ACHIRALITY OF SPATIAL GRAPHS AND THE SIMON INVARIANT

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In this short article we report that for any odd integer m there exist an achiral spatial complete graph on 5 vertices and an achiral spatial complete bipartite graph on $3+3$ vertices whose Simon invariants are equal to m.

Keywords: Spatial graph; Achirality; Simon invariant.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $\mathbb{S}^3$ be the unit 3-sphere in $\mathbb{R}^4$ centered at the origin. An embedding f of a finite graph G into $\mathbb{S}^3$ is called a *spatial embedding of* G or simply a *spatial graph*. If G is homeomorphic to a disjoint union of n cycles then spatial embeddings of G are none other than n-component links. Two spatial embeddings f and g of G are said to be *ambient isotopic* if there exists an orientation-preserving homeomorphism $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\Phi \circ f = g$.

Let us consider an orientation-reversing homeomorphism $\varphi : \mathbb{S}^3 \to \mathbb{S}^3$ defined by $\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$. For a spatial embedding f of G, we call $f! = \varphi \circ f$ the *mirror image embedding* of f. In this article we say that a spatial embedding f of G is *achiral* if there exists an orientation-preserving homeomorphism $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\Phi(f(G)) = f!(G)$. Note that there is no necessity for f and $f!$ to be ambient isotopic, namely we may forget the labels of vertices and edges of G. Achirality of spatial graphs is not only an interesting theme in geometric topology as a generalization of amphicheirality of knots and links but also an important research object from a standpoint of application to macromolecular chemistry, what is called *molecular topology*. We refer the reader to [5] for a pioneer work.

For the case of 2-component links, it is known that achirality of a 2-

component oriented link depends on its homological information. Kirk and Livingston showed that a 2-component oriented link is achiral only if its *linking number* is not congruent to 2 modulo 4 [2, 6.1 COROLLARY] (an elementary proof was also given in [3], and also see Kidwell [1] for recent progress). On the other hand, let K_5 and $K_{3,3}$ be a *complete graph* on five vertices and a *complete bipartite graph* on $3+3$ vertices respectively that are known as two obstruction graphs for Kuratowski's planarity criterion. For spatial embeddings of K_5 and $K_{3,3}$, the *Simon invariant* is defined [6, §4], that is an odd integer valued homological invariant calculated from their regular diagrams, like the linking number. Moreover it is known that a non-trivial homological invariant for spatial embeddings of G exists if and only if G is non-planar or contains a pair of disjoint cycles [6, THEOREM C], and the Simon invariant is essentially unique as a homological invariant [7]. Therefore it is natural to ask whether achirality of a spatial embedding of K_5 or $K_{3,3}$ depends on its Simon invariant or not. In the conference in Hiroshima, the author claimed in his talk that a spatial embedding of K_5 or $K_{3,3}$ is achiral only if its Simon invariant is not congruent to -3 and 3 modulo 8. But he found a gap for his proof after the conference. What he claimed turned out to be wrong in the end, namely we have the following.

Theorem 1.1. *Let G be K_5 or $K_{3,3}$. For any odd integer m, there exists an achiral spatial embedding of G whose Simon invariant is equal to m.*

Actually Taniyama informed me that such a spatial embedding can be constructed. The author is grateful to him and sorry for the mistake. In the next section we recall the definition of the Simon invariant and demonstrate Taniyama's construction of an achiral spatial embedding of K_5 or $K_{3,3}$ which realizes an arbitrary value of the Simon invariant.

2. Achiral spatial embedding of K_5 and $K_{3,3}$ with any value of the Simon invariant

First we recall the definition of the Simon invariant. For K_5 and $K_{3,3}$, we give a label to each of the vertices and edges (note that the vertices of $K_{3,3}$ are divided into the black vertices $\{1,2,3\}$ and the white vertices $\{1,2,3\}$), and an orientation to each of the edges as illustrated in Fig. 1. For a pair of disjoint edges (x,y) of K_5, we define the sign $\varepsilon(x,y)=\varepsilon(y,x)$ by $\varepsilon(e_i,e_j)=1$, $\varepsilon(d_k,d_l)=-1$ and $\varepsilon(e_i,d_k)=-1$. For a pair of disjoint edges (x,y) of $K_{3,3}$, we also define the sign $\varepsilon(x,y)=\varepsilon(y,x)$ by $\varepsilon(c_i,c_j)=1$, $\varepsilon(b_k,b_l)=1$ and $\varepsilon(c_i,b_k)=1$ if c_i and b_k are parallel in Fig. 1 and -1 if c_i and b_k are not parallel in Fig. 1.

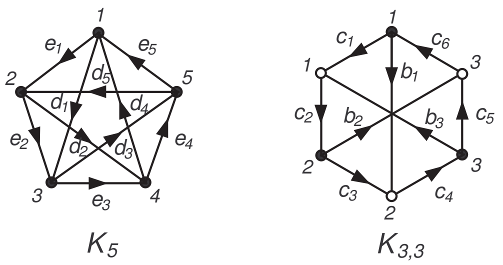

Fig. 1.

For a spatial embedding f of K_5 or $K_{3,3}$, we fix a regular diagram of f and denote the sum of the signs of the crossing points between $f(x)$ and $f(y)$ by $l(f(x), f(y))$, where (x, y) is a pair of disjoint edges. Then an integer

$$\mathcal{L}(f) = \sum_{x \cap y = \emptyset} \varepsilon(x, y) l(f(x), f(y))$$

is called the *Simon invariant* of f. Actually we can see that $\mathcal{L}(f)$ is an odd integer valued ambient isotopy invariant by observing the variation of $\mathcal{L}(f)$ by each of the (generelized) Reidemeister moves. We also have $\mathcal{L}(f!) = -\mathcal{L}(f)$ immediately by the definition. Hence any spatial embedding of K_5 or $K_{3,3}$ is not ambient isotopic to its mirror image embedding. But the embedding can be achiral, see Fig. 2.

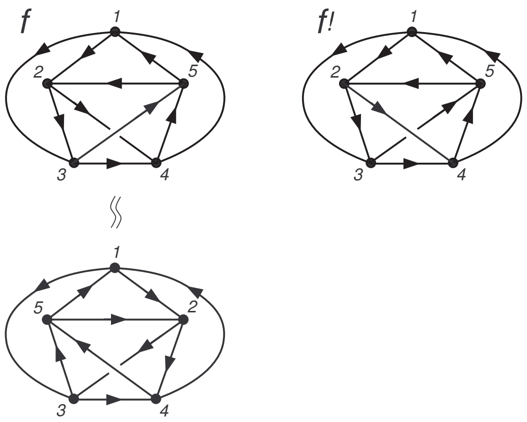

Fig. 2.

Now let us consider a spatial embedding $f_{m,q}$ ($m \in \mathbb{Z},\ q = \pm 1$) of K_5 or $K_{3,3}$ as illustrated in Fig. 3, where m denotes m full twists and q denotes a positive (resp. negative) crossing point if $q = 1$ (resp. -1) in Fig. 3. By applying a counter clockwise $\pi/4$-rotation of the diagram, it can be easily seen that $f_{m,q}$ is achiral. On the other hand, by a direct calculation we have that $\mathcal{L}(f_{m,q}) = 4m + q$. Thus Simon invariants of $f_{m,q}$ can cover all odd integers. This completes the proof of Theorem 1.1.

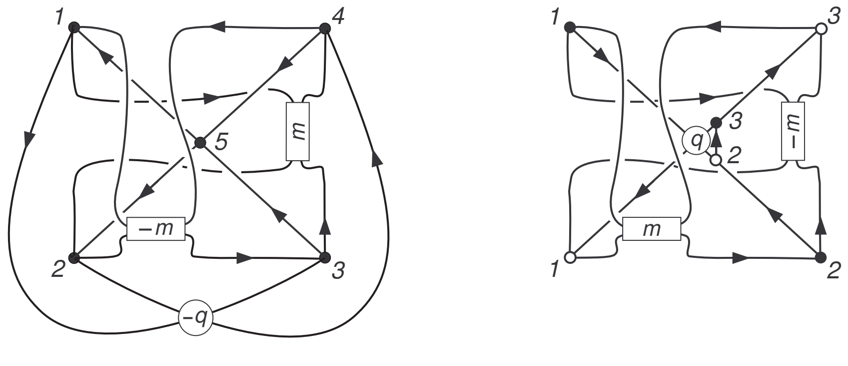

Fig. 3.

Let $\mathrm{Aut}(G)$ be the *automorphism group* of a graph G. Note that $\mathrm{Aut}(K_5)$ is isomorphic to $\mathfrak{S}_5$ and $\mathrm{Aut}(K_{3,3})$ is isomorphic to the wreath product $\mathfrak{S}_2[\mathfrak{S}_3]$, where $\mathfrak{S}_n$ denotes the *symmetric group* of degree n. Achirality of $f_{m,q}$ of K_5 as above means that there exists an orientation-reversing homeomorphism $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\Phi \circ f_{m,q} = f_{m,q} \circ (1\ 2\ 3\ 4)$, where $(1\ 2\ 3\ 4) \in \mathfrak{S}_5 \cong \mathrm{Aut}(K_5)$. In the same way, the achirality of $f_{m,q}$ of $K_{3,3}$ as above means that there exists an orientation-reversing homeomorphism $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\Phi \circ f_{m,q} = f_{m,q} \circ ((1\ 2); (1\ 3\ 2), (2\ 3))$, where $((1\ 2); (1\ 3\ 2), (2\ 3)) \in \mathfrak{S}_2[\mathfrak{S}_3] \cong \mathrm{Aut}(K_{3,3})$. Then it is natural to generalize our main theme as follows: For any $\sigma \in \mathrm{Aut}(K_5)$ (resp. $\mathrm{Aut}(K_{3,3})$) and any odd integer m, do there exist a spatial embedding f of K_5 (resp. $K_{3,3}$) and a homeomorphism $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\mathcal{L}(f) = m$ and $\Phi \circ f = f \circ \sigma$? This work is now in progress by the author and Taniyama [4], and the details will appear elsewhere.

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