Localization, local cohomology, and the \( b \)-function of a \( D \)-module with respect to a polynomial

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Abstract.

Given a \( D \)-module \( M \) generated by a single element, and a polynomial \( f \), one can construct several \( D \)-modules attached to \( M \) and \( f \) and can define the notion of the (generalized) \( b \)-function following M. Kashiwara. These modules are closely related to the localization and the local cohomology of \( M \). We show that the \( b \)-function, if it exists, controls these modules and present general algorithms for computing these modules and the \( b \)-function if it exists without any further assumptions. We also give some examples of multiplicity computation of such \( D \)-modules including a possibly well-known explicit formula for the localization of the polynomial ring by a hyperplane arrangement.

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§1. Introduction

Let \( K \) be an algebraically closed field of characteristic zero and \( K[x] = K[x_1, \ldots, x_n] \) be the polynomial ring with \( x = (x_1, \ldots, x_n) \). Let \( D_n = K[x] \langle \partial \rangle = K[x] \langle \partial_1, \ldots, \partial_n \rangle \) be the \( n \)-th Weyl algebra, i.e., the ring

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of differential operators with polynomial coefficients with respect to the variables \( x \), where we denote \( \partial = (\partial_1, \ldots, \partial_x) \) with \( \partial_i = \partial_x / \partial x_i \) being the derivation with respect to \( x_i \). An arbitrary element \( P \) of \( D_n \) is written in a finite sum

\[
P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha \quad \text{with} \quad a_\alpha(x) \in K[x],
\]

where we denote \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_x^{\alpha_x} \) for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) with \( \mathbb{N} \) being the set of non-negative integers. One can define the dimension of a finitely generated left \( D_n \)-module \( M \); J. Bernstein [3], [4] proved that the dimension of \( M \) is not less than \( n \) unless \( M \) is the zero module. A finitely generated left \( D_n \)-module is called holonomic if its dimension is \( n \) or else it is the zero module.

Let \( M \) be a finitely generated left \( D_n \)-module and \( f \in K[x] \) be a non-constant polynomial. Then the localization \( M[f^{-1}] \) and the local cohomology groups \( H^j_{(f)}(M) \) have natural structures of left \( D_n \)-module and are holonomic if so is \( M \), as was shown by Kashiwara [13]. More generally, one can construct a left \( D_n[s] \)-module

\[
M(u, f, s) = D_n[s](u \otimes f^s)
\]

with an indeterminate \( s \). Suppose that \( M \) is generated by \( u \) over \( D_n \). Then the (generalized) \( b \)-function for \( u \) and \( f \) is defined to be the univariate (and monic) polynomial \( b_{u,f}(s) \) of the least degree such that

\[
b_{u,f}(s)(u \otimes f^s) \in D_n[s](u \otimes f^{s+1})
\]

holds. The existence of \( b_{u,f}(s) \) was proved by Kashiwara [13] under the assumption that \( M \) is holonomic outside of the hypersurface \( f = 0 \). If \( M \) is the polynomial ring \( K[x] \) with \( u = 1 \), then \( b_{u,f}(s) \) is nothing but the classical Bernstein-Sato polynomial, or simply the \( b \)-function, of \( f \). In the same way as the Bernstein-Sato polynomial controls the localization of the polynomial ring as a \( D_n \)-module, the \( b \)-function controls the localization \( M[f^{-1}] \) or its generalization \( D_n(u \otimes f^\lambda) \).

On the other hand, algorithms to compute \( M(u, f, s) \) and the \( b \)-function if it exists were introduced in [17] under the assumption that \( M \) is \( f \)-torsion free. These algorithms are based on various Gröbner bases over the ring of differential operators as is presented, e.g., in [23] and [18]. Torrelli [24] studied the \( b \)-function \( b_{u,f}(s) \) systematically when \( M \) is the local cohomology group \( H^k_{(f_1, \ldots, f_k)}(K[x]) \) under the assumption that \( f_1, \ldots, f_k, f \) define a quasi-homogeneous non-isolated singularity, together with the general property of \( M(u, f, s) \) under the assumption that \( M \) is holonomic without \( f \)-torsion.
The purpose of our study on the $b$-function and $M(u, f, s)$ is twofold: first, we want to clarify how the $b$-function controls the module $M(u, f, s)$ and the localization $M[f^{-1}]$ as well as the local cohomology $H^1_{(f)}(M)$. This will be performed in Sections 2 and 5. These results should be more or less well-known under some stronger conditions. See, e.g., [24] and Chapter VI of [5], where $M$ is assumed to be $f$-torsion free, or regular holonomic. The second purpose is to remove the assumption of $f$-saturatedness from our former algorithms in [17]. For this purpose, we reinterpret the algorithm introduced in [21] for the localization $M[f^{-1}]$ in Section 3. Our algorithms work at least if $M$ is holonomic outside of $f = 0$ without any further assumptions.

In the latter half of this article (Sections 5 and 6), we study the multiplicity (in the sense of Bernstein [3]) and the length of a holonomic $D$-module, as the most fundamental numerical invariants. This can be also used to prove a relation between $b_{u,f}(s)$ and $M(u, f, \lambda)$. We also give some examples of the multiplicity computation of the localization or the local cohomology. In the last section, we present, together with an elementary proof, a possibly well-known formula on the length and the multiplicity of the localization of the polynomial ring by a polynomial $f$ which defines a hyperplane arrangement. The result is that the length and the multiplicity of $K[x, f^{-1}]$ both coincide with $\pi(1)$, where $\pi(t)$ is what is called the Poincaré polynomial of the hyperplane arrangement.

We use computer algebra system Risa/Asir [16] for computation of Gröbner bases over the ring of differential operators, and in particular, for computation of $D$-module theoretic integration, which is needed in the localization algorithm.

We would like to thank the organizers of MSJ SI 2015 for the invitation both to the conference and to the proceedings. We would be pleased if we could convince the reader who is interested in $D$-module theory of the usefulness of Gröbner bases, which are the main theme of MSJ SI 2015, over the ring of differential operators in our case. This work was supported by JSPS Grant-in-Aid for Scientific Research (C) 26400123.

§2. The $b$-function for a $D$-module and a polynomial

Let $K$ be an arbitrary field of characteristic zero and $X = K^n$ be the $n$-dimensional affine space over $K$. We denote by $D_X$ the $n$-th Weyl algebra $D_n$ over $K$. Let $M$ be a left $D_X$-module and $f \in K[x]$ a non-constant polynomial. We can associate several $D_X$-modules with $M$ and $f$ by translating the definitions by Kashiwara [13] for analytic $D$-modules to algebraic setting. First, the localization $M[f^{-1}] := M \otimes_{K[x]} K[x, f^{-1}]$
and the local cohomology groups $H^j_{(R)}(M) \ (j = 0, 1)$ are defined with $M$ being regarded as a $K[x]$-module; they become again left $D_X$-modules.

Introducing an indeterminate $s$, let

$$\mathcal{L}_f := K[x, f^{-1}, s]^f$$

be the free $K[x, f^{-1}, s]$-module with a free generator $f^s$. Then $\mathcal{L}_f$ has a natural structure of left $D_X[s]$-module through the action of $\partial_{x_i}$ on $\mathcal{L}_f$ defined by

$$\partial_{x_i}(a(x, s)f^{-k}f^s) = \left( \frac{\partial a(x, s)}{\partial x_i} f^{-k} + (s - k)f a(x, s)f^{-k-1} \right) f^s$$

for $j = 1, \ldots, n$ with $f_i := \partial f/\partial x_i$. Sometimes $f^{-k}f^s$ is abbreviated to $f^{s-k}$.

The tensor product $M \otimes_{K[x]} \mathcal{L}_f$ has a natural structure of left $D_X[s]$-module induced by

$$\partial_{x_i}(u \otimes a(x, s)f^s) = (\partial_{x_i}u) \otimes a(x, s)f^s + u \otimes \partial_{x_i}(a(x, s)f^s) \quad (1 \leq i \leq n)$$

for $u \in M$ and $a(x, s) \in K[x, s]$. In what follows, we fix an arbitrary nonzero element $u$ of $M$. Let

$$M(u, f, s) := D_X[s](u \otimes f^s)$$

be the left $D_X[s]$-submodule of $M \otimes_{K[x]} \mathcal{L}_f$ generated by $u \otimes f^s$. In a special case where $M = K[x]$ and $u = 1$, let us denote by

$$\mathcal{N}_f = K[x](1, f, s) = D_X[s]^f$$

the left $D_X[s]$-submodule of $\mathcal{L}_f$ generated by $f^s$. Set

$$I(u, f) := \{ b(s) \in K[x] \mid b(s)(u \otimes f^s) \in D_u[s](fu \otimes f^s) \}.$$ 

If $I(u, f) \neq \{0\}$, then the (monic) generator $b_{u,f}(s)$ of $I(u, f)$ is called the (generalized) $b$-function for $u$ and $f$. It was defined by Kashiwara [13] with the following existence theorem.

**Theorem 2.1** (Kashiwara [13]). Let $D_X$ be defined over an algebraically closed field $K$ of characteristic zero. If a left $D_X$-module $M$ is holonomic on $X_f = \{ x \in X \mid f(x) \neq 0 \}$, then one has $I(u, f) \neq \{0\}$ for any $u \in M$.

When $M = K[x]$ and $u = 1$, the $b$-function $b_{1,f}(s)$ is nothing but what is called the Bernstein-Sato polynomial, or the $b$-function, associated with $f$. In fact, Kashiwara proved this theorem for a module
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$M$ over the ring of differential operators with analytic coefficients and a complex analytic function $f$. This corresponds to what is called the local $b$-function. The coincidence of the local $b$-functions in the algebraic setting and in the analytic setting is noticed, e.g., as Corollary 8.6 of [17]. It will turn out in what follows that the $b$-function ‘controls’ the $D$-modules associated with $M$ and $f$.

The $b$-function can exist even if $M$ is not holonomic on $X_f$.

**Example 2.2.** Set $n = 2$, $x_1 = x$, $x_2 = y$, and $P = x\partial_x^2 + \partial_y$. Then $M := D_X/D_X P = D_X u$ with $u$ being the residue class of $1$ is not holonomic even outside of $x = 0$ (the dimension of $M$ is three), but has the $b$-functions $b_{u,x}(s) = (s + 1)(s + 2)$ and $b_{u,y}(s) = s + 1$. In fact, one has
\[
-x\partial_x^2 + 2(s + 1)\partial_x - \partial_y)(u \otimes x^{s+1}) = (s + 1)(s + 2)u \otimes x^s,
\]
\[
P(u \otimes y^{s+1}) = (s + 1)u \otimes y^s
\]
in $M \otimes_{K[x,y]} K[x, y, x^{-1}]x^s$ and in $M \otimes_{K[x,y]} K[x, y, y^{-1}]y^s$ respectively.

**Definition 2.3.** A left $D_X$-module $M$ is said to be $f$-saturated or $f$-torsion free if the homomorphism $f : M \to M$ is injective. This is equivalent to $R^0_{(f)}(M) = 0$.

An algorithm to determine if there exists the $b$-function and to compute it if it exists was given in [17] under the assumption that $M = D_X u$ is $f$-torsion free.

Let us define a $D_X$-automorphism $t : \mathcal{L}_f \to \mathcal{L}_f$ by
\[
t((a(x, s)f^{-k}f^s)) = a(x, s + 1)f^{-k+1}f^s
\]
for $a(x, s) \in K[x, s]$ and $k \in \mathbb{N}$. The inverse $t^{-1}$ is defined by
\[
t^{-1}(a(x, s)f^{-k}f^s)) = a(x, s - 1)f^{-k-1}f^s.
\]
It induces a $D_X$-automorphism
\[
t : M \otimes_{K[x]} \mathcal{L}_f \longrightarrow M \otimes_{K[x]} \mathcal{L}_f
\]
which also induces a $D_X$-endomorphism of $M(u, f, s)$. Note that the actions of $t$ and $s$ on $M(u, f, s)$ satisfies the commutation relation $st = t(s - 1)$. It follows that $tM(u, f, s)$ is a left $D_X[s]$-module. It also follows from the definition that $b_{u,f}(s)$ is the minimal polynomial of $s$ acting on the left $D_X$-module $M(u, f, s)/tM(u, f, s)$ since $P(s)(fu \otimes f^s) = t(P(s - 1)(u \otimes f^s))$. 

Let $\lambda \in K$ be a constant. Then specializing the parameter $s$ to $\lambda$, we obtain left $D_X$-modules

$$\mathcal{L}_f(\lambda) := \mathcal{L}_f/(s-\lambda)\mathcal{L}_f, \quad \mathcal{N}_f(\lambda) := \mathcal{N}_f/(s-\lambda)\mathcal{N}_f.$$ 

Let us denote by $f^\lambda$ and $f^s|_{s=\lambda}$ the residue class of $f^s$ in $\mathcal{L}_f(\lambda)$, and that of $f^s$ in $\mathcal{N}_f(\lambda)$ respectively. In particular, $\mathcal{L}_f(\lambda) = K[x, f^{-1}]f^\lambda$ is a free $K[x, f^{-1}]$-module generated by $f^\lambda$. In the same way, we define a left $D_X$-module

$$M(u, f, \lambda) = M(u, f, s)/(s-\lambda)M(u, f, s)$$

and denote the residue class of $u \otimes f^s$ in $M(u, f, \lambda)$ by $(u \otimes f^s)|_{s=\lambda}$.

Kashiwara also proved the following fundamental fact, to which we shall give an elementary proof in Section 5.

**Theorem 2.4 (Kashiwara [13]).** Let $K$ be algebraically closed. If $M$ is holonomic on $X_f$, then $M(u, f, \lambda)$ is a holonomic $D_X$-module for any $u \in M$ and $\lambda \in K$.

Let us define the specialization homomorphism

$$\rho_\lambda : M \otimes_{K[x]} \mathcal{L}_f \longrightarrow M \otimes_{K[x]} \mathcal{L}_f(\lambda)$$

by

$$\rho_\lambda(v \otimes a(x, s)f^{-k} f^s) = v \otimes a(x, \lambda)f^{-k}f^\lambda$$

for $v \in M$, $a(x, s) \in K[x, s]$, and $k \in \mathbb{N}$. Then $\rho_\lambda(P(s)w) = P(\lambda)\rho_\lambda(w)$ holds for any $w \in M \otimes_{K[x]} \mathcal{L}_f$ and $P(s) \in D_X[s]$. Since any element of $(s-\lambda)M(u, f, s)$ is sent by $\rho_\lambda$ to zero, $\rho_\lambda$ induces a surjective $D_X$-homomorphism

$$\bar{\rho}_\lambda : M(u, f, \lambda) \longrightarrow D_X(u \otimes f^\lambda) \subset M \otimes_{K[x]} \mathcal{L}_f(\lambda),$$

which sends $(u \otimes f^s)|_{s=\lambda}$ to $u \otimes f^\lambda$. It is not injective in general even if $M = K[x]$ and $u = 1$. For example, $\partial_x x^0 = 0$ holds in $\mathcal{L}_x(0)$ but $\partial_x(x^s|_{s=0})$ does not vanish in $\mathcal{N}_x(0)$.

**Lemma 2.5.** Let $M = D_X u$ be a left $D_X$-module generated by $u$.

1. Every element of $M \otimes_{K[x]} \mathcal{L}_f$ can be expressed as $Q(s)(u \otimes f^{s-k})$ with some $Q(s) \in D_X[s]$ and $k \in \mathbb{N}$.
2. Let $\lambda$ be an arbitrary element of $K$. Then every element of $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ can be expressed as $Q(u \otimes f^{\lambda-k})$ with some $Q \in D_X$ and $k \in \mathbb{N}$. 
Proof. From the identity
\[ \partial_x(v \otimes f^{s-k}) = (\partial_x v) \otimes f^{s-k} + v \otimes (s - k)f, f^{s-k-1} \]
for any \( v \in M \) and \( k \in \mathbb{Z} \), we get
\[ (\partial_x v) \otimes f^{s-k} = (\partial_x f - (s - k)f)(v \otimes f^{s-k-1}). \]
By induction, we can show that for any multi-index \( \alpha \in \mathbb{N}^n \) and \( k \in \mathbb{Z} \), there exists \( Q_\alpha(s) \in D_X[s] \) such that
\[ (\partial_x^\alpha v) \otimes f^{s-k} = Q_\alpha(s)(v \otimes f^{s-k-|\alpha|}). \]
This proves the statement (1). The statement (2) can be proved similarly.
Q.E.D.

The following proposition should be well-known; see, e.g., Propositions 7.1 and 7.4 of [17]. The case \( M = K[x] \) and \( f = 1 \) was proved by Kashiwara [12].

**Proposition 2.6.** Let \( M \) be a left \( D_X \)-module generated by \( u \in M \) and assume that there exists the b-function \( b_{u,f}(s) \). Let \( \lambda \) be an element of \( K \) and suppose that \( b_{u,f}(\lambda - k) \neq 0 \) for any positive integer \( k \). Then

1. The image \( \rho_\lambda(M(u,f,s)) = D_X(u \otimes f^\lambda) \) coincides with \( M \otimes_{K[x]} L_f(\lambda) \). In other words, \( M \otimes_{K[x]} L_f(\lambda) \) is generated by \( u \otimes f^\lambda \) over \( D_X \).
2. \( \ker \rho_\lambda \cap M(u,f,s) \) coincides with \( (s - \lambda)M(u,f,s) \). Hence \( \rho_\lambda : M(u,f,\lambda) \to D_X(u \otimes f^\lambda) \) is an isomorphism of left \( D_X \)-modules.

Proof. (1) In view of Lemma 2.5, we have only to show that \( u \otimes f^{s-k} \) belongs to \( \rho_\lambda(M(u,f,s)) \) for any \( k \in \mathbb{N} \). This is obvious for \( k = 0 \) since \( \rho_\lambda(u \otimes f^s) = u \otimes f^\lambda \).

Let us show that \( u \otimes f^{s-k} \) belongs to \( \rho_\lambda(M(u,f,s)) \). Suppose \( k \geq 1 \). There exists \( P(s) \in D_X[s] \) such that \( P(s)(u \otimes f^{s+1}) = b_{u,f}(s)(u \otimes f^s) \). Applying \( t^{-k} \), we get
\[ P(s - k)(u \otimes f^{s+1-k}) = b_{u,f}(s - k)(u \otimes f^{s-k}) \]
in \( M \otimes_{K[x]} L_f \). Proceeding inductively, we see that there exists \( \tilde{P}(s) \in D_X[s] \) such that
\[ \tilde{P}(s)(u \otimes f^s) = b_{u,f}(s - 1) \cdots b_{u,f}(s - k)u \otimes f^{s-k} \]
holds in \( M \otimes_{K[x]} L_f \). The homomorphism \( \rho_\lambda \) gives an identity
\[ \tilde{P}(\lambda)(u \otimes f^\lambda) = b_{u,f}(\lambda - 1) \cdots b_{u,f}(\lambda - k)u \otimes f^{\lambda-k} \]
in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$. Since $b_{u,f}(\lambda - j) \neq 0$ for $j = 1, \ldots, k$ by the assumption, it follows that

$$u \otimes f^{\lambda-k} = \frac{1}{b_{u,f}(\lambda - 1) \cdots b_{u,f}(\lambda - k)} \tilde{P}(\lambda)(u \otimes f^\lambda).$$

The right-hand side belongs to $\rho_\lambda(M(u,f,s))$. This completes the proof of (1).

(2) Assume $\rho_\lambda(Q(s)(u \otimes f^s)) = 0$ with $Q(s) \in D_X[s]$. There exist $l \in \mathbb{N}$ and $Q_j \in D_X$ which are zero except finitely many indices $j$ such that

$$Q(s)(u \otimes f^s) = \sum_{j \geq 0} (Q_j u) \otimes (s - \lambda)^j f^{s-l}.$$

By the assumption, $\rho_\lambda(Q(s)(u \otimes f^s)) = (Q_0 u) \otimes f^{\lambda-l}$ vanishes in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$, which means that $(Q_0 u) \otimes f^{-l}$ vanishes in $M \otimes_{K[x]} K[x,f^{-1}]$. It follows that $(Q_0 u) \otimes 1 = f^l(Q_0 u) \otimes f^{-l} = 0$ in $M \otimes_{K[x]} K[x,f^{-1}]$. Consequently, $(Q_0 u) \otimes f^s$ vanishes in $M \otimes_{K[x]} \mathcal{L}_f$. Thus we have

$$Q(s)(u \otimes f^s) = (s - \lambda) \sum_{j \geq 1} (Q_j u) \otimes (s - \lambda)^j f^{s-l} = (s - \lambda)Q'(s)(u \otimes f^{s-k})$$

with some $k \in \mathbb{N}$ and $Q'(s) \in D_X[s]$ in view of the proof of Lemma 2.5. By using (1) we obtain

$$b_{u,f}(s - 1) \cdots b_{u,f}(s - k)Q(s)(u \otimes f^s) = (s - \lambda)Q'(s)\tilde{P}(s)(u \otimes f^s).$$

Hence $b_{u,f}(\lambda - 1) \cdots b_{u,f}(\lambda - k)Q(s)(u \otimes f^s)$ belongs to $(s - \lambda)M(u,f,s)$. This completes the proof of (2).

Q.E.D.

The following proposition extends Lemma 1.3 of Walther [27] for the case $M = K[x]$ and $u = 1$ almost verbatim.

**Lemma 2.7.** Under the same assumption as in the preceding proposition, assume moreover that $b_{u,f}(\lambda) = 0$. Then one has

$$D_X(fu \otimes f^k) \subseteq D_X(u \otimes f^\lambda)$$

in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$. In particular, $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ is generated by $u \otimes f^\lambda$, but not by $u \otimes f^{\lambda+1} = fu \otimes f^\lambda$, over $D_X$.

**Proof.** There exists $P(s) \in D_X[s]$ such that $P(s)(fu \otimes f^s) = b_{u,f}(s)(u \otimes f^s)$. Assume $D_X(fu \otimes f^\lambda) = D_X(u \otimes f^\lambda)$. Then there exists $A \in D_X$ such that $(1 - Af)(u \otimes f^\lambda) = 0$. By virtue of (2) of the preceding proposition, there exist $Q(s), R(s) \in D_X[s]$ such that

$$1 - Af = Q(s) + (s - \lambda)R(s), \quad Q(s)(u \otimes f^s) = 0.$$
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It follows that
\[
\frac{b_{u,f}(s)}{s - \lambda} (u \otimes f^s) = \frac{b_{u,f}(s)}{s - \lambda} A f(u \otimes f^s) + b_{u,f}(s) R(s)(u \otimes f^s)
\]
\[
= \left( \frac{b_{u,f}(s)}{s - \lambda} A + R(s) P(s) \right) (f u \otimes f^s).
\]

This means that \( b_{u,f}(s)/(s - \lambda) \) belongs to the ideal \( I(u, f) \), which contradicts the definition of \( b_{u,f}(s) \). This completes the proof. Q.E.D.

Summing up we obtain

**Theorem 2.8.** Let \( M = D_X u \) be a left \( D_X \)-module generated by \( u \in M \) and \( f \in K[x] \) be a non-constant polynomial. Assume that there exists the \( b \)-function \( b_{u,f}(s) \) for \( u \) and \( f \). Then the following conditions on \( \lambda \in K \) are equivalent:

1. \( b_{u,f}(\lambda - k) \neq 0 \) for any positive integer \( k \).
2. \( M \otimes_{K[x]} \mathcal{L}_f(\lambda) \) is generated by \( u \otimes f^\lambda \) over \( D_X \).

**Proof.** Assume \( b_{u,f}(\lambda - k) = 0 \) for some positive integer \( k \) and let \( k_0 \) be the maximum among such \( k \). Then by (1) of Proposition 2.6 and Lemma 2.7, we have

\[
\mathcal{L}_f(\lambda) = \mathcal{L}_f(\lambda - k_0) = D_X(u \otimes f^{\lambda - k_0})
\]
\[
\supseteq D_X(u \otimes f^{\lambda - k_0 + 1}) \supset D_X(u \otimes f^\lambda).
\]

Hence \( \mathcal{L}_f(\lambda) \) is not generated by \( u \otimes f^\lambda \). Q.E.D.

The converse of the statement (2) of Proposition 2.6 will be given in Theorem 5.9 of Section 5 under the additional assumption that \( M \) is holonomic on \( X_f \).

Let us recall local cohomology of \( D \)-modules. Let \( M \) be a finitely generated left \( D_X \)-module, and \( I \) be an ideal of \( K[x] \). Then the \( k \)-th local cohomology group \( H^k_I(M) \) supported by \( I \) is defined to be the \( k \)-th derived functor of the functor

\[
M \mapsto H^0_I(M) = \{ u \in M \mid I^k u = 0 \text{ for some } k \in \mathbb{N} \}.
\]

They have natural structure of left \( D_X \)-module, and they are holonomic if so is \( M \) as was proved by Kashiwara [13] in the analytic category.

If \( I \) is the principal ideal \((f)\) generated by \( f \in K[x] \), then there exists an exact sequence

\[
0 \rightarrow H^0_{(f)}(M) \rightarrow M \overset{i}{ightarrow} M[f^{-1}] \rightarrow H^1_{(f)}(M) \rightarrow 0
\]
of left $D_X$-modules, where $\iota$ stands for the natural homomorphism such that $\iota(v) = v \otimes 1$ in $M[f^{-1}] = M \otimes_{K[x]} K[x, f^{-1}]$ for $v \in M$. Hence there is an isomorphism $H^1_{(f)}(M) \cong M[f^{-1}]/\iota(M)$ as left $D_X$-module.

In general, algorithms to compute $H^1_{(f)}(M)$ as left $D_X$-module were given in [17] for the case $I$ is principal, and in [26] and [20] for general $I$, under the condition that $M$ is holonomic. See also [25], where related topics such as associated primes and the Weyl closure are also discussed.

**Corollary 2.9.** Let $M = D_Xu$ be a left $D_X$-module generated by $u \in M$ and $f \in K[x]$ be a non-constant polynomial. Assume that there exists the $b$-function $b_{u,f}(s)$ for $u$ and $f$. Then the following conditions are equivalent:

1. $b_{u,f}(j) \neq 0$ for any integer $j < k$.
2. The localization $M[f^{-1}]$ is generated by $u \otimes f^{-k}$ over $D_X$.
3. The local cohomology group $H^1_{(f)}(M)$ is generated by the cohomology class $[u \otimes f^{-k}]$ over $D_X$.

**Proof.** The equivalence of (1) and (2) is a special case of Theorem 2.8. In general, if $M[f^{-1}]$ is generated by $u \otimes f^{-k}$, then $H^1_{(f)}(M) = M[f^{-1}]/\iota(M)$ is generated by its residue class. Conversely, assume that $M[f^{-1}]/\iota(M)$ is generated by $[u \otimes f^{-k}]$. Then for any $w \in M[f^{-1}]$, there exist $P, Q \in D_X$ such that

$$w = P(u \otimes f^{-k}) + (Qu) \otimes 1 = (P + Qf^k)(u \otimes f^{-k}).$$

Hence $M[f^{-1}]$ is generated by $u \otimes f^{-k}$. Q.E.D.

### §3. Localization algorithm revisited

Let $X = K^n$ be the $n$-dimensional affine space over $K$. Let $M$ be a left module over $D_X = D_u$ and $f \in K[x]$ be a non-constant polynomial. Then $X_f := \{ x \in X \mid f(x) \neq 0 \}$ is an affine open subset of $X$. Our purpose is to reformulate the algorithm given in [21] for computing the localization $M[f^{-1}] := M \otimes_{K[x]} K[x, f^{-1}]$ as left $D_X$-module by using local cohomology, hoping to clarify the meaning of the algorithm as well as to make the canonical homomorphism $\iota : M \to M[f^{-1}]$ more explicit.

We assume in what follows, as well as in [21], that $M$ is a holonomic $D_X$-module, or else it is holonomic on $X_f$ with $K$ being algebraically closed; i.e., $\text{Char}(M) \cap \pi^{-1}(X_f)$ is an $n$-dimensional algebraic set of $\pi^{-1}(X_f)$, where $\text{Char}(M)$ is the characteristic variety of $M$, which is an algebraic set of the cotangent bundle $T^*X = \{(x, \xi) \in K^n \times K^n \}$ and $\pi : T^*X \to X$ is the projection (see e.g., 2.4 of [18]).
Localization of D-modules

Introducing a new variable $t$, set $Y = X \times K \ni (x, t)$ and

$$Z := \{(x, t) \in Y \mid tf(x) = 1\}.$$ 

Then $Z$ is an affine subset of $Y$ which is isomorphic to $X_f$. Let

$$B_{Z|Y} := H^1_{(tf(x) - 1)}(K[x, t]) = K[x, t, (tf - 1)^{-1}]/K[x, t]$$

be the first local cohomology group of $K[x, t]$ with support in $Z$, which we regard as a left $D_Y$-module. An arbitrary element of $B_{Z|Y}$ is expressed as

$$\left[ \frac{a(x, t)}{(tf(x) - 1)^{k+l}} \right] \quad (k \in \mathbb{N}, a(x, t) \in K[x, t]),$$

where the bracket denotes the residue class in $B_{Z|Y}$.

Set $f_i = \partial f/\partial x_i$ for $i = 1, \ldots, n$ and define

$$\delta^{(k, l)} := \left[ \frac{tf^{l+1}}{(tf - 1)^{k+1}} \right]$$

for $k, l \in \mathbb{Z}$ with $l \geq -1$. Note that $\delta^{(k, l)} = 0$ by the definition if $k < 0$. As left $K[x, t]$-module, $B_{Z|Y}$ is generated by $\delta^{(k, -1)}$ with $k \in \mathbb{N}$.

We have the following identities for $k, l \geq 0$:

$$\partial_t \delta^{(k, l)} = -(k + 1) \left[ \frac{f^{l+2}}{(tf - 1)^{k+2}} \right] = -(k + 1) \delta^{(k+1, l+1)},$$

$$\partial_{x_i} \delta^{(k, l)} = (l + 1) \left[ \frac{f_i f^l}{(tf - 1)^{k+1}} \right] - (k + 1) \left[ \frac{tf_i f^{l+1}}{(tf - 1)^{k+2}} \right]$$

$$= (l + 1) f_i \delta^{(k, l-1)} - (k + 1) \left[ f_i (tf - 1 + 1) f^l \right] \left[ (tf - 1)^{k+2} \right]$$

$$= (l + 1) f_i \delta^{(k, l-1)} - (k + 1) f_i \delta^{(k+1, l-1)}$$

$$= (l - k) f_i \delta^{(k, l-1)} - (k + 1) f_i \delta^{(k+1, l-1)},$$

$$t \delta^{(k, l)} = \left[ \frac{tf^{l+1}}{(tf - 1)^{k+1}} \right] = \left[ \frac{(tf - 1 + 1) f^l}{(tf - 1)^{k+1}} \right] = \delta^{(k-1, l)} + \delta^{(k, l-1)}.$$ 

In particular, we have

$$(\partial_t + k) \delta^{(k, l)} = -(k + 1) \delta^{(k+1, l)},$$

$$t \delta^{(0, 0)} = \delta^{0, -1}.$$ 

Hence $B_{Z|Y}$ is generated by $\delta^{(0, 0)} = [f(tf - 1)]$ as a left $D_Y$-module.

**Lemma 3.1.** One has $(tf - 1) \delta^{(0, 0)} = 0$ and $(\partial_{x_i} - f_i \partial_t) \delta^{(0, 0)} = 0$ for $i = 1, \ldots, n.$
Proof. The first equality follows immediately from the definition. The second equality follows from
\[ \partial_t^2 \delta^{(0,0)} = \partial_t \delta^{(0,-1)} = -\delta^{(1,-1)} \]
in view of the formulae above. Q.E.D.

Let us regard \( B_{Z/Y} \) as a module over the subring \( K[x] \) of \( D_Y \) and consider the localization
\[
B_{Z/Y}[f^{-1}] := B_{Z/Y} \otimes_{K[x]} K[x, f^{-1}]
\]
with respect to \( f \). Let us denote the residue class in \( B_{Z/Y}[f^{-1}] \) by \([ \bullet ]'\) in order to distinguish it from the residue class in \( B_{Z/Y} \) which is denoted \([ \bullet ]\).

Lemma 3.2. The natural homomorphism
\[
\iota' : B_{Z/Y} \ni \frac{a(x,t)}{(tf-1)^{k+1}} \mapsto \frac{a(x,t)}{(tf-1)^{k+1}}' \in B_{Z/Y}[f^{-1}]
\]
is an isomorphism of left \( D_Y \)-modules.

Proof. Assume \( \iota'([a(x,t)(tf-1)^{-k-1}]) = 0 \) with \( a(x,t) \in K[x,t] \). Then there exists an integer \( l \) such that \( f^l a(x,t) \) is divisible by \( (tf-1)^{k+1} \) in \( K[x,t] \). Since \( f \) and \( tf-1 \) are relatively prime, \( a(x,t) \) must be divisible by \( (1-tf)^{k+1} \). This proves that \( \iota' \) is injective.

Let us show that \( \iota' \) is surjective. It suffices to show that
\[
[f^{-m}(tf-1)^{-k-1}]' \in \iota'(B_{Z/Y})
\]
for any \( k, m \in \mathbb{N} \) by induction on \( k + m \), which obviously holds for \( k = m = 0 \). Suppose \( k + m \geq 1 \). We have
\[
\begin{pmatrix}
1 + (tf-1) \\
(tf-1)^{k+1}
\end{pmatrix}
= \begin{pmatrix}
1 \\
(tf-1)^{k+1}
\end{pmatrix} + \begin{pmatrix}
1 \\
(tf-1)^{k+1}
\end{pmatrix}.
\]
It follows that
\[
\begin{pmatrix}
f^{-m} \\
(tf-1)^{k+1}
\end{pmatrix}' = \begin{pmatrix}
tf^{1-m} \\
(tf-1)^{k+1}
\end{pmatrix}' - \begin{pmatrix}
f^{-m} \\
(tf-1)^{k}
\end{pmatrix}'.
\]
By the induction hypothesis, the right-hand side belongs to the image of \( \iota' \). This completes the proof. Q.E.D.
Localization of $D$-modules

**Proposition 3.3.** Let $M$ be a finitely generated left $D_X$-module. Then the homomorphism

$$B_{Z[Y]} \otimes_{K[x]} M \overset{\sim}{\longrightarrow} B_{Z[Y]} \otimes_{K[x]} M[f^{-1}]$$

of left $D_Y$-modules, which is induced by the natural homomorphism $\iota : M \rightarrow M[f^{-1}]$ is an isomorphism.

**Proof.** We have

$$B_{Z[Y]} \otimes_{K[x]} M[f^{-1}] = B_{Z[Y]} \otimes_{K[x]} (K[x, f^{-1}] \otimes_{K[x]} M)$$

$$= (B_{Z[Y]} \otimes_{K[x]} K[x, f^{-1}]) \otimes_{K[x]} M$$

$$= B_{Z[Y]}[f^{-1}] \otimes_{K[x]} M.$$

Hence the isomorphism $\iota'$ induces an isomorphism

$$B_{Z[Y]} \otimes_{K[x]} M \overset{\sim}{\longrightarrow} B_{Z[Y]}[f^{-1}] \otimes_{K[x]} M = B_{Z[Y]} \otimes_{K[x]} M[f^{-1}].$$

Q.E.D.

**Proposition 3.4.** Let $M$ be a finitely generated left $D_X$-module. Then there exists an isomorphism

$$B_{Z[Y]} \otimes_{K[x]} M \overset{\sim}{\longrightarrow} B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$$

of left $D_Y$-modules.

**Proof.** We have

$$B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$$

$$= (B_{Z[Y]} \otimes_{K[x]} K[x, f^{-1}]) \otimes_{K[x,f^{-1}]} M[f^{-1}] = B_{Z[Y]} \otimes_{K[x]} M[f^{-1}].$$

This completes the proof combined with Proposition 3.4. Q.E.D.

Let $D_X[f^{-1}] := K[x, f^{-1}] \otimes_{K[x]} D_X$ and $D_Y[f^{-1}] := K[x, f^{-1}] \otimes_{K[x]} D_Y$ be the localization of $D_X$ and $D_Y$ by $f$, which can be regarded as ring extensions of $D_X$ and of $D_Y$ respectively. Then $B_{Z[Y]}[f^{-1}]$ and $B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ have natural structures of left $D_Y[f^{-1}]$-module.

**Definition 3.5.** We set $\delta^{(j)} = \iota'(\delta^{(j)})$ for $j \in \mathbb{N}$. We denote $\delta = \delta^{(0)}$.

**Lemma 3.6.** As an element of the left $D_Y[f^{-1}]$-module $B_{Z[Y]}[f^{-1}]$, the annihilator of $\delta$ coincides with the left ideal of $D_Y[f^{-1}]$ generated by

$$t - f^{-1}, \quad \partial_{x_i} - f_i \partial t^2 \quad (i = 1, \ldots, n).$$
Proposition 3.7. Any element of $B_{Z[Y][f^{-1}] \otimes K[x,f^{-1}] M[f^{-1}]}$ is uniquely expressed as a finite sum $\sum_{j \geq 0} \delta^{(j)} \otimes v_j$ with $v_j \in M[f^{-1}]$.

Proof. By Lemma 3.6, $B_{Z[Y][f^{-1}]}$ is isomorphic to $K[x,f^{-1},\partial_t]$ as left $K[x,f^{-1}]-$module. Hence $\delta^{(j)} = (-1)^{j} (1/j!) \partial_t^{j} \delta$ ($j \in \mathbb{N}$) constitute a free basis of $B_{Z[Y][f^{-1}]}$ over $K[x,f^{-1}]$. This implies the assertion of the proposition.

Q.E.D.

Definition 3.8. Set $\vartheta_i := \partial x_i - f_i \partial t^2$ for $i = 1, \ldots, n$. Define a ring homomorphism $\tau$ from $D_X$ to $D_Y$ by

$$\tau : D_X \ni P(x, \partial_x) \mapsto P(x, \vartheta_1, \ldots, \vartheta_n) \in D_Y.$$  

Since $\vartheta_1, \ldots, \vartheta_n$ commute with each other, and $[\vartheta_i, x_j] = \delta_{ij}$, this substitution is a well-defined ring homomorphism.

Lemma 3.9. One has

$$\delta^{0,0} \otimes P v = \tau(P)(\delta^{0,0} \otimes v), \quad \delta \otimes P v' = \tau(P)(\delta \otimes v')$$

in $B_{Z[Y] \otimes K[\epsilon]} M$ and in $B_{Z[Y][f^{-1}] \otimes K[x,f^{-1}] M[f^{-1}]}$ respectively for any $v \in M, v' \in M[f^{-1}]$ and $P \in D_X$. 

Proof. By Lemma 3.1, these operators annihilate $\delta$. Assume $P \in D_Y[f^{-1}]$ annihilates $\delta$. There exist elements $Q_i$ and $R$ of $D_Y[f^{-1}]$ such that

$$P = Q_0(t - f^{-1}) + \sum_{i=1}^{n} Q_i(\partial x_i - f_i \partial t^2) + R$$

and that $R$ belongs to $K[x, f^{-1}, \partial_t]$. Writing $R$ in a finite sum

$$R = \sum_{j=0}^{l} r_j(x) f^{-k} \partial_t^j$$

with $r_j(x) \in K[x]$ and $k, l \in \mathbb{N}$, we have

$$0 = R \delta = \sum_{j=0}^{l} f^{-k} r_j(x) \partial_t^j \delta = \sum_{j=0}^{l} (-1)^{j} j! f^{-k} r_j(x) \delta^{(j)}$$

$$= f^{-k} \left[ \sum_{j=0}^{l} (-1)^{j} j! \left((t - 1)^{l-j} r_j(x) \right) \right] .$$

Since $l'$ is injective, this implies that $r_j(x) = 0$ for any $j \geq 0$, that is, $R = 0$.

Q.E.D.
Localization of $D$-modules

\textbf{Proof.} We have only to show the first equality. Since
\[(\partial_x - f_i \partial_i t^2)\delta(0,0) = 0,\]
we have
\[
\tau(\partial_x)(\delta(0,0) \otimes v) = (\partial_x - f_i \partial_i t^2)\delta(0,0) \otimes v + \delta(0,0) \otimes \partial_x v
\]
\[
= \delta(0,0) \otimes \partial_x v.
\]
We can verify that
\[
\tau(P)(\delta(0,0) \otimes v) = \delta(0,0) \otimes (Pv)
\]
holds by induction on the order of $P$. Q.E.D.

\textbf{Proposition 3.10.} Let $v \in M[f^{-1}]$, $P \in DX$, and $k \in \mathbb{N}$. Then $P(f^{-k}v) = 0$ holds in $M[f^{-1}]$ if and only if $\tau(P) t^k (\delta \otimes v) = 0$ holds in $B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$.

\textbf{Proof.} Since $t^k f^k \delta = (1 + t^k f^k - 1)\delta = \delta$, we have by Lemma 3.9
\[
\delta \otimes P(f^{-k}v) = \tau(P)(\delta \otimes f^{-k}v)
\]
\[
= \tau(P)t^k f^k (\delta \otimes f^{-k}v) = \tau(P)t^k (\delta \otimes v).
\]
This vanishes if and only if $P(f^{-k}v) = 0$ by Proposition 3.7. Q.E.D.

Summing up we obtain

\textbf{Theorem 3.11.} Let $M = DXu$ be a left $DX$-module generated by $u$ and $I = \text{Ann}_{DX} u$ the annihilator of $u$ so that $M = DX/I$. Let $\iota : M \to M[f^{-1}]$ be the canonical homomorphism which sends $u \in M$ to $u \otimes 1$. Let $G$ be a finite set of generators of $I$, and $J$ be the left ideal of $DY$ generated by $\{\tau(P) \mid P \in G\}$ and $tf - 1$. Then

1. $J$ coincides with the annihilator $\text{Ann}_{DY}(\delta \otimes \iota(u))$ of $\delta \otimes \iota(u)$ in $B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$.
2. $B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ is generated by $\delta \otimes \iota(u)$ as a left $DY$-module.
3. As a left $DY$-module, $B_{Z[Y]} \otimes_{K[x]} M$ is isomorphic to $DY/J$.

\textbf{Proof.} (1) It is obvious that $J$ is contained in $\text{Ann}_{DY}(\delta \otimes \iota(u))$. Suppose $P(\delta \otimes \iota(u)) = 0$ with $P \in DY$. Then there exist $R \in DY[f^{-1}]$ and $a_{\alpha,j}(x) \in K[x, f^{-1}]$ which are zero except finitely many $(\alpha, j) \in \mathbb{N}^n \times \mathbb{N}$
such that

\[ P = \sum_{\alpha \in \mathbb{N}^n, j \geq 0} a_{\alpha,j}(x) \partial_j^\alpha (\partial_{x_1} - f_1 \partial_t^2)^{\alpha_1} \cdots (\partial_{x_n} - f_n \partial_t^2)^{\alpha_n} + R(t - f^{-1}) = \sum_{j \geq 0} \partial_j^\alpha \tau(Q_j) + R(t - f^{-1}) \]

with \( Q_j := \sum_{a \in \mathbb{N}^n} a_{\alpha,j}(x) \partial_x^a \in D_X[f^{-1}] \). Then we have

\[ 0 = P(\delta \otimes \iota(u)) = \sum_{j \geq 0} \partial_j^\alpha \tau(Q_j)(\delta \otimes \iota(u)) = \sum_{j \geq 0} (-1)^j \delta^{(j)} \otimes Q_j \iota(u) \]

and consequently \( Q_j \iota(u) = 0 \) for each \( j \geq 0 \) by Proposition 3.7. This implies that \( f^l Q_j u = 0 \) holds in \( M \), that is, \( f^l Q_j \) belongs to \( I \), for some \( l \in \mathbb{N} \) independent of \( j \). We may also assume that \( f^l R \) belongs to \( D_Y f \). Hence \( f^l P = \sum_{j \geq 0} \partial_j^\alpha \tau(f^l Q_j) + f^l R(t - f^{-1}) \) belongs to \( J \). Since \( (1 - t^l f^l)^k P \) belongs to \( D_Y (1 - t^l f^l) \), and hence to \( J \), if we take \( k \in \mathbb{N} \) sufficiently large, and \( t^l f^l P \) belongs to \( J \), we conclude that \( P \) itself belongs to \( J \).

(2) By the assumption, Lemma 2.5, and Proposition 3.7, an arbitrary element of \( B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}] \) is expressed as a finite sum

\[ \sum_{j \geq 0} \partial_j^\alpha \delta \otimes P_j(u \otimes f^{-k}) \]

with \( P_j \in D_X \) and \( k \in \mathbb{N} \). We get

\[ \sum_{j \geq 0} \partial_j^\alpha \delta \otimes P_j(u \otimes f^{-k}) = \sum_{j \geq 0} \partial_j^\alpha \tau(P_j)(\delta \otimes (u \otimes f^{-k})) = \sum_{j \geq 0} \partial_j^\alpha \tau(P_j)t^k f^k(\delta \otimes (u \otimes f^{-k})) = \sum_{j \geq 0} \partial_j^\alpha \tau(P_j)t^k(\delta \otimes \iota(u)). \]

This completes the proof of (2).

(3) follows from (1), (2) and Proposition 3.4. Q.E.D.

**Definition 3.12.** For the sake of simplicity of the notation, let us set

\[ \tilde{M} := B_{Z[Y]}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}] \cong B_{Z[Y]} \otimes_{K[x]} M \]
and define a homomorphism $\varphi : M[f^{-1}] \to \tilde{M}/\partial_t \tilde{M}$ by

$$\varphi(v) = \delta \otimes v \mod \partial_t \tilde{M}.$$ 

for $v \in M[f^{-1}]$.

**Theorem 3.13.** The homomorphism $\varphi : M[f^{-1}] \to \tilde{M}/\partial_t \tilde{M}$ is an isomorphism of left $D_X[f^{-1}]$-modules, and consequently of $D_X$-modules.

**Proof.** By Proposition 3.7 one has direct sum decompositions

$$\tilde{M} = (\delta \otimes M[f^{-1}]) \oplus (\delta^{(1)} \otimes M[f^{-1}]) \oplus (\delta^{(2)} \otimes M[f^{-1}]) \oplus \cdots,$$

$$\partial_t \tilde{M} = (\delta^{(1)} \otimes M[f^{-1}]) \oplus (\delta^{(2)} \otimes M[f^{-1}]) \oplus \cdots$$

as $K[x, f^{-1}]$-modules. Hence $\varphi$ is an isomorphism of $K[x, f^{-1}]$-modules. For $v \in M[f^{-1}]$ and $P \in D_X[f^{-1}]$, one has

$$P(\delta \otimes v) \equiv \tau(P)(\delta \otimes v) = \delta \otimes Pv \mod \partial_t \tilde{M}$$

since $P - \tau(P)$ belongs to $\partial_t D_Y$. Hence $\varphi$ is an isomorphism of left $D_X[f^{-1}]$-modules. Q.E.D.

**Theorem 3.14.** Assume $K$ is algebraically closed. If $M$ is holonomic on $X_f$, i.e., if $\text{Char}(M) \cap \pi^{-1}(X_f)$ is an $n$-dimensional algebraic set, then $D_Y/J$ is a holonomic $D_Y$-module.

**Proof.** We may assume $M = D_X/I$. By the definition, we have

$$\text{Char}(D_Y/J) = \{(x, t; \xi, \tau) \in K^{2n+2} | \sigma(P)(x, \xi_1 - f_1 t^2 \tau, \ldots, \xi_n - f_n t^2 \tau) \quad (\forall P \in I), \quad tf(x) = 1\}$$

$$= \{(x, t; \xi, \tau) | (x, \xi_1 - f_1 t^2 \tau, \ldots, \xi_n - f_n t^2 \tau) \in \text{Char}(M), \quad tf(x) = 1\}.$$ 

Hence there is a bijection

$$(\text{Char}(M) \cap \pi^{-1}(X_f)) \times K \ni (x, \xi, \tau)$$

$$\mapsto (x, 1/tf(x); \xi_1 + f_1 t^2 \tau, \ldots, \xi_n + f_n t^2 \tau) \in \text{Char}(D_Y/J).$$

This implies that $\text{Char}(D_Y/J)$ is of dimension $n + 1$. Q.E.D.

The $D_X$-module $\tilde{M}/\partial_t \tilde{M}$ is nothing but the integral of the $D_Y$-module $\tilde{M}$ with respect to $t$, and $\tilde{M}$ is isomorphic to $D_Y/J$ by Theorem 3.11. Suppose that $M$ is holonomic on $X_f$. Then $\tilde{M} = D_Y/J$ is a holonomic $D_Y$-module by the theorem above. Hence $\tilde{M}/\partial_t \tilde{M}$ is also a holonomic $D_X$-module. In particular, there exists $k_0 \in \mathbb{N}$, or else
The relations among these generators, i.e., a presentation of the $D_X$-module $\bar{M}/\partial_t \bar{M}$ can be computed by the integration algorithm for $D$-modules under the assumption that what is called the $b$-function $b(s)$ with respect to the weight vector $w = (0, ..., 0, 1; 0, ..., 0, -1)$ for $(x_1, \ldots, x_n, t; \partial_{x_1}, \ldots, \partial_{x_n}, \partial_t)$ exists. The integer $k_0$ above can be taken as the maximum integer root of $b(s)$. See [19], [23], [18] for details. This condition is fulfilled if $B_{X,Y} \otimes_{K[x]} M = D_Y/J$ is holonomic, which is the case if $M$ is holonomic on $X_f$.

In conclusion, we get

**Algorithm 3.15** (localization). Input: A set $G_0$ of generators of a left ideal $I = \text{Ann}_{D_X} u$ of $D_X$ such that $M = D_X u$.
Output: a presentation of $M[f^{-1}]$.

1. Let $J$ be the left ideal of $D_Y$ generated by $\tau(G_0) \cup \{tf - 1\}$.
2. Compute the $b$-function $b(s)$ of $D_Y/J$ with respect to the weight vector $(0, ..., 0, 1; 0, ..., 0, -1)$ by e.g., Algorithm 5.6 of [18]. Quit if $b(s)$ does not exist; the computation fails.
3. Let $k_0$ be the maximum integer root of $b(s)$ if any. Then $D_Y/(J + \partial_t D_Y)$, which is isomorphic to $\bar{M}/\partial_t \bar{M}$, is generated by the residue classes $[t^j]$ with $0 \leq j \leq k_0$. If $k_0 < 0$ or $b(s)$ does not have any integer root, then $M[f^{-1}] = 0$; quit.
4. Compute a set $G_1$ of generators of the left $D_X$-submodule

$$N := \left\{ (P_0, P_1, \ldots, P_{k_0}) \in (D_X)^{k_0+1} \mid \sum_{j=0}^{k_0} P_j [t^j] = 0 \right\}$$

of $(D_X)^{k_0+1}$ by using the integration algorithm (e.g., Algorithm 5.10 of [18]). Then one has isomorphisms

$$M[f^{-1}] \cong D_Y/(J + \partial_t D_Y) \cong (D_X)^{k_0+1}/N,$$

by which $u \otimes 1, \ldots, u \otimes f^{-k_0}$ correspond to the residue classes $[[1, 0, \ldots, 0]], \ldots, [[0, \ldots, 0, 1]]$.

In [21], the homomorphism $\tau$ above (denoted $\phi$ in [21]) is defined with $\partial_t$ replaced by $\partial_t - f_t t^2 \partial_t$. This induces the homomorphism $M \to M[f^{-1}]$ which sends $u$ to $u \otimes f^{-2}$ instead of $u \otimes 1$.

The algorithm above and the isomorphisms

$$H^0_{(f)} (M) = \ker \iota \cong \text{Ann}_{D_X} \iota(u)/\text{Ann}_{D_X} u, \quad H^1_{(f)} (M) = M[f^{-1}]/\iota(M)$$
also provide us with algorithms to compute $H^{j}_{(f)}(M)$ for $j = 0, 1$, which work at least if $M$ is holonomic on $X_f$. We remark that H. Tsai gave an algorithm for $H^{0}_{(f)}(M)$ without any assumption on $M$; see Algorithms 4.3 and 4.5 of [25].

**Algorithm 3.16 ($\iota(M)$ and $H^{2}_{(f)}(M)$).** Input: A set $G_0$ of generators of a left ideal $I = \text{Ann}_{D_X}u$ of $D_X$ such that $M = D_X u$. Output: presentations of $\iota(M)$, $H^{1}_{(f)}(M)$, and $H^{0}_{(f)}(M)$ as $D_X$-modules.

1. Compute the integer $k_0$ and a set $G_1$ of generators of the module $N$ of Algorithm 3.15.

2. Compute a set $G_2$ of generators of the left ideal $\tilde{I} := \{ P \in D_X \mid (P, 0, \ldots, 0) \in N \} = \text{Ann}_{D_X}\iota(u)$ of $D_X$ by using a Gröbner basis of $N$ with respect to a POT (position-over-term) order (see e.g., Chapter 5 of [9]). Then one has $\iota(M) \cong D_X / \tilde{I}$ with the correspondence $\iota(u) \leftrightarrow 1$.

3. Set $G'_1 := \{(P_1, \ldots, P_{k_0}) \mid (P_0, P_1, \ldots, P_{k_0}) \in G_1 (\exists P_0 \in D_X)\}$ and $N'$ be the left $D_X$-submodule of $(D_X)^{k_0}$ generated by $G'_1$. Compute a set $G_3$ of generators of the left ideal $I_1 := \{ P \in D_X \mid (0, \ldots, 0, P) \in N' \} = \text{Ann}_{D_X}[f^{-k_0}\iota(u)]$ of $D_X$ by using a Gröbner basis of $N'$ with respect to a POT order. Then one has $H^{1}_{(f)}(M) \cong D_X / I_1$ with the correspondence $[f^{-k_0}\iota(u)] \leftrightarrow \tilde{I}$.

4. Set $G'_2 := \{ P_1, \ldots, P_l \}$. Compute a set of generators of the left $D_X$-module

$$N_0 := \left\{ (Q_1, \ldots, Q_l) \in (D_X)^l \mid \sum_{j=1}^{l} Q_j P_j \in I \right\}$$

through syzygies among $\{ P_1, \ldots, P_l \} \cup G_0$. Then one has

$$H^{0}_{(f)}(M) \cong \tilde{I} / I \cong (D_X)^l / N_0$$

with the correspondence $P_1 u \leftrightarrow (1, 0, \ldots, 0)$ mod $N_0$, ..., $P_l u \leftrightarrow (0, \ldots, 0, 1)$ mod $N_0$. 


In what follows, we freely use the notation and the terminology introduced in Chapters 2, 3, 5 of [18] concerning weight vectors, Gröbner bases, and b-functions.

Example 3.17. Set $n = 1$ and $x = x_1$. It is easy to see that
\[ N_x = D_X x^s \cong D_X / D_X (x \partial_x - s). \]
Since the $b$-function of $x$ is $s + 1$, $N_x(\lambda)$ is isomorphic to the submodule $D_X x^{b_1}$ of $L_\omega(\lambda)$ and hence $x$-saturated, if $\lambda \not\in \mathbb{N}$. So let us consider $M = N_f(0) = D_X / D_X x \partial_x$. Let $u$ be the residue class of 1 in $M$. The left ideal $J$ of $D_Y$ defined in Theorem 3.11 is generated by $tx - 1$ and $\partial_x - \partial_t t^2$. A Gröbner basis of $J$ with respect to a monomial order adapted to the weight vector $(1, 0; -1, 0)$ for $(t, x, \partial_t, \partial_x)$ is
\[ tx - 1, \quad x_2 \partial_x - \partial_t, \quad t \partial_x - x \partial_x + 1 = \partial_t - x \partial_x, \quad t^2 \partial_x - \partial_x + 2t = \partial_t - \partial_x. \]
The $b$-function of $J$ with respect to the weight vector above (see 5.2 of [18]) is divisible by $s(s - 1)$ by virtue of the operator $\partial_t t^2 - \partial_x$. Hence the integration module $M / \partial_t \tilde{M} = D_Y / (J + \partial_t D_Y)$ is generated by the residue classes $[u]$ and $[tu]$, which correspond to $u \otimes 1$ and $u \otimes x^{-1}$ in $M[x^{-1}]$ respectively, over $D_X$. The fundamental relations among the generators can be read off from the Gröbner basis above as follows (see Algorithm 5.10 of [18]):
\[ x[tx] - [u] = 0, \quad \partial_x[u] = 0, \quad x^2 \partial_x[tu] + [u] = 0, \quad (x \partial_x + 1)[tu] = 0. \]
We translate these relations to vectors
\[-1, x, \quad (\partial_x, 0), \quad (1, x^2 \partial_x), \quad (0, x \partial_x + 1) \]
in the free module $D_X^2$. Let $N$ be the left $D_X$-submodule of $D_X^2$ generated by these vectors. By using Gröbner bases of $N$ with respect to POT orders we can confirm that $\text{Ann}_{D_X} [u] = D_X \partial_x$ and $\text{Ann}_{D_X} [tu] = D_X (x \partial_x + 1)$. Hence $M[x^{-1}]$ is generated by $u \otimes x^{-1}$ and isomorphic to $D_X / D_X (x \partial_x + 1) \cong K[x, x^{-1}]$ with the correspondence $u \otimes x^{-1} \leftrightarrow \mathbb{T}$. The image $\imath(M)$ of $\imath : M \to M[x^{-1}]$ is isomorphic to $D_X / D_X \partial_x \cong K[x]$ with the correspondence $u \otimes 1 \leftrightarrow \mathbb{T}$. Finally we get
\[ H^0_{(x)}(M) = \ker \imath = D_X \partial_x u = K[\partial_x]u \cong D_X / D_X x, \]
\[ H^1_{(x)}(M) = D_X [tu] / D_X [u] \cong D_X / D_X x. \]

The following is an example of non-holonomic $M$:
Example 3.18. Set $n = 2$, $x_1 = x$, $x_2 = y$, $P = x\partial_x^2 + \partial_y$, and $M = D_X/D_XP = D_Xu$ as in Example 2.2. Then $M$ is $x$- and $y$-saturated. The localizations of $M$ are

$$M[x^{-1}] = D_X(u \otimes x^{-2}) = D_X/(x^2\partial_x^2 + 4x\partial_x + x\partial_y + 2),$$

$$M[y^{-1}] = D_X(u \otimes y^{-1}) = D_X/(xy\partial_x^2 + y\partial_y + 1).$$

The first local cohomology groups are

$$H^1_x(M) = D_X[u \otimes x^{-2}] = D_X/(D_X x^2 + D_X x\partial_y),$$

$$H^1_y(M) = D_X[u \otimes y^{-1}] = D_X/D_X y,$$

both of which are not holonomic.

Example 3.19. Set $n = 2$, $x_1 = x$, $x_2 = y$, and $f = x^3 - y^2$. Let us consider

$$M = D_X/(D_X \partial_x f + D_X \partial_y f) = D_X u \quad (u = 1).$$

The characteristic variety of $M$ is

$$\text{Char}(M) = \{(x, y; \xi, \eta) \in K^4 \mid x^3 - y^2 = 0\} \cup \{(x, y; 0, 0) \mid (x, y) \in X\}.$$ 

Hence $M$ is holonomic on $X_f$ but not on $X$. The localization $M[f^{-1}]$ is given by

$$M[f^{-1}] = D_X/(D_X (2x\partial_x + 3y\partial_y + 6) + D_X (2y\partial_x + 3x^2\partial_y))$$

with the correspondence $u \otimes 1 \leftrightarrow 1$ and is holonomic on $X$. Hence we have $\iota(M) = M[f^{-1}]$ and $H^1_{(f)}(M) = 0$. Algorithm 3.16 also gives a presentation of $H^0_{(f)}(M)$, which is rather complicated. We can verify that its characteristic variety is

$$\text{Char}(H^0_{(f)}(M)) = \{(x, y; \xi, \eta) \in K^4 \mid x^3 - y^2 = 0\}$$

as is expected.

§4. Algorithms for $u \otimes f^*$ and the $b$-function

The purpose here is to give algorithms to compute $M(u, f, s)$ and $M(u, f, \lambda)$ as well as the $b$-function $b_{u, f}(s)$ for an arbitrary $D_X$-module $M = D_X u$ that is holonomic on $X_f$, and an arbitrary non-constant polynomial $f$. Algorithms for these objects were already given in [17] under the additional assumption that $M$ is $f$-saturated. We remove this assumption by using the localization algorithm.
Set $X = K^n$ and $Y = K^{n+1}$ with coordinates $x = (x_1, \ldots, x_n)$ of $X$ and $(x, t)$ of $Y$. Let $f = f(x) \in K[x]$ be a non-constant polynomial and let $Z$ be the affine subset $Z = \{(x, t) \mid t = f(x)\}$ of $Y$. (Note that $Z$ is different from what was defined in the previous section.) We regard the local cohomology group

$$B_{Z|Y} := H^1_Z(K[x,t]) = K[x, t, (t - f)^{-1}]/K[x, t]$$

as a left $D_Y$-module. For $k \in \mathbb{N}$, let

$$\delta^{(k)}(t - f(x)) = \left[ \frac{(-1)^kk!}{(t - f(x))^{k+1}} \right]$$

be the residue class in $B_{Z|X}$ and denote $\delta(t - f) = \delta^{(0)}(t - f)$. Then $\delta(t - f)$ satisfies a holonomic system

$$(t - f)\delta(t - f) = (\partial_{x_i} + f_i\partial_t)\delta(t - f) = 0 \quad (1 \leq i \leq n)$$

with $f_i = \partial f/\partial x_i$. Hence there exists an isomorphism $B_{Z|Y} \cong D_Y/J_0$ with

$$J_0 := D_Y(t - f) + D_Y(\partial_{x_1} + f_1\partial_t) + \cdots + D_Y(\partial_{x_n} + f_n\partial_t)$$

as left $D_Y$-module since $J_0$ is maximal. In particular, $\delta^{(k)}(t - f)$ $(k \in \mathbb{N})$ constitute a free basis of $B_{Z|Y}$ over $K[x]$.

Following Malgrange [15], let us give $\mathcal{L}_f = D_X[s]f^s$ a structure of left $D_Y$-module by

$$t(a(x, s)f^{-k}f^s) = a(x, s+1)f^{-k+1}f^s,$$

$$\partial_t(a(x, s)f^{-k}f^s) = -sa(x, s-1)f^{-k-1}f^s$$

for $a(x, s) \in K[x,s]$ and $k \in \mathbb{N}$. The actions of $t$ and $\partial_t$ on $\mathcal{L}_f$ satisfy $[\partial_t, t] = 1$, and they commute with $x_i, \partial_{x_i}$. Hence the definition above extends to the action of $D_Y$ on $\mathcal{L}_f$. In particular, $\partial_t f^s = -sf^s$ holds, which will play an important role in what follows.

With respect to this action, we can regard $B_{Z|Y}$ as a left $D_Y$-submodule of $\mathcal{L}_f$ by identifying $\delta^{(k)}(t - f)$ with $(-1)^k s(s - 1) \cdots (s - k + 1)f^{s-k}$ in $\mathcal{L}_f$. In fact, we have

$$J_0 = \text{Ann}_{D_Y} f^s = \text{Ann}_{D_Y} \delta(t - f)$$

since $J_0$ annihilates $f^s$ as well as $\delta(t - f)$ and $J_0$ is a maximal left ideal.

For a left $D_X$-module $M = D_Xu$, let us consider the tensor product $M \otimes_{K[x]} B_{Z|Y}$, which is a left $D_Y$-module.
**Definition 4.1.** Set \( \vartheta'_i := \partial_{x_i} + f_i \partial_t \) for \( i = 1, \ldots, n \). Define a ring homomorphism \( \tau' \) from \( D_X \) to \( D_Y \) by

\[
\tau' : D_X \ni P(x, \partial_x) \mapsto P(x, \vartheta'_1, \ldots, \vartheta'_n) \in D_Y.
\]

Since \( \vartheta'_1, \ldots, \vartheta'_n \) commute with each other, and \( [\vartheta'_i, x_j] = \delta_{ij} \), this substitution is a well-defined ring homomorphism.

Since \( \vartheta'_i \delta(t - f) = 0 \) for \( 1 \leq i \leq n \),

\[
\tau'(P)(v \otimes \delta(t - f)) = P v \otimes \delta(t - f)
\]

holds for any \( v \in M \) and \( P \in D_X \). Hence we have

\[
Pu \otimes \partial^k \delta(t - f) = \partial^k \tau'(P)(u \otimes \delta(t - f)).
\]

This proves

**Lemma 4.2.** If \( M = D_Xu \), then the left \( D_Y \)-module \( M \otimes_{K[x]} \mathcal{B}_{Z|Y} \) is generated by \( u \otimes \delta(t - f) \).

**Theorem 4.3.** Let \( M = D_Xu \) be a left \( D_X \)-module generated by \( u \) and \( f \in K[x] \). Let \( G \) be a finite set of generators of \( I := \operatorname{Ann}_{D_X}u \) and let \( J \) be the left ideal of \( D_Y \) generated by \( \{ \tau'(P) \mid P \in G \} \cup \{ t - f \} \). Then \( J \) coincides with the annihilator \( \operatorname{Ann}_{D_Y}(u \otimes \delta(t - f)) \). Moreover, if \( M \) is a holonomic \( D_X \)-module, then \( M \otimes_{K[x]} \mathcal{B}_{Z|Y} \) is a holonomic \( D_Y \)-module.

The first part of this proposition was proved by Walther [26]. The proof is almost the same as the proof of Theorem 3.11. The last assertion can be proved in the same way as Theorem 3.14.

Thus we have an algorithm to compute \( M \otimes_{K[x]} \mathcal{B}_{Z|Y} \). The inclusion \( \mathcal{B}_{Z|X} \subset \mathcal{L}_f \) induces a homomorphism

\[
\psi : M \otimes_{K[x]} \mathcal{B}_{Z|Y} \longrightarrow M \otimes_{K[x]} \mathcal{L}_f
\]

of left \( D_Y \)-modules. Our main aim is to compute the \( D_X[s] \)-submodule \( M(u, f, s) = D_X[s](u \otimes f^*) \) of \( M \otimes_{K[x]} \mathcal{L}_f \), which is the image of the submodule \( D_X[s](u \otimes f^*) \) of \( M \otimes_{K[x]} \mathcal{B}_{Z|Y} \) by \( \psi \). The following lemma was proved in [17] as Proposition 6.13.

**Lemma 4.4.** The homomorphism \( \psi \) above is injective if and only if \( M \) is \( f \)-saturated.

**Proof.** An arbitrary element of \( M \otimes_{K[x]} \mathcal{B}_{Z|Y} \) is expressed uniquely as

\[
w = \sum_{j=0}^{k} v_j \otimes \delta^{(j)}(t - f)
\]
with \( k \in \mathbb{N} \) and \( v_j \in M \). Then

\[
\psi(u) = \sum_{j=0}^{k} (-1)^j v_j \otimes (s(s-1) \cdots (s-j+1)f^{-j}f^s)
\]

vanishes if and only if each \( v_j \otimes f^{-j} \) vanishes in \( M \otimes_{K[x]} K[x, f^{-1}] \), which is equivalent to \( v_j \in H^0_f(M) \). Q.E.D.

**Lemma 4.5.** Let \( M = D_Xu \) be a left \( D_X \)-module generated by \( u \) and \( f \in K[x] \) be a non-constant polynomial. Let \( \iota : M \to M[f^{-1}] \) be the canonical homomorphism. Then \( \iota \) induces isomorphisms

\[
M \otimes_{K[x]} \mathcal{L}_f \xrightarrow{\sim} \iota(M) \otimes_{K[x]} \mathcal{L}_f,
\]

\[
M(u, f, s) \xrightarrow{\sim} \iota(M)(u, f, s)
\]

of left \( D_X[s] \)-modules.

**Proof.** Since \( \mathcal{L}_f \) is isomorphic to \( K[x, f^{-1}, s] \) as a \( K[x, s] \)-module, we have only to show that the natural homomorphism

\[
M[f^{-1}] \to \iota(M)[f^{-1}]
\]

is an isomorphism, which is obvious by the definition. The second isomorphism follows from the first. Q.E.D.

Summing up we obtain

**Algorithm 4.6** \((M(u, f, s) \text{ and } M(u, f, \lambda))\). Input: A finite set of generators of a left ideal \( I \) of \( D_X \) so that \( M = D_Xu = D_X/I \), a non-constant \( f \in K[x] \), and \( \lambda \in K \).

Output: presentation of \( M(u, f, s) \) and of \( M(u, f, \lambda) \).

1. Compute \( \iota(M) = D_X/\text{Ann}_{D_Xu}(\iota(u)) \) by Algorithm 3.16.
2. Compute \( J = \text{Ann}_{D_X}(\iota(u) \otimes \delta(t-f)) \) by using Theorem 4.3.
3. Compute \( J \cap D_X[s] \), which is equal to \( \text{Ann}_{D_X[s]}(\iota(u) \otimes f^s) = \text{Ann}_{D_X[u]}(u \otimes f^s) \).
4. The substitution \( s = \lambda \) for generators of \( J \cap D_X[s] \) gives a set of generators of \( \text{Ann}_{D_X}[u \otimes f^s] \mid_{s=\lambda} \).

**Proposition 4.7.** Let \( M = D_Xu \) be a left \( D_X \)-module generated by \( u \) and \( f \in K[x] \) a non-constant polynomial. Then the \( b \)-function \( b_{u,f}(x) \) exists if and only if there exists a nonzero polynomial \( b(s) \in K[s] \) such that

1. \( b(-\partial_i t)(u \otimes \delta(t-f)) \in tD_X[\iota(\delta_i)[u \otimes \delta(t-f)]] = D_X[t\delta_i t(u \otimes \delta(t-f))]. \)

If \( M \) is \( f \)-saturated and such \( b(s) \) exists, then \( b_{u,f}(s) \) is the monic polynomial of the minimum degree among such \( b(s) \).
Proof. Assume that (2) holds. Then there exists \( P(s) \in D_X[s] \) such that
\[
b(-\partial_t)(u \otimes \delta(t - f)) = P(-\partial_t)t(u \otimes \delta(t - f)) = P(-\partial_t)f(u \otimes \delta(t - f)).
\]
Applying the homomorphism \( \psi \) we get \( b(s)(u \otimes f^k) = P(s)(u \otimes f^{k+1}) \).
Hence \( b_{u,f}(s) \) exists and divides \( b(s) \).

On the other hand, assume that there exist nonzero \( b(s) \in K[s] \)
and \( P(s) \in D_X[s] \) such that \( b(s)(u \otimes f^k) = P(s)(u \otimes f^{k+1}) \) holds in \( M \otimes_{K[x]} \mathcal{L}_f \). Then as is seen by the proof of Lemma 4.4, there exists \( k \in \mathbb{N} \) such that
\[
f^k b(-\partial_t)(u \otimes \delta(t - f)) = f^k P(-\partial_t)f(u \otimes \delta(t - f)).
\]
Since
\[
f^k b(-\partial_t)(u \otimes \delta(t - f)) = b(-\partial_t)f^k(u \otimes \delta(t - f)) = b(-\partial_t)t^k(u \otimes \delta(t - f))
\]
holds, we get
\[
\partial^k_t b(-\partial_t)f^k(u \otimes \delta(t - f)) = \partial^k_t f^k P(-\partial_t)f(u \otimes \delta(t - f)) = D_X[t\partial_t]t(u \otimes \delta(t - f)).
\]
This completes the proof because there exists \( c(s) \in K[x] \) such that
\[
\partial^k_t b(-\partial_t)f^k = c(-\partial_t). \quad \text{Q.E.D.}
\]

Now we obtain an algorithm to determine whether the \( b \)-function exists and to compute it if it does:

**Algorithm 4.8** \((b_{u,f}(s))\). Input: \( M = D_Xu = D_X/I \) with a finite set of generators of \( I \) and a non-constant \( f \in K[x] \).
Output: the \( b \)-function \( b_{u,f}(s) \) if it exists, ‘No’ if it does not exist.

1. Compute \( J' := \text{Ann}_{D_X}(u \otimes \delta(t - f)) \) by using Theorem 4.3.
2. Compute \( I' = (J' + D_X[s]f) \cap K[s] \) by elimination. If \( I' \neq \{0\} \), then there exists \( b_{u,f}(s) \). Otherwise, the \( b \)-function does not exist; output ‘No’ and quit.
3. Compute a set of generators of \( J := \text{Ann}_{D_X[s]}(u \otimes f^s) \) by Algorithm 4.6.
4. Compute \( I(u,f) = (J + D_X[s]f) \cap K[s] \) by elimination. Let \( b_{u,f}(s) \) be the monic generator of \( I(u,f) \).

**Algorithm 4.9** \((D_X(u \otimes f^\lambda))\). Input: \( M = D_Xu = D_X/I \) with a finite set of generators of \( I \), a non-constant \( f \in K[x] \), and \( \lambda \in K \).
Output: presentation of \( D_X(u \otimes f^\lambda) \), i.e., \( \text{Ann}_{D_X}(u \otimes f^\lambda) \) if \( b_{u,f}(s) \) exists.
In fact, we obtain
\[ \text{Algorithm 3.15 shows that } (x, y) = A_{\text{ann}}(u) \left( x, y, s \right) = A_{\text{ann}}(u) \left( x, y + 1 \right) = 0. \]
\[ \text{By Algorithm 3.16, we have } \] 
\[ \text{Example 4.10.} \]
\[ \text{Set } n = 2, \] 
\[ x_1 = x, \quad x_2 = y, \quad P = x\partial_x^2 + \partial_y, \quad \text{and} \quad M = D_X/D_X P = D_X u \] 
\[ \text{as in Example 2.2. By Algorithms 4.6 and 4.8 we obtain } b_{u,x}(s) = (s+1)(s+2), \]
\[ b_{u,y}(s) = s + 1, \] 
\[ \text{and} \]
\[ M(u, x, s) := D_X[s](u \otimes x^s) \]
\[ = D_X[s]/D_X[s](x^2\partial_x^2 - 2sx\partial_x + x\partial_y + s^2 + s), \]
\[ M(u, y, s) := D_X[s](u \otimes y^s) = D_X[s]/D_X[s](xy\partial_x^2 + y\partial_y - s). \]
\[ \text{Example 4.11.} \]
\[ \text{Setting } n = 2 \] 
\[ \text{and } x_1 = x, \quad x_2 = y, \] 
\[ \text{let us consider } f(x, y) = x^3 - y^2 \] 
\[ \text{and} \]
\[ N_f = D_X[s]f^s, \quad M := N_f \left( \frac{1}{6} \right) = N_f \left( s - \frac{1}{6} \right) N_f. \]
We have \( M = D_X/I = D_X u \) with \( u := f^s|_{s=1/6} \) and \( I = \text{Ann}_{D_X} u = D_X(2x\partial_x + 3y\partial_y - 1) + D_X(2y\partial_x + 3x^2\partial_y). \)
\[ \text{Then Algorithm 3.15 shows that } M[f^{-1}] \text{ is generated by } u \otimes f^{-1} \] 
\[ \text{and is isomorphic to } D_X/J \text{ with} \]
\[ J = \text{Ann}_{D_X} u \otimes f^{-1} = D_X(2x\partial_x + 3y\partial_y + 5) + D_X(2y\partial_x + 3x^2\partial_y). \]
In fact, \( M[f^{-1}] \) is isomorphic to \( L_f(1/6) = K[x, f^{-1}]f^{1/6} = D_X f^{-5/6}, \) 
and \( \iota(M) \) to the submodule \( D_X f^{1/6}. \)
\[ \text{By Algorithm 3.16, we have } \iota(M) = D_X/J_0 \text{ with the left ideal } J_0 \]
generated by 
\[ 2x\partial_x + 3y\partial_y - 1, \quad 2y\partial_x + 3x^2\partial_y, \quad 8\partial_x^3 + 27y\partial_y^3 + 9\partial_y^2. \]
Note that \( J_0 \) is strictly larger than \( I \) because of the last generator. Algorithm 3.16 also yields presentations of the local cohomology groups

\[
H^0_{(f)}(M) \cong D_X / (D_X x + D_X y) \quad \text{with} \quad (8 \partial_x^2 + 27y \partial_y^3 + 9 \partial_x^2)u \leftrightarrow T,
\]

\[
H^1_{(f)}(M) \cong D_X / (D_X x + D_X y) \quad \text{with} \quad [u \otimes f^{-1}] \leftrightarrow T.
\]

The \( b \)-function for \( u \) and \( f \) is

\[
b_{u,f}(s) = (s + 1) \left( s + \frac{4}{3} \right) \left( s + \frac{7}{6} \right) = b_f \left( s + \frac{1}{6} \right),
\]

where \( b_f(s) = (s + 1)(s + 5/6)(s + 7/6) \) is the \( b \)-function of \( f \).

**Example 4.12.** Set \( n = 2 \) and \( x_1 = x, x_2 = y \) and consider

\[
M = H^1_{(xy)}(K[x, y]) = D_X / (D_X xy + D_X(x \partial_x + 1) + D_X(y \partial_y + 1)) = D_X u
\]

with \( u = [(xy)^{-1}] \). Then \( M[x^{-1}] \) is generated by \( u \otimes 1 \), and hence \( M[1^{-1}] = \mathcal{I}(M) \) and \( H^1_{(xy)}(M) = 0 \). We have

\[
M[x^{-1}] = D_X / (D_X(x \partial_x + 1) + D_X y),
\]

\[
H^0_{(xy)}(M) \cong D_X / (D_X x + D_X \partial_y)
\]

with the correspondence \( u \otimes 1 \leftrightarrow \mathcal{T} \) and \( yu \leftrightarrow \mathcal{T} \) respectively. The \( b \)-function of \( u := [(xy)^{-1}] \) and \( x \) is \( b_{u,x}(s) = s \). The module \( M(u, x, s) \) is

\[
M(u, x, s) = D_X / (D_X s(x \partial_x - s + 1) + D_X s y).
\]

**Example 4.13.** Set \( n = 3, x_1 = x, x_2 = y, x_3 = z, \) and \( f = x^3 - y^2 z^2 \). Let us consider

\[
M = H^1_{(xf)}(K[x, y]) = D_X u \quad (u = [(xf)^{-1}]).
\]

The localizations \( M[x^{-1}] \) and \( M[f^{-1}] \) are given by

\[
M[x^{-1}] = D_X (u \otimes 1)
\]

\[
= D_X / (D_X f + D_X(x \partial_x + 3z \partial_z + 8) + D_X(y \partial_y - z \partial_z),
\]

\[
M[f^{-1}] = D_X (u \otimes 1)
\]

\[
= D_X / (D_X x + D_X(y \partial_y + 2) + D_X(z \partial_z + 2).
\]

In particular, we have \( H^1_{(xy)}(M) = H^1_{(f)}(M) = 0 \). The zeroth cohomology groups are

\[
H^0_{(xy)}(M) \cong D_X / (D_X x + D_X \partial_y + D_X \partial_z).
\]
with the correspondence \( fu \leftrightarrow \mathfrak{T} \), and \( H^0(f)(M) \cong \mathcal{D}_X/I \) with the correspondence \((y\partial_y + 2)u \leftrightarrow \mathfrak{T} \), where \( I \) is the left ideal generated by
\[
\begin{align*}
&f^2, \quad y\partial_y - z\partial_z, \quad 2x\partial_x + 3z\partial_z + 8, \quad x^3\partial_y - z^3y\partial_z - 4z^2y, \\
&f\partial_z - 4zy^2, \quad x^3\partial_y^2 - z^4\partial_z^2 - 6z^3\partial_z - 4z^2, \\
&8z^2y\partial_x^3 + (27z^2\partial_z^2 + 81z\partial_z + 24)\partial_y.
\end{align*}
\]
We also have
\[
\begin{align*}
M(u, x, s) &= \mathcal{D}_X[s]/(\mathcal{D}_X[s]f + \mathcal{D}_X[s](2x\partial_x + 3y\partial_y - 2s + 8) \\
&\quad + \mathcal{D}_X[s](y\partial_y - z\partial_z)), \\
M(u, f, s) &= \mathcal{D}_X[s]/(\mathcal{D}_X[s]x + \mathcal{D}_X[s](y\partial_y - 2s - 2) \\
&\quad + \mathcal{D}_X[s](y\partial_y - z\partial_z)).
\end{align*}
\]
The corresponding \( b \)-functions are
\[
b_{u,x}(s) = s^2 \left( s - \frac{3}{2} \right)^2, \quad b_{u,f}(s) = s^2 \left( s - \frac{1}{2} \right)^2.
\]

§5. Length and multiplicity of \( D \)-modules

We set \( X = K^n \) as in the preceding sections. First let us recall basic facts about the length and the multiplicity of a left \( \mathcal{D}_X \)-module following J. Bernstein ([3],[4]). Let \( M \) be a finitely generated left \( \mathcal{D}_X \)-module. A composition series of \( M \) of length \( k \) is a sequence
\[
M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0
\]
of left \( \mathcal{D}_n \)-submodules such that \( M_i/M_{i-1} \) is a nonzero simple left \( \mathcal{D}_X \)-module (i.e. having no proper left \( \mathcal{D}_X \)-submodule other than 0) for \( i = 1, \ldots, k \). The length of \( M \), which we denote by \( \text{length} \ M \), is the least length of composition series (if any) of \( M \). If there is no composition series, the length of \( M \) is defined to be infinite. The length is additive in the sense that if
\[
0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0
\]
is an exact sequence of left \( \mathcal{D}_X \)-modules of finite length, then \( \text{length} \ M = \text{length} \ N + \text{length} \ L \) holds.

For each integer \( k \), set
\[
F_k(\mathcal{D}_X) = \left\{ \sum_{|\alpha| + |\beta| \leq k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in K \right\}.
\]
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In particular, we have $F_k(D_X) = 0$ for $k < 0$ and $F_0(D_X) = K$. The filtration $\{F_k(D_X)\}_{k \in \mathbb{Z}}$ is called the Bernstein filtration on $D_X$.

Let $M$ be a finitely generated left $D_X$-module. A family $\{F_k(M)\}_{k \in \mathbb{Z}}$ of $K$-subspaces of $M$ is called a Bernstein filtration on $M$ if it satisfies

1. $F_k(M) \subset F_{k+1}(M)$ (\forall k \in \mathbb{Z}),
2. $\bigcup_{k \in \mathbb{Z}} F_k(M) = M$,
3. $F_j(D_X)F_k(M) \subset F_{j+k}(M)$ (\forall j, k \in \mathbb{Z}).

Moreover, $\{F_k(M)\}$ is called a good Bernstein filtration if there exist $u_i \in F_k(M)$ ($i = 1, \ldots, m$) such that

$$F_k(M) = F_{k-h_1}(D_X)u_1 + \cdots + F_{k-h_m}(D_X)u_m \quad (\forall k \in \mathbb{Z}).$$

If $\{F_k(M)\}$ is a good Bernstein filtration, then each $F_k(M)$ is a finite dimensional vector space over $K$ and $F_k(M) = 0$ for $k \ll 0$ (see e.g., 2.3 of [18]).

Let $\{F_k(M)\}$ be a good Bernstein filtration on $M$. Then there exists a polynomial $p(T) = c_dT^d + c_{d-1}T^{d-1} + \cdots + c_0 \in \mathbb{Q}[T]$ such that

$$\dim_K F_k(M) = p(k) \quad (k \gg 0)$$

and $d|c_d$ is a positive integer. We call $p(T)$ the Hilbert polynomial of $M$ with respect to the filtration $\{F_k(M)\}$. The leading term of $p(T)$ does not depend on the choice of a good Bernstein filtration $\{F_k(M)\}$. The degree $d$ of the Hilbert polynomial $p(T)$ is called the dimension of $M$ and denoted $\dim M$. The multiplicity of $M$, denoted $\text{mult} M$ is defined to be the positive integer $d|c_d$. The dimension and the multiplicity are invariants of a finitely generated left $D_X$-module.

If $M \neq 0$, then the dimension of $M$ is not less than $n$ (Bernstein’s inequality). By definition, $M$ is holonomic if $M = 0$ or $\dim M = n$. If $M$ is a holonomic left $D_X$-module, we have an inequality length $M \leq \text{mult} M$ and hence $M$ is of finite length in particular. Moreover, the multiplicity is additive for holonomic left $D_X$-modules.

We can compute the dimension and the multiplicity of a given finitely generated (not necessarily holonomic) $D_X$-module by using a Gröbner basis with respect to a term order compatible with the Bernstein filtration.

**Example 5.1.** Let $M$ be the $D_X$-module with $X = K^2$ defined in Example 4.11. We get exact sequences

$$0 \longrightarrow H^0_{(f)}(M) \longrightarrow M \longrightarrow \iota(M) \longrightarrow 0,$$

$$0 \longrightarrow H^0_{(f)}(M) \longrightarrow M \longrightarrow M[f^{-1}] \longrightarrow H^1_{(f)}(M) \longrightarrow 0.$$
with \( H^0_{(f)}(M) \cong H^2_{(x,y)}(K[x,y]) \cong H^1_{(f)}(M) \). We have
\[
\text{mult } M = \text{mult } M[f^{-1}] = 6, \quad \text{mult } \iota(M) = 5, \\
\text{mult } H^0_{(f)}(M) = \text{mult } H^1_{(f)}(M) = 1.
\]

The following two propositions are easy and should be well-known.

**Proposition 5.2.** Let \( f \) be a non-constant polynomial in \( x_1, \ldots, x_n \). Then the multiplicity of \( K[x, f^{-1}] \) is at most \((\deg f + 1)^n\), where \( \deg f \) stands for the total degree of \( f \).

**Proof.** Let \( d \) be the degree of \( f \). Then
\[
F_k(K[x, f^{-1}]) := \left\{ \frac{a}{f^{k+1}} \mid a \in K[x_1, \ldots, x_n], \deg a \leq (d + 1)k \right\}
\]
for \( k \in \mathbb{Z} \) constitute a (not necessarily good) Bernstein filtration on \( K[x, f^{-1}] \), which is finitely generated over \( D_n \). Since
\[
\dim_K F_k(K[x, f^{-1}]) = \binom{n + (d + 1)k}{n},
\]
we have \( \dim_K K[x, f^{-1}] = n \) and \( \text{mult } M \leq (d + 1)^n \). Q.E.D.

**Proposition 5.3.** Let \( n = 1 \) and \( f \in K[x] = K[x_1] \) be non-constant square free. Then one has \( \text{mult } K[x, f^{-1}] = \deg f + 1 \).

**Proof.** Set \( M := H^1_{(f)}(K[x]) \). Then \( M \) is isomorphic to \( D_X/D_X f \) since \( f \) is square-free. Hence
\[
F_k(M) := F_k(D_1)[f^{-1}] \cong F_k(D_1)/F_{k-d}(D_1)f
\]
with \( d := \deg f \) constitute a good Bernstein filtration on \( M \). Since
\[
\dim F_k(M) = \dim F_k(D_1) - \dim F_{k-d}(D_1) = \binom{k + 2}{2} - \binom{k - d + 2}{2} = dk - \frac{1}{2}d(d - 3)
\]
holds for \( k \geq d \), the multiplicity of \( M \) is \( d \). Q.E.D.

We shall give two examples in two variables.

**Proposition 5.4.** Set \( X = K^2 \) and write \( x_1 = x, x_2 = y \). Set \( f = x^m + y^l \) with positive integers \( l, m \). Then the multiplicity of \( K[x, y, f^{-1}] \) equals \( 2 \max\{l, m\} \).
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Proof. We may assume \( m \leq l \). Set \( M := H^1_f(K[x,y]) \). Since the \( b \)-function \( b_f(s) \) of \( f \) does not have any negative integer \( \leq -2 \) as a root (see e.g., 6.4 of [14]), \( M \) is generated by \( u := f^{-1} \in M \) over \( D_X \). The annihilator \( \text{Ann}_{D_X} u \) is generated by

\[
\begin{align*}
f, & \quad E := lx \partial_x + my \partial_y + ml, \\
P & := ly^{l-1} \partial_y - mx^{m-1} \partial_y
\end{align*}
\]

(see also 6.4 of [14]). A Gröbner basis of \( \text{Ann}_{D_X} [f^{-1}] \) with respect to a total-degree reverse lexicographic order \( \prec \) such that \( x \succ y \succ \xi \succ \eta \) is \( G = \{ f, E, P \} \), where \( \xi \) and \( \eta \) are the commutative variables corresponding to \( \partial_x \) and \( \partial_y \) respectively. In fact, in case \( m < l \) the \( S \)-pairs (see Chapter 3 of [18]) are divisible by \( G \):

\[
\begin{align*}
\text{sp}_\prec(f, E) & = lx \partial_x f - y^lE = x^mE - my \partial_y f, \\
\text{sp}_\prec(f, P) & = l \partial_x f - yP = x^{m-1}E, \\
\text{sp}_\prec(E, P) & = y^{l-1}E - xP = m \partial_y f.
\end{align*}
\]

The initial monomials of the Gröbner basis \( G \) are \( \text{in}_\prec(f) = y^l \), \( \text{in}_\prec(E) = x^l \), \( \text{in}_\prec(P) = y^{l-1} \). Hence for \( k \geq l \) we obtain

\[
\begin{align*}
\dim_K F_k(D_X)/(\text{Ann}_{D_X} [f^{-1}] \cap F_k(D_X)) & = 2^l \binom{x y^j \xi^i \eta^\nu}{i+j+\mu+\nu \leq k} \setminus \langle y^l, x^l, y^{l-1} \rangle \\
& = 2^l \binom{x y^j \xi^i \eta^\nu}{i+j+\nu \leq k, j \leq l-1} \\
& \quad + 2^l \binom{x y^j \xi^i \eta^\nu}{i+\mu+\nu \leq k, j \leq l-2, \mu \geq 1} \\
& = \sum_{j=0}^{l-1} \binom{2+k-j}{2} + \sum_{j=0}^{l-2} \binom{2+k-j-1}{2} = \frac{2l-1}{2} k^2 + \cdots .
\end{align*}
\]

On the other hand, in case \( m = l \) we have

\[
\begin{align*}
\text{sp}_\prec(f, E) & = l \partial_x f - x^{l-1}E = yP, \\
\text{sp}_\prec(f, P) & = ly^{l-1} \partial_y f - x^lP = y^lP + lx^{l-1} \partial_y f, \\
\text{sp}_\prec(E, P) & = y^{l-1}E - xP = l \partial_y f.
\end{align*}
\]
The initial monomials are in$_{\mathcal{I}}(f) = x^l$, in$_{\mathcal{I}}(E) = x\xi$, in$_{\mathcal{I}}(P) = y^{l-1}\xi$. (Note that $y^{l-1}\xi > x_{l-1}\eta$ holds.) Hence for $k \geq l$ we obtain
\[
\dim_K F_k(D_X)/(\Ann_{D_X}[f^{-1}] \cap F_k(D_X))
= \sharp\{x^i y^j \xi^k \eta^k | i + j + \mu + \nu \leq N\} \setminus \{x^l, y^{l-1}\xi\}
= \sharp\{x^i y^j \xi^k \eta^k | i + j + \nu \leq k, i \leq l - 1\}
+ \sharp\{y^j \xi^k \eta^k | j + \mu + \nu \leq k, j \leq l - 2, \mu \geq 1\}
= \sum_{i=0}^{l-1} \binom{2 + k - i}{2} + \sum_{j=0}^{l-2} \binom{2 + k - j - 1}{2}
= \frac{2l - 1}{2} k^2 + \ldots.
\]
Hence the multiplicity of $M$ is $2l - 1$ in both cases. This proves the assertion. Q.E.D.

**Proposition 5.5.** Set $X = K^2$ with $x_1 = x$ and $x_2 = y$. Set $f = x^m + y^{l+1}$ with positive integers $l, m$. Then the multiplicity of $K[x, y, f^{-1}]$ equals $lm + |l - m| + 1$.

**Proof.** We may assume $m \leq l$. Set $M := \mathcal{H}_{(f)}(K[x, y])$. Since the curve $f = 0$ is non-singular, the $b$-function is $b_f(s) = s + 1$. Hence $M$ is generated by $u := [f^{-1}]$. The annihilator $\Ann_{D_X} u$ is generated by $f$ and $P := ly^{l-1}\partial_x - mx^{m-1}\partial_y$ since $f = 0$ is non-singular.

In case $l = m$, $G = \{f, P\}$ is a Gröbner basis of $\Ann_{D_X}[f^{-1}]$ with respect to a total-degree reverse lexicographic order $\prec$ such that $x > y \succ \xi > \eta$. In fact, we have
\[
\sp_{\mathcal{I}}(f, P) = ly^{l-1}\partial_x f - x^l P = ly^{l-1}\partial_x f + x^l P.
\]
Since in$_{\mathcal{I}}(f) = x^l$ and in$_{\mathcal{I}}(P) = y^{l-1}\xi$, we have for $k \geq 2l$
\[
\dim_K F_k(D_X)/(\Ann_{D_X}[f^{-1}] \cap F_k(D_X))
= \sharp\{x^i y^j \xi^k \eta^k | i + j + \mu + \nu \leq N\} \setminus \{x^l, y^{l-1}\xi\}
= \sharp\{x^i y^j \xi^k \eta^k | i + j + \nu \leq k, i \leq l - 1\}
+ \sharp\{y^j \xi^k \eta^k | j + \mu + \nu \leq k, j \leq l - 2, \mu \geq 1\}
= \sum_{i=0}^{l-1} \binom{2 + k - i}{2} + \sum_{j=0}^{l-2} \sum_{i=0}^{l-1} \binom{2 + k - i - j - 1}{2}
= \frac{l^2}{2} k^2 + \ldots.
\]
In case $m < l$, the Gröbner basis of $\Ann_{D_X}[f^{-1}]$ with respect to the same order as above is $G = \{f, P, Q\}$ with
\[
Q := l(x^m + 1)\partial_x + mx^{m-1} y\partial_y + mlx^{m-1}.
\]
In fact, we have
\[
\begin{align*}
\text{sp}_K(f, P) &= l\partial_x f - yP = Q, \\
\text{sp}_K(f, Q) &= lx^n\partial_x f - y^jQ = -mx^{n-1}y\partial_y f - yP + x^nQ, \\
\text{sp}_K(P, Q) &= x^nP - y^{j-1}Q = -mx^{n-1}\partial_y f - P.
\end{align*}
\]
Since \(\text{in}_K(f) = y^l\), \(\text{in}_K(P) = y^{l-1}\xi\), \(\text{in}_K(Q) = x^m\xi\), we have for \(k \geq l+m\),
\[
\dim_K F_k(D_X)/(\text{Ann}_{D_X}[f^{-1}] \cap F_k(D_X)) = \sharp\{(x^i y^j \xi^\mu \eta^\nu \mid i + j + \mu + \nu \leq k \} \setminus \{(y^l, y^{l-1}\xi, x^m\xi)\} + \sharp\{(x^i y^j \xi^\mu \eta^\nu \mid i + j + \mu + \nu \leq k, i \leq l-1\} + \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} \binom{2+k-i}{2} \binom{2+k-i-j-1}{2} = \frac{l + m(l-1)}{2} k^2 + \cdots .
\]
Hence the multiplicity of \(M\) is \(l + m(l-1) = ml + l - m\). Q.E.D.

Now let us resume the study on \(M(u, f, s)\) for a \(D_X\)-module \(M = D_X u\) and a polynomial \(f\). As in the preceding section, set \(Y = X \times K\) and \(Z = \{(x, t) \in Y \mid t = f(x)\}\).

**Lemma 5.6.** Let \(M = D_X u\) be a left \(D_X\)-module generated by \(u\). For any \(\lambda \in K\), the endomorphism of \(M(u, f, s)\) defined by \(s - \lambda\) is injective. Hence the sequence
\[
0 \longrightarrow M(u, f, s) \xrightarrow{s-\lambda} M(u, f, s) \longrightarrow M(u, f, \lambda) \longrightarrow 0
\]
of left \(D_X\)-modules is exact.

**Proof.** We may assume that \(M\) is \(f\)-saturated as was seen in the previous section. The homomorphism \(\psi : M \otimes_{K[x]} B_{Z[Y]} \rightarrow M \otimes_{K[x]} \mathcal{L}_f\) is injective by Lemma 4.4.

Hence we have only to show that \(s - \lambda = -\partial_t t - \lambda\) is an injective endomorphism of \(M \otimes_{K[x]} B_{Z[Y]}\). Let
\[
v = \sum_{j=0}^k v_j \otimes \delta^{(j)}(t - f)
\]
be an arbitrary element of $M \otimes_{K[x]} B_{Z[Y]}$ with $k \in \mathbb{N}$ and $v_j \in M$. Then one has

$$(s - \lambda)v = -\sum_{j=0}^{k} v_j \otimes (t \partial_t + \lambda + 1) \delta^{(j)}(t - f)$$

$$= -\sum_{j=0}^{k} v_j \otimes (f \delta^{(j+1)}(t - f) + (\lambda - j) \delta^{(j)}(t - f))$$

$$= -\lambda v_0 \otimes \delta(t - f) - \sum_{j=1}^{k} (fv_{j-1} + (\lambda - j)v_j) \otimes \delta^{(j)}(t - f)$$

$$= f v_k \otimes \delta^{(k+1)}(t - f).$$

Thus $(s - \lambda)v = 0$ is equivalent to

$$\lambda v_0 = f v_k = fv_{j-1} + (\lambda - j)v_j = 0 \quad (1 \leq j \leq k),$$

which implies $v_k = v_{k-1} = \cdots = v_0 = 0$ since $M$ is $f$-saturated. Q.E.D.

**Theorem 5.7.** Let $f \in K[x]$ be a non-constant polynomial with $K$ being algebraically closed. Let $M = D_X u$ be a left $D_X$-module generated by $u$ which is holonomic on $X_f := \{x \in X \mid f(x) \neq 0\}$. Then $M(u, f, \lambda)$ and $M(u, f, s)/tM(u, f, s)$ are holonomic $D_X$-modules for any $\lambda \in K$.

**Proof.** Since $M(u, f, s) = \iota(M)(\iota(u), f, s)$, we may assume $M$ to be a nonzero holonomic $D_X$-module and $f$-saturated replacing $M$ by $\iota(M)$. Set

$$N := M \otimes_{K[x]} B_{Z[Y]}, \quad F_k(N) := F_k(D_Y)\delta(t - f(x)) \quad (k \in \mathbb{Z}).$$

Since $N$ is holonomic, there exists a polynomial $p(k)$ of degree $n + 1$ such that $p(k) = \dim_K F_k(N)$ for any sufficiently large $k$. Let us define a filtration $F_k(D_X[s])$ on the ring $D_X[s]$ by

$$F_k(D_X[s]) = \left\{ \sum_{\alpha, \beta, j} a_{\alpha, \beta, j} x^\alpha \partial_x^\beta s^j \mid |\alpha| + |\beta| + 2j \leq k \right\}.$$

Set

$$F_k(M(u, f, s)) := F_k(D_X[s])(u \otimes f^s) \quad (k \in \mathbb{Z}).$$

Applying a well-known fact in commutative algebra (see e.g., Theorem 4.4.3 in [7]) to the graded module $gr(M(u, f, s))$ over the graded ring $gr(D_X[s])$, we can show that there exist two polynomials $q_1(k)$ and $q_2(k)$ of the same degree $d$ such that

$$\dim_K F_{2k}(M(u, f, s)) = q_1(2k), \quad \dim_K F_{2k+1}(M(u, f, s)) = q_2(2k + 1)$$
for any sufficiently large $k$. We have $d \leq n + 1$ since $F_k(M(u, f, s))$ is contained in $\psi(F_k(N))$ and $\psi$ is an injective homomorphism from $M \otimes_{K[x]} B_{Z[F]}$ to $M \otimes_{K[x]} L_f$.

Set $\mathcal{M} = M(u, f, s)/tM(u, f, s)$ and

$F_k(\mathcal{M}) = F_k(M(u, f, s))/(tM(u, f, s) \cap F_k(M(u, f, s)).$

Then $\{F_k(\mathcal{M})\}$ is a Bernstein filtration on the left $D_X$-module $\mathcal{M}$ (i.e., the action of $s$ being ignored) although we do not know at this stage whether $\mathcal{M}$ is finitely generated over $D_X$ or not.

Since $t : M(u, f, s) \to M(u, f, s)$ is injective, we have

$$\dim_K F_k(\mathcal{M}) = \dim_K F_k(M(u, f, s)) - \dim_K (tM(u, f, s) \cap F_k(M(u, f, s))) \leq \dim_K F_k(M(u, f, s)) - \dim_K t^2 F_{k-2}(M(u, f, s)) = \dim_K F_k(M(u, f, s)) - \dim_K F_{k-2}(M(u, f, s))$$

$$= \begin{cases} q_1(k) - q_1(k - 2) & \text{if } k \gg 0 \text{ is even}, \\ q_2(k) - q_2(k - 2) & \text{if } k \gg 0 \text{ is odd}. \end{cases}$$

Since the degree of $q_i(k) - q_i(k - 2)$ ($i = 1, 2$) is $d - 1 \leq n$, this inequality implies that an arbitrary finitely generated $D_X$-submodule of $\mathcal{M}$ is holonomic and its multiplicity is bounded in terms of the leading coefficients of $q_1(k)$ and $q_2(k)$. Hence we conclude that $\mathcal{M}$ itself is holonomic.

We can prove the holonomicity of $M(u, f, \lambda)$ in the same way replacing $t^2$ by $s - \lambda$. This fact is a special case of Theorem 6.10 in [18].

Q.E.D.

The first statement of the following theorem is given in 6.5 of [14] for the case $\mathcal{M} = K[x]$ and $u = 1$.

**Theorem 5.8.** Let $M = D_X u$ be a $D_X$-module generated by $u$ and $f \in K[x]$ be a non-constant polynomial. Assume that the $b$-function $b_{u, f}(s)$ exists. For $\lambda \in K$ let $\varphi_\lambda : M(u, f, \lambda + 1) \to M(u, f, \lambda)$ be the $D_X$-homomorphism induced by $t$, which sends $(u \otimes f^s)|_{s = \lambda + 1}$ to $(fu \otimes f^s)|_{s = \lambda}$.

1. $\varphi_\lambda$ is an isomorphism if and only if $b_{u,f}(\lambda) \neq 0$.

2. Assume that $M$ is holonomic on $X_f$ with $K$ being algebraically closed. Then one has

$$\text{mult} M(u, f, \lambda + k) = \text{mult} M(u, f, \lambda),$$

$$\text{length} M(u, f, \lambda + k) = \text{length} M(u, f, \lambda)$$

for any $\lambda \in K$ and any integer $k$. In particular, one has

$$\text{mult} M[f^{-1}] = \text{mult} M(u, f, k), \quad \text{length} M[f^{-1}] = \text{length} M(u, f, k)$$
for any integer $k$.

Proof. There exists a commutative diagram

```
0 → M(u, f, s) ↑t ↓s−λ−1 M(u, f, s) → M → 0
   ↓                ↓                ↓                ↓
0 → M(u, f, s) ↑t ↓s−λ M(u, f, s) → M → 0
   ↓                ↓                ↓                ↓
M(u, f, λ + 1) ↑ϕλ ↓M(u, f, λ) → K1(λ) → 0
```

with $M = M(u, f, s)/tM(u, f, s)$ and some left $D_X$-modules $K_0(\lambda)$, $K_1(\lambda)$, where the three vertical sequences and the upper two horizontal sequences are exact in view of Lemma 5.6. Hence by the snake lemma we obtain an exact sequence

(3) $0 \rightarrow K_0(\lambda) \rightarrow M(u, f, \lambda + 1) \rightarrow M(u, f, \lambda) \rightarrow K_1(\lambda) \rightarrow 0$

of left $D_X$-modules.

(1) Assume $b_{u, f}(\lambda) \neq 0$. Then there exist $a(s), c(s) \in K[s]$ such that $a(s)(s - \lambda) + c(s)b_{u, f}(s) = 1$. Hence for any $Q(s) \in D_X[s]$, $Q(s)(u \otimes f^s) = Q(s)c(s)b_{u, f}(s)(u \otimes f^s) + (s - \lambda)Q(s)a(s)(u \otimes f^s)$ belongs to $tM(u, f, s) + (s - \lambda)M(u, f, s)$. If $(s - \lambda)Q(s)(u \otimes f^s)$ belongs to $tM(u, f, s)$, then $Q(s)(u \otimes f^s)$ also belongs to $tM(u, f, s)$. Hence $s - \lambda$ is an automorphism of $M$.

Conversely, assume that $s - \lambda$ is an automorphism of $M$. Then the minimal polynomial $b_{u, f}(s)$ of $s$ on this module cannot be a multiple of $s - \lambda$. Summing up we have shown that $b_{u, f}(\lambda) \neq 0$ if and only if $K_0(\lambda) = K_1(\lambda) = 0$. In view of the exact sequence (3), this is also equivalent to $\varphi_\lambda$ being an isomorphism.
(2) We may assume that $M$ is a holonomic $D_X$-module and that $M$ is $f$-saturated replacing $M$ by $\iota(M)$. Since $M$ is holonomic by Theorem 5.7, the length (and the multiplicity) of $K_0(\lambda)$ and the length (and the multiplicity respectively) of $K_1(\lambda)$ are the same in view of the rightmost vertical exact sequence. Combined with this fact the exact sequence (3) proves the statement (2).

This theorem provides us with an algorithm to compute the multiplicity of $M[f^{-1}]$ without any information on $b_{u,f}(s)$; we have only to compute a Gröbner basis, e.g., of $M(u, f, 0)$ with respect to a term order compatible with the Bernstein filtration.

**Theorem 5.9.** Let $M = D_X u$ be a $D_X$-module generated by $u$ and $f \in K[x]$ be a non-constant polynomial with $K$ being algebraically closed. Assume that $M$ is holonomic on $X_f$. Then the homomorphism $\tilde{\rho}_\lambda : M(u, f, \lambda) \to D_X (u \otimes f^\lambda)$ is an isomorphism if and only if $b_{u,f}(\lambda - k) \neq 0$ for any positive integer $k$.

**Proof.** If $b_{u,f}(\lambda - k) \neq 0$ for any positive integer $k$, then $\tilde{\rho}_\lambda$ is an isomorphism by virtue of Proposition 2.6. Now suppose $b_{u,f}(\lambda - k) = 0$ holds for some positive integer $k$ and let $k_0$ be the maximum among such $k$. Then Proposition 2.6 and Lemma 2.7 imply that $\tilde{\rho}_{\lambda-k_0}$ is an isomorphism and that $D_X (u \otimes f^{\lambda-k_0+1}) \subseteq D_X (u \otimes f^{\lambda-k_0})$. Hence by (2) of Theorem 5.8 we have

$$\text{length } M(u, f, \lambda) = \text{length } M(u, f, \lambda - k_0) = \text{length } D_X (u \otimes f^{\lambda-k_0}) > \text{length } D_X (u \otimes f^{\lambda-k_0+1}) \geq \text{length } D_X (u \otimes f^{\lambda}).$$

Thus $\tilde{\rho}_\lambda$ is not an isomorphism. Q.E.D.

**Corollary 5.10.** Under the same assumptions as in Theorem 5.9, $M(u, f, \lambda)$ is $f$-saturated if and only if $b_{u,f}(\lambda - k) \neq 0$ for any positive integer $k$. In general, $\iota(M(u, f, \lambda))$ is isomorphic to $D_X (u \otimes f^\lambda)$.

**Proof.** We may assume $M$ to be $f$-saturated. First note that $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ is $f$-saturated for any $\lambda \in K$ since it is isomorphic to $M[f^{-1}]$ as $K[x]$-module. Hence $M(u, f, \lambda) \simeq D_X (u \otimes f^\lambda)$ is also $f$-saturated under the assumption on $b_{u,f}(s)$.

Now assume $b_{u,f}(\lambda - k) = 0$ for some positive integer $k$. Then $\tilde{\rho}_\lambda$ is not injective. Thus there exists $P \in D_X$ such that $P(u \otimes f^s)|_{s=\lambda} \neq 0$ but $P(u \otimes f^\lambda) = 0$. There exist $P_j \in D_X$ and $k, l \in \mathbb{N}$ such that

$$P(u \otimes f^s) = \sum_{j=0}^{k} (P_j u) \otimes (s-\lambda)^j f^{s-l}.$$
Then the equality $P(u \otimes u^\lambda) = 0$ means $P_0 u = 0$. Hence by Lemma 2.5, there exist $Q(s) \in D_X[s]$ and $m \in \mathbb{N}$ such that

$$P(u \otimes f^s) = (s - \lambda)Q(s)(u \otimes f^{s-m}).$$

Take a sufficiently large $l \in \mathbb{N}$ so that $f^lQ(s)f^{-m}$ belongs to $D_X[s]$. Then we have

$$f^lP(u \otimes f^s) = (s - \lambda)(f^lQ(s)f^{-m})(u \otimes f^s),$$

and consequently $f^lP((u \otimes f^s)|_{s=\lambda}) = 0$. Hence $M(u, f, \lambda)$ is not $f$-saturated. The last statement also follows from this argument. Q.E.D.

**Example 5.11.** Set $n = 2$ and write $x_1 = x$, $x_2 = y$. Let $u$ be the residue class of 1 in $M = D_X/I$ with $I$ being the left ideal of $D_X$ generated by two operators

$$P_1 = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y, \quad P_2 = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y.$$ 

This is Appell’s hypergeometric system $F_1$ with all parameters equal to zero. The singular locus of $M$ is a line arrangement defined by

$$f := x(x-1)y(y-1)(x-y) = 0.$$ 

Let $\iota : M \to M[f^{-1}]$ be the canonical homomorphism. Then $M[f^{-1}]$ is generated by $f^{-2}\iota(u)$ and $\iota(M)$ is given by

$$\iota(M) = D_X\iota(u)$$

$$= D_X/(D_X\partial_x\partial_y + D_X((1-x)\partial_x^2 - \partial_x) + D_X((1-y)\partial_y^2 - \partial_y)).$$

The $b$-function with respect to $u$ and $f$ is

$$b_{u,f}(s) = (s+1)^3(s+2)^2(s+\frac{2}{3})^2(s+\frac{4}{3})^2(s+\frac{5}{3}).$$

As to multiplicities we have

$$\text{mult } M = 10, \quad \text{mult } \iota(M) = 5, \quad \text{mult } M[f^{-1}] = 36,$$

$$\text{mult } H^1_{(f)}(M) = 31, \quad \text{mult } H^0_{(f)}(M) = 5.$$
The characteristic varieties are
\[
\text{Char}(M) = \{ (x, y; \xi, \eta) \mid x = y = 0 \} \cup \{ x = y = 1 \} \cup \{ x = \eta = 0 \}
\]
\[
\cup \{ x - 1 = \eta = 0 \} \cup \{ y = \xi = 0 \} \cup \{ y - 1 = \xi = 0 \}
\]
\[
\cup \{ x = y = x = 0 \} \cup \{ \xi = \eta = 0 \},
\]
\[
\text{Char}(\iota(M)) = \{ x - 1 = \eta = 0 \} \cup \{ y - 1 = \xi = 0 \} \cup \{ \xi = \eta = 0 \},
\]
\[
\text{Char}(M[f^{-1}]) = \{ x = y = 0 \} \cup \{ x - 1 = y = 0 \} \cup \{ x = y - 1 = 0 \}
\]
\[
\cup \{ x = y = 1 \} \cup \{ x = \eta = \xi = \eta = 0 \}
\]
\[
\cup \{ y = \xi = 0 \} \cup \{ y - 1 = \xi = 0 \} \cup \{ x - y = \xi + \eta = 0 \}
\]
\[
\cup \{ \xi = \eta = 0 \},
\]
\[
\text{Char}(H^1_{(f)}(M)) = \{ x = y = 0 \} \cup \{ x - 1 = y = 0 \} \cup \{ x = y - 1 = 0 \}
\]
\[
\cup \{ x = y = 1 \} \cup \{ x = \eta = \xi = y = 0 \}
\]
\[
\cup \{ y = \xi = 0 \} \cup \{ y - 1 = \xi = 0 \} \cup \{ x - y = \xi + \eta = 0 \},
\]
\[
\text{Char}(H^0_{(f)}(M)) = \{ x = y = 0 \} \cup \{ x = y = 1 \} \cup \{ x = \eta = 0 \}
\]
\[
\cup \{ y = \xi = 0 \} \cup \{ x - y = \xi + \eta = 0 \}. 
\]

**Example 5.12.** Set \( X = k^4 \) and consider the \( A \)-hypergeometric system associated with the matrix \( A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \), which is taken from Example 4.3.9 of [23]. It is the left \( D_X \)-module \( M_A(b_1, b_2) = D_X/H_A(b_1, b_2) \) with the left ideal \( H_A(b_1, b_2) \) of \( D_X \) generated by operators
\[
x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - b_1, \quad x_2 \partial_2 + 3x_3 \partial_3 + 4x_4 \partial_4 - b_2,
\]
\[
x_1 \partial_1 \partial_2^2 - \partial_3^3, \quad \partial_1 \partial_2 - \partial_2 \partial_3, \quad \partial_2 \partial_3^2 - \partial_1 \partial_3^2, \quad \partial_1 \partial_3^3 - \partial_2^3
\]
with parameters \( b_1, b_2 \in K \). Computing a Gröbner basis of the left ideal of \( D_X[b_1, b_2] \) (i.e., regarding \( b_1 \) and \( b_2 \) as indeterminates) generated by these operators with respect to a term order \( < \) such that
\[
|\alpha| + |\beta| < |\alpha'| + |\beta'| \Rightarrow \delta_1^i b_1^j x^{\alpha} \partial_x^{\beta} < \delta_1^i b_1^j x^{\alpha'} \partial_x^{\beta'}
\]
and that \( \delta_1^i b_1^j x^{\alpha} \partial_x^{\beta} \) for any \( i, j, \alpha, \beta \), we can verify that the multiplicity of \( M_A(b_1, b_2) \) is 16 unless \( b_1 = 1 \) and \( b_2 = 2 \), while the multiplicity of \( M_A(1, 2) \) is 17. A similar phenomenon with respect to the holonomic rank was shown in [23].
On the other hand, the characteristic variety of \( M_A(b_1, b_2) \) does not depend on \( b_1, b_2 \) and its singular locus is the zero set of
\[
g(x) = x_1x_4(256x_1^3x_3^3 + (-192x_1^2x_2x_3 - 27x_2^3)x_4^2
- 6x_1x_2^3x_4 - 27x_1^2x_3^4 - 4x_2^2x_4^3).
\]
For example, the \( b \)-functions of \( u := T \in M_A(1, 2) \) and \( x_1 \) is \( b_{u,x_1}(s) = s(s + 1)(s + 2) \), while the \( b \)-functions of \( v := T \in M_A(0, 0) \) and \( x_1 \) is \( b_{v,x_1}(s) = (s + 1)^2 \). Algorithm 3.16 ensures that \( M_A(0, 0) \) and \( M_A(1, 2) \) are \( x_1 \)- and \( x_4 \)-saturated. The computation of the localization with respect to \( g \) is intractable.

§6. Hyperplane arrangements

Let us prove a formula on the multiplicity and the length of the local cohomology \( H^i_f(K[x]) \) or the localization \( K[x, f^{-1}] \) of the polynomial ring when \( f \) defines a hyperplane arrangement in the affine space \( X = K^n \). We set \( R = K[x] \) in what follows.

The length of such modules was studied e.g., in [1], [27]. By the characteristic cycle of the local cohomology with respect to an arrangement of linear subvarieties was studied in [2]. Although not explicitly stated, Corollary 1.3 of [2] should yield main results of this section. Nero Budur informed the author that Theorem 6.4 below follows from results in Section 1.7 of [8]. We give an elementary direct proof with a hope to make both the statement and the proof more accessible.

**Lemma 6.1.** Let \( h_0 = h_0(x) \in K[x] \) be a linear polynomial and \( I \) be an ideal of \( R = K[x] \). Let \( R' := R/Rh_0 \) be the affine ring associated with the hyperplane \( h_0(x) = 0 \) and set \( I' = (I + Rh_0)/Rh_0 \). Then we have
\[
\begin{align*}
\text{length } H^i_{I' + Rh_0}(R) &= \text{length } H^{i-1}_{I'}(R'), \\
\text{mult } H^i_{I' + Rh_0}(R) &= \text{mult } H^{i-1}_{I'}(R')
\end{align*}
\]
for any integer \( i \).

**Proof.** Since \( H^i_{Rh_0}(R) = 0 \) for \( i \neq 1 \), there is an isomorphism
\[
H^i_{I' + Rh_0}(R) \cong H^{i-1}_{I'}(Rh_0)(H^1_{Rh_0}(R)).
\]
We may assume by an affine coordinate transformation, which preserves the Bernstein filtration, that \( h_0(x) = x_n \). Then we may regard \( R' = K[x_1, \ldots, x_{n-1}] \) and have an isomorphism
\[
H^1_{Rh_0}(R) \cong R' \otimes_K H^1_{(x_n)}(K[x_n]),
\]
where the tensor product on the right-hand side is a left module over $D_n = D_{n-1} \otimes K D_1$ with $D_1$ being the ring of differential operators in the variable $x_n$.

Let $\{f_1, \ldots, f_r\}$ be a set of generators of $I$. We may assume that $f_1, \ldots, f_r$ belong to $R'$. (We replace $f_i$ with its homomorphic image in $R'$.) Then for $0 \leq i_1 < \cdots < i_k \leq r$ with $k \in \mathbb{N}$, the localization by $f_{i_1} \cdots f_{i_k}$ yields

$$(H^1_{R_{n_0}}(R))_{f_{i_1} \cdots f_{i_k}} := H^1_{R_{n_0}}(R)[(f_{i_1} \cdots f_{i_k})^{-1}] = R'_{f_{i_1} \cdots f_{i_k}} \otimes_K H^1_{(x_n)}(K[x_n]).$$

On the other hand, we have

$$(H^1_{R_{n_0}}(R))_{x_n} := H^1_{R_{n_0}}(R)[x_n^{-1}] = R' \otimes_K (H^1_{(x_n)}(K[x_n]))_{x_n} = 0.$$

Hence $H^{i-1}_{1+n_{0}}(H^1_{R_{n_0}}(R))$ is the $(i-1)$-th cohomology group of the Čech complex

$$0 \longrightarrow R' \otimes_K H^1_{(x_n)}(K[x_n]) \longrightarrow \bigoplus_{1 \leq i \leq r} R'_{f_i} \otimes_K H^1_{(x_n)}(K[x_n])$$

$$\longrightarrow \bigoplus_{1 \leq i < j \leq r} R'_{f_i f_j} \otimes_K H^1_{(x_n)}(K[x_n]) \longrightarrow$$

$$\cdots \longrightarrow R'_{f_1 \cdots f_r} \otimes_K H^1_{(x_n)}(K[x_n]) \longrightarrow 0,$$

which is isomorphic to $H^{i-1}_{1'}(R') \otimes_K H^1_{(x_n)}(K[x_n])$ (see e.g., Theorem 7.13 in [11]). This implies

$$H^1_{1+n_{0}}(R) \cong H^{i-1}_{1'}(R') \otimes_K H^1_{(x_n)}(K[x_n])$$

$$\cong H^{i-1}_{1'}(R') \otimes_K (D_1/D_n x_n)$$

$$\cong (D_n/D_n x_n) \otimes_{D_{n-1}} H^{i-1}_{1'}(R'),$$

where $D_n/D_n x_n$ is regarded as a $(D_n, D_{n-1})$-bimodule. The rightmost term is the $D$-module theoretic direct image of $H^{i-1}_{1'}(R')$ with respect to the inclusion $H_0 := \{ x \in X \mid x_n = 0 \} \to X$. In view of Kashiwara’s equivalence in the category of algebraic $D$-modules (see e.g., Theorem 7.11 of [6] or Theorem 1.6.1 of [10]), there is a one-to-one correspondence between the $D_{n-1}$-submodules $M$ of $H^{i-1}_{1'}(R')$ and the $D_n$-submodules $M \otimes_K H^1_{(x_n)}(K[x_n])$ of $H^1_{1+n_{0}}(R)$. This implies

$$\text{length } H^1_{1+n_{0}}(R) = \text{length } H^{i-1}_{1'}(R').$$
Next, let us show
\[ \text{mult } H^I_{i+\delta b}(R) = \text{mult } H^I_{i-1}(R') \] Let \( \{F_k\} \) be a good Bernstein filtration on \( H^I_{i-1}(R') \) and set \( m = \text{mult } H^I_{i-1}(R') \). Then there exists a polynomial \( p(k) \) and \( k_1 \in \mathbb{Z} \) such that
\[ \dim_K F_k = p(k) = \frac{m}{(n-1)!} k^{n-1} + \text{(terms with degree } < n - 1) \]
holds for \( k \geq k_1 \). Define a filtration \( \{G_k\} \) on \( H^I_{i-1}(R') \otimes_K H^1_{(x_n)}(K[x_n]) \) by
\[ G_k := \sum_{j=0}^{k} F_j \otimes_K (K[x_n^{-1}] + \cdots + K[x_n^{-(k-j)-1}]) = \bigoplus_{j=0}^{k} F_j \otimes_K K[x_n^{-(k-j)-1}] \]
where \( K[x_n^{-j}] \) denotes the \( K \)-space spanned by the residue class of \( x_n^{-j} \).
It is easy to see that \( \{G_k\} \) is a good Bernstein filtration. Hence we have
\[ \dim_K G_k = \sum_{j=0}^{k} \dim_K F_j = \sum_{j=0}^{k} \dim_K F_j + k \sum_{j=k_1}^{k} p(j) \]
By the assumption, there exists a polynomial \( q(k) \) of degree \( \leq n - 2 \) such that
\[ p(j) = \frac{m}{(n-1)!} j(j+1) \cdots (j+n-2) + q(j) \]
Since
\[ \sum_{j=k_1}^{k} j(j+1) \cdots (j+n-2) \]
\[ = \frac{1}{n} \{ k(k+1) \cdots (k+n-1) - (k_1 - 1)k_1 \cdots (k_1 + n - 2) \}, \]
we have
\[ \dim_K G_k = \frac{m}{n!} k^n + \text{(terms with degree } < n) \quad (\forall k \geq k_1). \]
Thus we also have \( \text{mult } H^I_{i+\delta x_n}(R) = m \). This completes the proof.
Q.E.D.
Theorem 6.2. Let \( f \in K[x] \) be a multiple of essentially distinct linear polynomials and \( A \) be the hyperplane arrangement in \( X = K^n \) defined by \( f \). Let \( H_0 \) be an element of \( A \). Set \( A' := A \setminus \{H_0\} \) and let \( f' \) be the product of the defining polynomials of hyperplanes belonging to \( A' \). Let us regard
\[
A'' := \{ H \cap H_0 \mid H \in A', H \cap H_0 \neq \emptyset \}
\]
as a hyperplane arrangement in the affine space \( H_0 \). Let \( R' = R/Rh_0 \) be the affine ring of \( H_0 \), where \( h_0 \) is a polynomial of first degree defining \( H_0 \). Let \( f'' \in R' \) be the product of the defining polynomials of the elements of \( A'' \). (Set \( f'' = 1 \) if \( A'' = \emptyset \).) Then one has
\[
\begin{align*}
\text{length} H^1_{(f')(R)} &= \text{length} H^1_{(f')(R)} + \text{length} H^1_{(f')(R')} + 1, \\
\text{mult} H^1_{(f')(R)} &= \text{mult} H^1_{(f')(R)} + \text{mult} H^1_{(f')(R')} + 1.
\end{align*}
\]

**Proof.** By the Mayer-Vietoris exact sequence (see e.g., Theorem 15.1 in [11]), we get an exact sequence
\[
0 \longrightarrow H^1_{(f')(R)} \oplus H^1_{(h_0)}(R) \longrightarrow H^1_{(f')(R)} \longrightarrow H^2_{(f')(R) + (h_0)}(R) \longrightarrow 0
\]
of holonomic left \( D_n \)-modules because \( H^1_{(f')(R) + (h_0)}(R) = 0 \). Since the length and the multiplicity of \( H^1_{(h_0)}(R) \) are both one, it follows that
\[
\begin{align*}
\text{length} H^1_{(f')(R)} &= \text{length} H^1_{(f')(R)} + \text{length} H^2_{(f')(R) + (h_0)}(R) + 1, \\
\text{mult} H^1_{(f')(R)} &= \text{mult} H^1_{(f')(R)} + \text{mult} H^2_{(f')(R) + (h_0)}(R) + 1.
\end{align*}
\]
(4)
Since \( f'' = R' f'' \equiv (R f' + Rh_0)/Rh_0 \), Lemma 6.1 implies
\[
\begin{align*}
\text{mult} H^2_{(f')(R) + (h_0)}(R) &= \text{mult} H^1_{(f')(R')}, \\
\text{length} H^2_{(f')(R) + (h_0)}(R) &= \text{length} H^1_{(f')(R')}.
\end{align*}
\]
This completes the proof in view of (4). Q.E.D.

**Corollary 6.3.** \( \text{length} H^1_{(f')(R)} = \text{mult} H^1_{(f')(R)}. \)

**Proof.** This can be easily proved by induction on \( xA \) by using Theorem 6.2. Q.E.D.

The intersection poset \( L(A) \) is the set of the non-empty intersections of elements of \( A \) including \( X \). For \( Y, Z \in L(A) \), the Möbius function \( \mu(Y,Z) \) is defined recursively by
\[
\mu(Y,Z) = \begin{cases} 
- \sum_{Z \subseteq W \subseteq Y} \mu(Y,W) & \text{if } Z \subsetneq Y \\
1 & \text{if } Z = Y \\
0 & \text{otherwise.}
\end{cases}
\]
Set $\mu(Y) = \mu(X,Y)$. Then $(-1)^{\text{codim} X} \mu(X)$ is positive (see e.g. Theorem 2.47 of [22]). The Poincaré polynomial of the arrangement $\mathcal{A}$ is defined by

$$\pi(\mathcal{A}, t) = \sum_{Y \in L(\mathcal{A})} \mu(Y)(-t)^{\text{codim} Y}.$$ 

**Theorem 6.4.** Let $\mathcal{A}$ be a hyperplane arrangement in $X = K^n$ defined by a polynomial $f \in R = K[x]$. Then the length of $H^1_{(f)}(R)$ is $\pi(\mathcal{A}, 1) - 1$.

**Proof.** Let $H_0$ be an element of $\mathcal{A}$ defined by a first degree polynomial $h_0$. Let us prove the equality by induction on $\# \mathcal{A}$. Since $H^1(h_0)(R)$ is a simple left $D_n$-module and $\pi(\{H_0\}, t) = t + 1$, the equality holds if $\mathcal{A} = \{H_0\}$. Let $\mathcal{A}', \mathcal{A}''$ be as in the proof of Theorem 6.2.

By the induction hypothesis, we have

$$\text{length } H^1_{(f')}(R) = \pi(\mathcal{A}', 1) - 1, \quad \text{length } H^1_{(f'')}(R') = \pi(\mathcal{A}'', 1) - 1.$$ 

Hence by Theorem 6.2 we get

$$\text{length } H^1_{(f)}(R) = \text{length } H^1_{(f')}(R) + \text{length } H^1_{(f'')}(R') + 1 = \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1.$$

On the other hand, $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + \pi(\mathcal{A}'', t)$ holds (see e.g., Theorem 2.56 of [22]). Thus we get

$$\text{length } H^1_{(f)}(R) = \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1 = \pi(\mathcal{A}, 1) - 1.$$

This completes the proof. Q.E.D.

**Corollary 6.5.** Let $\mathcal{A}$ be a hyperplane arrangement in $X = K^n$ defined by a polynomial $f \in R = K[x]$. Then the length of $R[f^{-1}]$ is $\pi(\mathcal{A}, 1)$.

Actual computation can be done effectively by using the recursive formula of Theorem 6.2. For example, we have $\pi(\mathcal{A}, t) = (2t + 1)(3t + 1)$ and hence $\pi(\mathcal{A}, 1) = 12$ if $f = xy(x - 1)(y - 1)(x - y)$ with $X = K^2$. If

$$f = xyz(x + y)(x - y)(x + z)(x - z)(y + z)(y - z)$$

with $X = K^3$, then we have $\pi(\mathcal{A}, t) = (t + 1)(3t + 1)(5t + 1)$ and hence $\pi(\mathcal{A}, 1) = 48$. For these relatively small examples, direct computation of the local cohomology group is also possible.
Localization of D-modules

References


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