$ of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$ is uniquely written in the form

$$w = \sum_{j=0}^{k} (\partial^j_t f^s) \otimes u_j$$

with $u_j \in M$ and $k \in \mathbb{N}$. Then

$$\iota(w) = \sum_{j=0}^{k} (-1)^j s(s - 1) \cdots (s - j + 1)f^{s-j} \otimes u_j$$

vanishes if and only if $f^{s-j} \otimes u_j = 0$, which is equivalent to $f^{m_j}u_j = 0$ with some $m_j \in \mathbb{N}$ by Lemma 2.3, for all $j = 0, 1, \ldots, k$. This completes the proof.

Let $\tilde{M}$ be the left $D_n$-submodule of the localization $M[f^{-1}] := \mathbb{C}[x, f^{-1}] \otimes_{\mathbb{C}[x]} M$ which is generated by $1 \otimes u$. Then $\tilde{M}$ is $f$-saturated and the natural homomorphism

$$L \otimes_{\mathbb{C}[x]} M \rightarrow L \otimes_{\mathbb{C}[x]} \tilde{M}$$

is an isomorphism by Lemma 2.3.

An algorithm to compute $M[f^{-1}]$ was presented in [7] under the assumption that $M$ is holonomic on $\mathbb{C}^n \setminus \{f = 0\}$. It provides us with an algorithm to compute $\tilde{M}$, i.e., the annihilator of $1 \otimes u \in M[f^{-1}]$. Hence we may assume, from the beginning, that $M$ is holonomic and $f$-saturated. Then $\iota' : N' \rightarrow N$ is an isomorphism by Proposition 3.1. The $f$-saturatedness is equivalent to the vanishing of the zeroth local cohomology group of $M$ with support in $f = 0$, which can be computed by algorithms presented in [3],[8],[6].

Thus we have only to give an algorithm to compute the structure of $N'$ assuming $\tilde{M}$ to be $f$-saturated. We follow an argument introduced by Walther [8]. Note that we gave in [3] an algorithm based on tensor product computation which is less efficient.

**Definition 3.2.** For a differential operator $P = P(x, \partial) \in D_n$, set

$$\tau(P) := P(x, \partial x_1 + f_1 \partial t, \ldots, \partial x_n + f_n \partial t) \in D_{n+1}$$

with $f_j = \partial f / \partial x_j$. This substitution is well-defined since the operators $\partial x_j + f_j \partial t$ commute with one another and $[\partial x_j + f_j \partial t, x_i] = \delta_{ij}$ holds.

Moreover, for a left ideal $I$ of $D_{n+1}$, let $\tau(I)$ be the left ideal of $D_{n+1}$ which is generated by the set $\{\tau(P) \mid P \in I\}$.

**Lemma 3.3.** $\tau(P)(f^s \otimes v) = f^s \otimes (Pv)$ holds in $L \otimes_{\mathbb{C}[x]} M$ for any $P \in D_n$ and $v \in M$. 

Proof. By the definition of the action of $D_{n+1}$ on $L \otimes_{\mathbb{C}[x]} M$ via the Mellin transform, we have

$$(\partial x_j + f_j \partial t)(f^s \otimes v) = sf^{-1}f_j f^s \otimes v + f^s \otimes (\partial x_j, v) - sf_j f^{-1} f^s \otimes v = f^s \otimes (\partial x_j, v).$$

This implies the conclusion of the lemma.

Proposition 3.4. Let $I$ be a left ideal of $D_n$ and set $M = D_n = I$ with $u \in M$ being the residue class of $1$ modulo $I$. Let $J$ be the left ideal of $D_{n+1}$ which is generated by $(I)[f^s(t - f(x))]$. Then $J$ coincides with the annihilator $\text{Ann}_{D_{n+1}}(f^s \otimes u)$ of $f^s \otimes u \in D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$.

Proof. We have only to show that for $P \in D_{n+1}$ the equivalence

$$P \in J \iff P(f^s \otimes u) = 0 \quad \text{in} \quad D_{n+1}f^s \otimes_{\mathbb{C}[x]} M.$$ 

Suppose $Q$ belongs to $J$. Then $P$ annihilates $f^s \otimes u$ by Lemma 3.3.

Conversely, suppose $P(f^s \otimes u) = 0$ in $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$. We can rewrite $P$ in the form

$$P = \sum_{\alpha \in \mathbb{N}^n, \nu \in \mathbb{N}} p_{\alpha, \nu}(x) \partial^\nu t \left( \partial x_1 + \frac{\partial f}{\partial x_1} \partial t \right)^{\alpha_1} \cdots \left( \partial x_n + \frac{\partial f}{\partial x_n} \partial t \right)^{\alpha_n} + Q \cdot (t - f(x))$$

with $p_{\alpha, \nu}(x) \in \mathbb{C}[x]$ and $Q \in D_{n+1}$. Setting $P_{\nu} := \sum_{\alpha \in \mathbb{N}^n} p_{\alpha, \nu}(x) \partial^\nu t$, we get

$$0 = P(f^s \otimes u) = \sum_{\nu = 0}^{\infty} (\partial^\nu t f^s) \otimes P_{\nu} u \in D_{n+1}f^s \otimes_{\mathbb{C}[x]} M.$$ 

It follows that each $P_{\nu}$ belongs to $I$ since $\{\partial^\nu t f^s\}$ constitutes a free basis of $D_{n+1}f^s$ over $\mathbb{C}[x]$. Hence we have

$$P = \sum_{\nu = 1}^{\infty} \partial^\nu t P_{\nu} + Q \cdot (t - f(x)) \in J.$$ 

This completes the proof.

In order to compute the structure of the $D_n[s]$-submodule $N' = D_n[s](f^s \otimes u)$ of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$, we have only to compute the annihilator

$$\text{Ann}_{D_n[s]}(f^s \otimes u) = D_n[s] \cap J,$$

where we regard $D_n[s]$ as a subring of $D_{n+1}$. This can be done as follows:

Introducing new variables $\sigma$ and $\tau$, for $P \in D_{n+1}$, let $h(P) \in D_{n+1}[\tau]$ be the homogenization of $P$ with respect to the weights.
Let $J'$ be the left ideal of $D_{n+1}[\sigma, \tau]$ generated by
\[ \{h(P) \mid P \in \tilde{G}\} \cup \{1 - \sigma \tau\}, \]
where $\tilde{G}$ is a set of generators of $J$.

Set $J'' = J' \cap D_{n+1}$. Since each element $P$ of $J''$ is homogeneous with respect to the above weights, there exists $P'(s) \in D_n[s]$ such that $P = SP'(-\partial_t t)$ with $S = t^\nu$ or $S = \partial_t^\nu$ with some integer $\nu \geq 0$. We set $P'(s) = \psi(P)(s)$. Then $\{\psi(P) \mid P \in J''\}$ generates the left ideal $J \cap D_n[s]$ of $D_n[s]$. This procedure can be done by using a Gröbner basis in $D_{n+1}[s]$. In conclusion, we have a set of generators of $J \cap D_n[s]$.

Then $N'$, and hence $N$ also if $M$ is $f$-saturated, is isomorphic to $D_n[s] = (J \cap D_n[s])$ as left $D_n[s]$-module.

The generalized $b$-function for $f$ and $u$ can be computed as the generator of the ideal
\[ \mathbb{C}[s] \cap (\text{Ann}_{D_n[s]} f^s \otimes u + D_n[s]f) \]
of $\mathbb{C}[s]$ by elimination via Gröbner basis computation in $D_n[s]$.

### § 3.3. Holonomic systems for the Laurent coefficients of $f^\lambda \varphi$

Let $\lambda_0$ be an arbitrary complex number. Our purpose is to compute a holonomic system of which each coefficient of the Laurent expansion of $f^\lambda \varphi$ is a solution.

Let $b_0(s)$ be the (global) $b$-function of $f$ and $u$. We can find a $P_0(s) \in D_n[s]$ such that
\[ P_0(s)(f^{s+1} \otimes u) = b_0(s)f^s \otimes u \]
holds in $N$ by, e.g., syzygy computation. Take $m \in \mathbb{N}$ such that $\text{Re} \lambda_0 + m \geq 0$ or $b_0(\lambda_0 + m + k) \neq 0 \ (\forall k \in \mathbb{N})$. Then $\lambda_0 + m$ is not a pole of $f^\lambda \varphi$.

We can find a nonzero polynomial $b(s)$ and $P(s) \in D_n[s]$ such that
\[ b(\lambda)f^\lambda = P(\lambda)f^{\lambda+m}. \]

In fact, we have only to set
\[ P(s) := P_0(s)P_0(s+1) \cdots P_0(s+m-1), \quad b(s) := b_0(s)b_0(s+1) \cdots b_0(s+m-1). \]

Factorize $b(s)$ as $b(s) = c(s)(s-\lambda_0)^l$ with $c(\lambda_0) \neq 0$. Then $f^\lambda \varphi$ has a Laurent expansion of the form
\[ f^\lambda \varphi = \sum_{k=-l}^{\infty} (\lambda - \lambda_0)^k \varphi_k \]
around $\lambda_0$, where $\varphi_k \in Db(U)$ is given by

$$\varphi_k = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \left( \frac{\partial}{\partial \lambda} \right)^{l+k} (c(\lambda)^{-1} P(\lambda) f_{\lambda}^{\lambda+m}) = \sum_{j=0}^{l+k} Q_{kj}(f_{\lambda_0}^{\lambda+m}(\log f)^j)$$

with

$$Q_{kj} := \frac{1}{j!(l+k-j)!} \left[ \left( \frac{\partial}{\partial \lambda} \right)^{l+k-j} (c(\lambda)^{-1} P(\lambda)) \right]_{\lambda = \lambda_0}. $$

Let

$$\hat{L} = \mathbb{C}[x, f^{-1}, s] f^s \oplus \mathbb{C}[x, f^{-1}, s] f^s \log f \oplus \mathbb{C}[x, f^{-1}, s] f^s (\log f)^2 \oplus \cdots$$

be the free $\mathbb{C}[x, f^{-1}, s]$-module with a natural structure of left $D_n(s, \partial_s)$-module. Consider the left $D_n[s]$-submodule

$$N[k] = D_n[s](f^s \otimes u) + D_n[s]((f^s \log f) \otimes u) + \cdots + D_n[s]((f^s (\log f)^k) \otimes u)$$

of $\hat{L} \otimes_{\mathbb{C}[x]} M$. For a complex number $\lambda_0$, set

$$N_{\lambda_0}[k] = N[k]/(s - \lambda_0)N[k].$$

Let us first give an algorithm to compute the structure of $N[k]$.

**Proposition 3.5.** Let $G_0$ be a set of generators of the annihilator $\text{Ann}_{D_n[s]}(f^s \otimes u) = J \cap D_n[s]$. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_{k+1} = (0, \ldots, 0, 1)$ be the canonical basis of $\mathbb{Z}^{k+1}$. For each $Q(s) \in G_0$ and an integer $j$ with $0 \leq j \leq k$, set

$$Q^{(j)}(s) := \sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} e_{i+1} \in (D_n[s])^{k+1}.$$

Let $J_k$ be the left $D_n[s]$-submodule of $(D_n[s])^{k+1}$ generated by $G_1 := \{Q^{(j)}(s) \mid Q(s) \in G_0, 0 \leq j \leq k\}$. Then $(D_n[s])^{k+1}/J_k$ is isomorphic to $N[k]$.

**Proof.** Let $\varphi : (D_n[s])^{k+1} \to N[k]$ be the canonical surjection. Let $Q(s)$ belong to $G_0$. Differentiating the equation $Q(s)(f^s \otimes u) = 0$ in $N[k]$ with respect to $s$, one gets

$$\sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} ((f^s (\log f)^i) \otimes u) = 0.$$

Hence $J_k$ is contained in the kernel of $\varphi$. Conversely, assume that

$$\tilde{Q}(s) = (Q_0(s), Q_1(s), \ldots, Q_k(s))$$
belongs to the kernel of \( \varpi \). This implies \( Q_k(s)(f^s \otimes u) = 0 \) since \( N[k]/N[k - 1] \) is isomorphic to \( N = D_n[s](f^s \otimes u) \). Hence \( \bar{Q}(s) - Q_k^{(k)}(s) \) belongs to the kernel of \( \varpi \), the last component of which is zero. We conclude that \( \bar{Q}(s) \) belongs to \( J_k \) by induction.

Thus we have

\[
N_{\lambda_0}[k] = (D_n)^{k+1}/J_k|_{s=\lambda_0}, \quad J_k|_{s=\lambda_0} := \{Q(\lambda_0) \mid Q(s) \in J_k\}.
\]

Set

\[
w := \sum_{j=0}^{l+k} Q_{kJ}(f^{\lambda_0+m}(\log f)^j \otimes u), \quad M_k := D_n w.
\]

Then we have

\[
Pw = 0 \iff P(Q_{k_0}, Q_{k_1}, \ldots, Q_{l+k}) \in J_{l+k}|_{s=\lambda_0+m}.
\]

Thus we can find a set of generators of \( \text{Ann}_{D_n} w \) by computation of syzygy or intersection. As was shown in \( \S \)2.4, \( \varphi_k \) is a solution of the holonomic system \( M_k \).

\[\text{§ 3.4. Difference equations for the local zeta function}\]

In the sequel, we assume that \( \varphi \) is a locally integrable function on \( \mathbb{R}^n \). As we have seen so far, \( f^\lambda \varphi \in \mathcal{F}(\mathbb{R}^n) \) is a solution of the holonomic \( D_{n+1} \)-module \( D_{n+1}/J \). Hence if the local zeta function \( Z(\lambda) := \int_{\mathbb{R}^n} f^\lambda \varphi \, dx \) is well-defined, e.g., if \( \varphi \) has compact support, or else is smooth on \( \mathbb{R}^n \) with all its derivatives rapidly decreasing on the set \( \{x \in \mathbb{R}^n \mid f(x) \geq 0 \} \), then \( Z(\lambda) \) is a solution of the integral module

\[
D_{n+1}/(J + \partial x_1 D_{n+1} + \cdots + \partial x_n D_{n+1})
\]

of \( D_{n+1}/J \), which is a holonomic module over \( D_1 = \mathbb{C}\langle t, \partial t \rangle \). This \( D_1 \)-module can be computed by the integration algorithm which is the ‘Fourier transform’ of the restriction algorithm given in [6] (see [5] for the integration algorithm). Then by Mellin transform we obtain linear difference equations for \( Z(\lambda) \). Thus we get

**Theorem 3.6.** Under the above assumptions, \( Z(\lambda) \) satisfies a non-trivial linear difference equation with polynomial coefficients in \( \lambda \).

**Example 3.7.** \( \Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} \, dx = \int_{-\infty}^\infty x^\lambda e^{-x} \, dx \) satisfies the difference equation

\[
(E_\lambda - (\lambda + 1))\Gamma(\lambda + 1) = 0,
\]

where \( E_\lambda : \lambda \mapsto \lambda + 1 \) is the shift operator.
§ 3.5. Examples

Let us present some examples computed by using algorithms introduced so far and their implementation in the computer algebra system Risa/Asir.

Example 3.8. Set \( f = x^3 - y^2 \in \mathbb{R}[x, y] \) and \( \varphi = 1 \). Since the \( b \)-function of \( f \) is \( b_f(s) = (s + 1)(6s + 5)(6s + 7) \), possible poles of \( f_+^l \) are \( -1 - \nu, -5/6 - \nu, -6/7 - \nu \) with \( \nu \in \mathbb{N} \) and they are at most simple poles. The residue \( \text{Res}_{x=y} f_+^l \) is a solution of

\[
D_2/(D_2(2x\partial_x + 3y\partial_y + 6) + D_2(2y\partial_x + 3x^2\partial_y) + D_2(x^3 - y^2)).
\]

\( \text{Res}_{x=y} f_+^l \) is a solution of \( D_2/(D_2x + D_2y) \). Hence it is a constant multiple of the delta function \( \delta(x, y) = \delta(x)\delta(y) \). \( \text{Res}_{x=y} f_+^{\lambda} \) is a solution of \( D_2/(D_2x^2 + D_2(x\partial_x + 2) + D_2y) \). Hence it is a constant multiple of \( \delta'(x)\delta(y) \).

Example 3.9. Set \( f = x^3 - y^2 \) and \( \varphi(x, y) = \exp(-x^2 - y^2) \). Then \( \varphi \) is a solution of a holonomic system \( M := D_2/(D_2(\partial_x + 2x) + D_2(\partial_y + 2y)) \) on \( \mathbb{R}^2 \), which is \( f \)-saturated since it is a simple \( D_2 \)-module. The generalized \( b \)-function for \( f \) and \( u := [1] \in M \) is \( b_f(s) = (s + 1)(6s + 5)(6s + 7) \). The local zeta function \( Z(\lambda) := \int_{\mathbb{R}^2} f_+^{\lambda} \varphi \, dx \, dy \) is annihilated by the difference operator

\[
32E_s^4 + 16(4s + 13)E_s^3 - 4(s + 3)(27s^2 + 154s + 211)E_s^2
- 6(s + 2)(s + 3)(36s^2 + 162s + 173)E_s - 3(s + 1)(s + 2)(s + 3)(6s + 5)(6s + 13),
\]

where \( s \) is an indeterminate corresponding to \( \lambda \). From this we see that \( -7/6 \) is not a pole of \( Z(\lambda) \).

Example 3.10. Set \( \varphi(x) = \exp(-x - 1/x) \) for \( x > 0 \) and \( \varphi(x) = 0 \) for \( x \leq 0 \). Then \( \varphi(x) \) belongs to the space \( S(\mathbb{R}) \) of rapidly decreasing functions on \( \mathbb{R} \) and satisfies a holonomic system

\[
M := D_1/D_1(x^2\partial_x + x^2 - 1),
\]

which is \( x \)-saturated. The generalized \( b \)-function for \( f = x \) and \( u := [1] \in M \) is \( s + 1 \). The local zeta function \( Z(\lambda) := \int_{\mathbb{R}} x_+^\lambda \varphi(x) \, dx \) is entire (i.e., without poles) and satisfies a difference equation

\[
(E_s^2 - (\lambda + 2)E_s - 1)Z(\lambda) = 0.
\]

This can also be deduced by integration by parts.

Example 3.11. Set \( \varphi_1(x) = \exp(-x - 1/x) \) for \( x > 0 \) and \( \varphi_1(x) = 0 \) for \( x \leq 0 \). Set \( \varphi(x, y) = \varphi_1(x)e^{-y} \). Then \( \varphi \) satisfies a holonomic system

\[
M := D_2/(D_2(x^2\partial_x + x^2 - 1) + D_2(\partial_y + 1)).
\]
The generalized $b$-function for $f := y^3 - x^2$ and $u = [1] \in M$ is $s + 1$. Moreover, we can confirm that $M$ is $f$-saturated by using the localization algorithm in [7]. The local zeta function $Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi \, dx dy$ is well-defined since $f(x, y) < 0$ if $y < 0$. It is annihilated by a difference operator of the form

$$E_{s+11} + a_{10}(s)E_{s+10} + \cdots + a_1(s)E_s + a_0(s),$$

$$a_0(s) = c(s + 1)(s + 2)(s + 3)(s + 4)(s + 5)(s + 6)(s + 7)(s + 8)(s + 9),$$

where $c$ is a positive rational number and $a_1(s), \ldots, a_{10}(s)$ are polynomials of $s$ with rational coefficients. Possible poles of $f_+^\lambda \varphi$ are the negative integers. For example, $-1$ is at most a simple pole of $f_+^\lambda \varphi$ and $\text{Res}_{\lambda = -1} f_+^\lambda \varphi$ is a solution of a holonomic system

$$D_2/(D_2(3x^2 \partial_x + 2xy \partial_y + 3x^2 + (2y + 6)x - 3) + D_2(y^3 - x^2)).$$

**Example 3.12.** Set $f = x^3 - y^2z^2$. The $b$-function of $f$ is $(s + 1)(3s + 4)(3s + 5)(6s + 5)^2(6s + 7)^2$. For example, its maximum root $-5/6$ is at most a pole of order 2 of $f_+^\lambda$. Let

$$f_+^\lambda = \left(\lambda + \frac{5}{6}\right)^2 \varphi_{-2} + \left(\lambda + \frac{5}{6}\right)^{1} \varphi_{-1} + \varphi_0 + \cdots$$

be the Laurent expansion. Then $\varphi_{-2}$ satisfies

$$x\varphi_{-2} = y\varphi_{-2} = z\varphi_{-2} = 0.$$ 

Hence $\varphi_{-2}$ is a constant multiple of $\delta(x, y)$. On the other hand, $\varphi_{-1}$ satisfies a holonomic system

$$x\varphi_{-1} = (y\partial_y - z\partial_z)\varphi_{-1} = yz\varphi_{-1} = (z^2 \partial_z - z)\varphi_{-1} = 0.$$ 

**References**


