An unknotting theorem for delta and sharp edge-homotopy

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Abstract

Two spatial embeddings of a graph are said to be delta (resp. sharp) edge-homotopic if they are transformed into each other by self delta (resp. sharp) moves and ambient isotopies. We show that any two spatial embeddings of a graph are delta (resp. sharp) edge-homotopic if and only if the graph does not contain a subgraph which is homeomorphic to the theta graph or the disjoint union of two 1-spheres, or equivalently G is homeomorphic to a bouquet.

Keywords: Spatial graph; Delta move; Sharp move; C_k-move
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1. Introduction

Let G be a finite graph which does not contain a free vertex. We consider G as a topological space in the usual way. An embedding $f : G \to \mathbb{S}^3$ is called a *spatial embedding of* G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding $G \to \mathbb{S}^2$. For a planar graph G, a spatial embedding of G is said to be *trivial* if it is ambient isotopic to an embedding $G \to \mathbb{S}^2 \subset \mathbb{S}^3$. We note that a trivial spatial embedding of a planar graph is unique up to ambient isotopy in $\mathbb{S}^3$ [6].

In the following we recall the three equivalence relations on spatial graphs generated by specific local moves as follows:

(1) A *crossing change* is a local move on a spatial graph as illustrated in Fig. 1.1. A crossing change is called a *self crossing change* if all two strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *edge-homotopic* [1] if they are transformed into each other by self crossing changes and ambient isotopies.

(2) A *delta move* [7], [11] is a local move on a spatial graph as illustrated in Fig. 1.2. A delta move is called a *self delta move* if all three strings in the move belong to the same spatial edge. Two spatial embeddings of a

[1] This equivalence relation was called simply a *homotopy* in [23].

graph are said to be *delta edge-homotopic* if they are transformed into each other by self delta moves and ambient isotopies.

(3) A *sharp move* [10] is a local move on a spatial *oriented* graph as illustrated in Fig. 1.3. A sharp move is called a *self sharp move* if all four strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *sharp edge-homotopic* if they are transformed into each other by self sharp moves and ambient isotopies. If we turn the orientations of all strings in a self sharp move the other way at once, then the concluded move is also a self sharp move. Therefore this equivalence relation does not depend on the edge orientations.

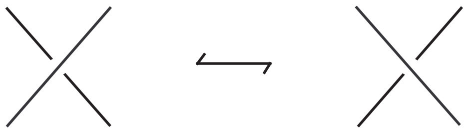


Fig. 1.1.

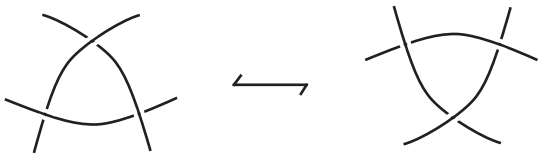


Fig. 1.2.

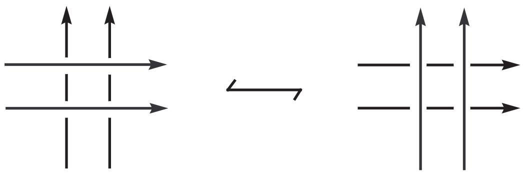


Fig. 1.3.

Edge-homotopy on spatial graphs was introduced by Taniyama in [23] as a generalization of *link homotopy* in the sense of Milnor [8]. Delta edge-homotopy and sharp edge-homotopy on spatial graphs were introduced by the author in [15] and [17] as generalizations of *self* Δ*-equivalence* [22] (or *delta link homotopy* [12]) and *self* $\sharp$*-equivalence* [20] on oriented links, respectively. It is known that the implication (2) $\Rightarrow$ (3) $\Rightarrow$ (1) holds [17, Theorem 1.1].

Obviously the crossing change is an *unknotting operation*, namely every knot can be undone by crossing changes and ambient isotopies. Besides it is well known that the delta move and the sharp move are also unknotting operations [7], [11], [10]. In general, if any two spatial embeddings of a graph G are transformed into each other by a finite sequence of specific local moves and ambient isotopies, then the local move is called a *uniforming operation* for the spatial embeddings of G [18]. We have already known when the self crossing change is a uniforming operation as follows:

Theorem 1.1. ([23, THEOREM B]) *For a graph G, the following are equivalent.*
(1) *Any two spatial embeddings of G are edge-homotopic.*
(2) *G does not contain a subgraph which is homeomorphic to K_4, D_3 or the disjoint union of two 1-spheres, where K_4 and D_3 are graphs as illustrated in Fig. 1.4.*
(3) *G is a generalized bouquet, namely there exists a vertex v of G such that $H_1(G - v; \mathbb{Z}) = 0$.* $\square$

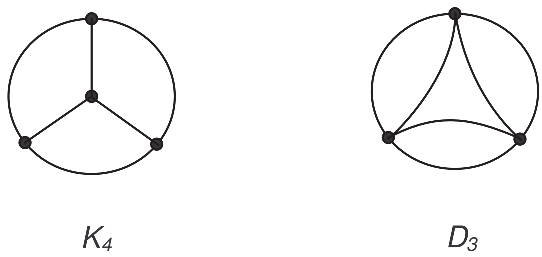


Fig. 1.4.

Therefore Theorem 1.1 is an unknotting theorem for edge-homotopy on spatial graphs. Our purpose in this paper is to determine when the self delta (resp. sharp) move is a uniforming operation, namely giving an unknotting theorem for delta (resp. sharp) edge-homotopy on spatial graphs. The following is our main result.

Theorem 1.2. *For a graph G, the following are equivalent.*
(1) *Any two spatial embeddings of G are delta edge-homotopic.*
(2) *Any two spatial embeddings of G are sharp edge-homotopic.*
(3) *G does not contain a subgraph which is homeomorphic to the graph Θ as illustrated in Fig. 1.5 or the disjoint union of two 1-spheres.*
(4) *G is a bouquet, namely there exists a positive integer m such that G is homeomorphic to the graph B_m as illustrated in Fig. 1.5.*

We remark here that all of the spatial embeddings of Θ and all of the spatial embeddings of the disjoint union of two 1-spheres, namely all *spatial*

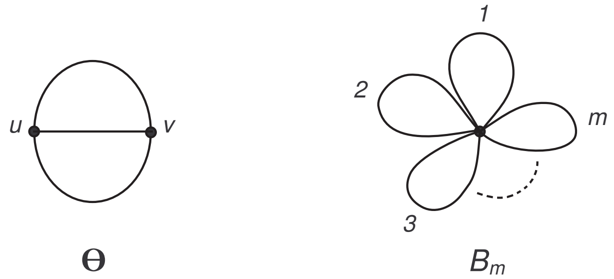


Fig. 1.5.

theta curves and all 2-component links, have been classified completely up to delta edge-homotopy [16], [13], and up to sharp edge-homotopy [17], [21]. In the next section we prove lemmas needed later. We prove Theorem 1.2 in section 3.

2. C_k-moves on spatial graphs

In this section, we prove lemmas concerning specific local moves needed later. A C_1*-move* is a crossing change and a C_k*-move* ($k \geq 2$) is a local move on a spatial graph as illustrated in Fig. 2.1. This move was introduced by Habiro as a local move on an oriented link [4], [2], and it was extended to spatial graphs by Taniyama and Yasuhara from a stand point of the "band description" [26] (see also [25], [19]). We note that the original definition of the C_k-move is different from the one above, but it is known that each of the original C_k-moves can be realized by local moves as illustrated in Fig. 2.1 and ambient isotopies [4]. We note that a C_2-move is equivalent to a delta move and a C_3-move is called a *clasp-pass move* [3].

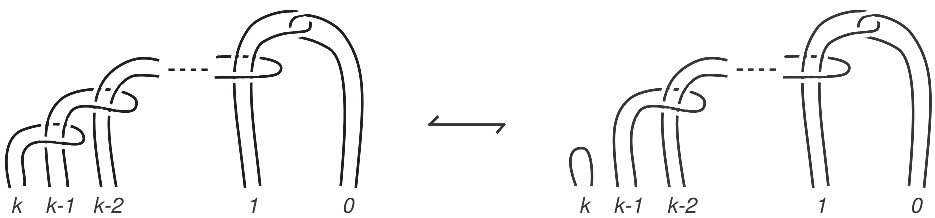


Fig. 2.1.

For a C_k-move and a self delta move, we have the following.

Lemma 2.1. *A C_k-move is realized by self delta moves and ambient isotopies if at least three of the $(k+1)$ strings in it belong to the same spatial* *edge.* □

The statement above was pointed out in [12, p. 179] and the case of $k = 4$ was applied to classify 2-component links and spatial theta curves up to delta edge-homotopy effectively [13], [16]. Actually we can show Lemma 2.1 by applying ambient isotopic transformations and C_2-moves on the same spatial edge to Fig. 2.1 directly. For example, see Fig. 2.2, where gray parts belong to the same spatial edge. We omit the details.

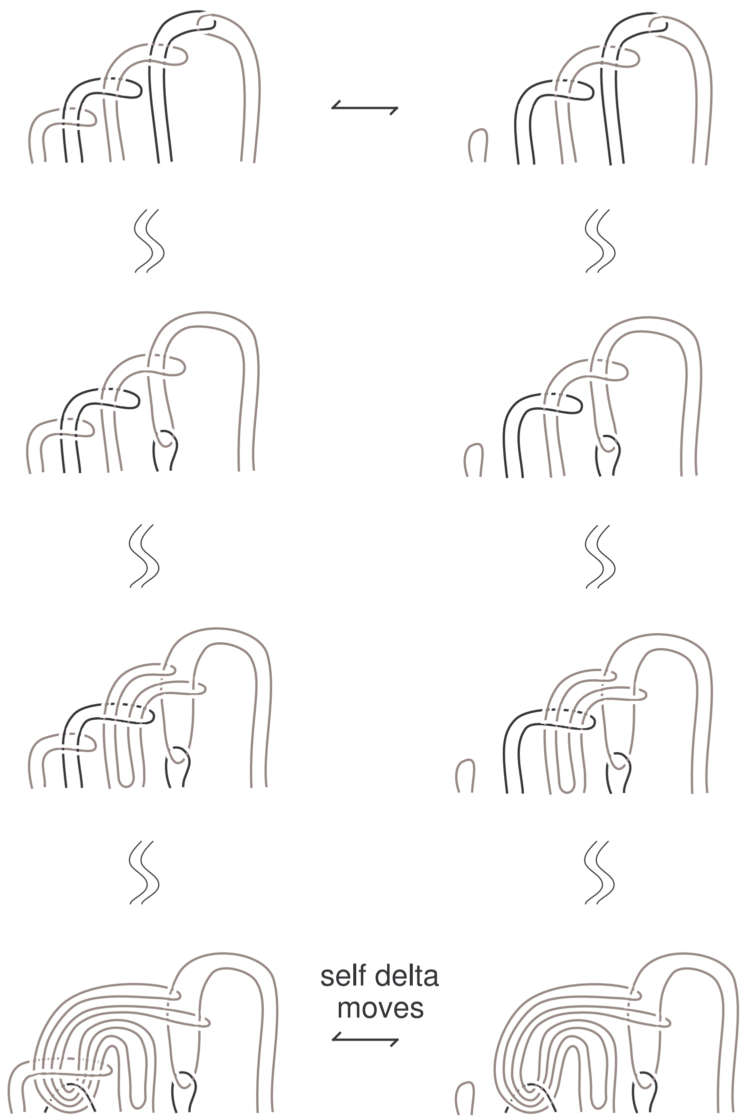


Fig. 2.2.

We note that a C_k-move can be realized by a *band sum of a* $(k+1)$-

component Milnor link [2] [8], see Fig. 2.3. A $(k+1)$-component Milnor link is one of the C_{k-1}*-links*, and it is known the following. We refer the reader to [25], [19], [26] for details.

Lemma 2.2. *Each of the local moves as illustrated in Fig. 2.4 (1), (2) and (3) is realized by* C_k*-moves and ambient isotopies, where* M_k *denotes a* k*-component Milnor link.* $\square$

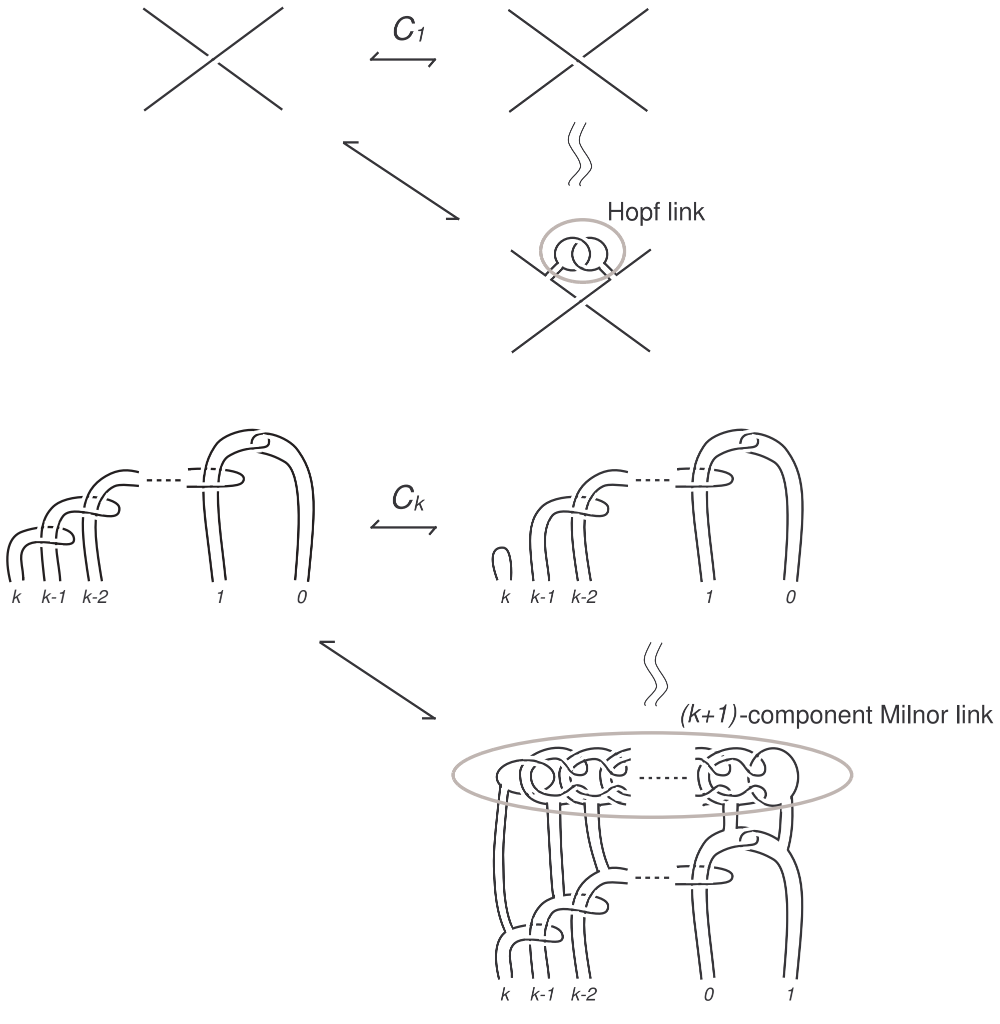


Fig. 2.3.

Therefore, a fusion band with a k-component Milnor link can leap over a spatial edge, and a root of a fusion band with a k-component Milnor link can pass through a root of a fusion band with another k-component Milnor

[2] A 2-component Milnor link is a Hopf link, and an $n(\geq 3)$-component Milnor link can be defined as one of the *iterated Bing doubles* of a Hopf link [1].

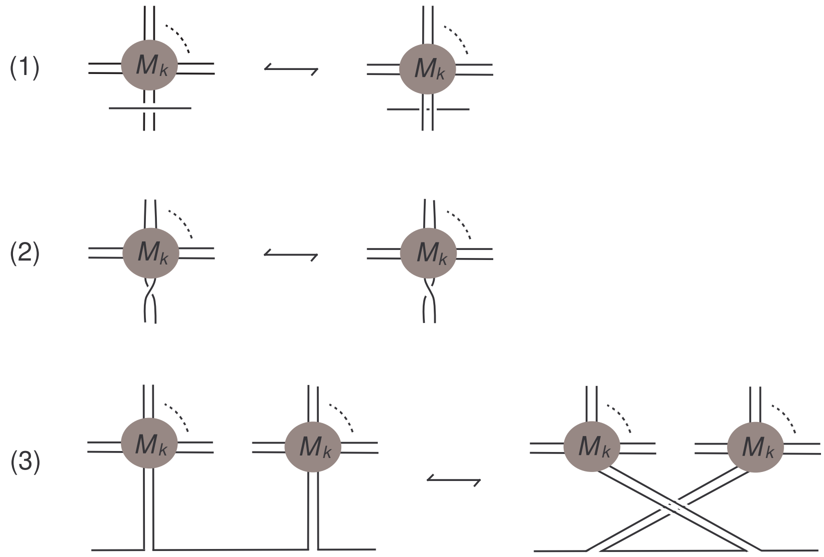


Fig. 2.4.

link by C_k-moves and ambient isotopies. We note that any of the full twists of a fusion band with a k-component Milnor link can be cancelled out by Fig. 2.4 (2).

A C_k-move is called an *adjacent C_k-move* if all $(k+1)$ strings in the move belong to exactly $(k+1)$ mutually adjacent spatial edges. Here we regard a loop as two mutually adjacent edges. The following lemma is a generalization of the facts that a crossing change between adjacent spatial edges is realized by delta moves and ambient isotopies [9] and a delta move between exactly three mutually adjacent spatial edges is realized by clasp-pass moves and ambient isotopies [24, Lemma 3.2].

Lemma 2.3. *An adjacent C_k-move is realized by C_{k+1}-moves and ambient isotopies.*

Proof. We note that any adjacent C_k-move can be realized by a band sum of a $(k+1)$-component Milnor link, where the roots of all of the fusion bands with the link belong to exactly $(k+1)$ mutually adjacent spatial edges. It is sufficient to show that this Milnor link can be cancelled out by C_{k+1}-moves and ambient isotopies. By Lemma 2.2, we can draw the link up sufficiently near by the shared vertex and deform it into the one on the left-hand side in Fig. 2.5 if $k=1$ and Fig. 2.6 if $k \geq 2$ identically by C_{k+1}-moves and ambient isotopies, where  =  or . In the case of $k=1$, this Hopf link is cancelled out up to ambient isotopy. Thus the result holds for $k=1$.

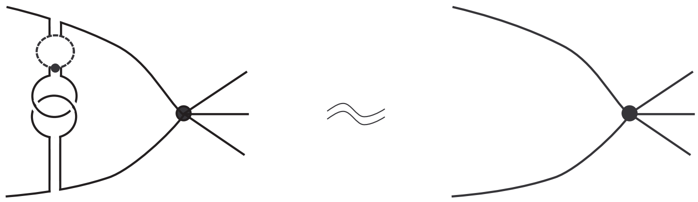


Fig. 2.5.

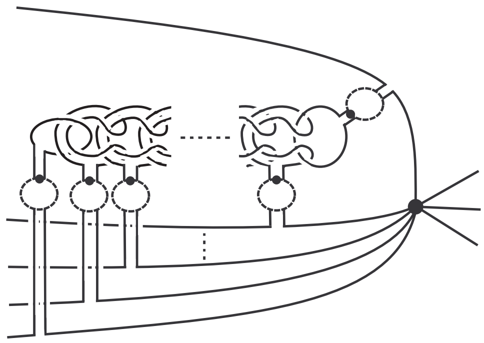
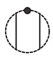


Fig. 2.6.

Next we show the case of $k \geq 2$. Let us consider the diagrams as illustrated in Fig. 2.7 which satisfies the following:

(1) $c_{i,1} =$  or  $(i \neq 1)$.

(2) If $c_{0,1} =$ , then $c_{1,1} =$  or .

(3) If $c_{0,1} =$ , then $c_{1,1} =$  or .

(4) If $c_{i,1}$= , then $c_{i,2}$= .

(5) If $c_{i,1}$= , then $c_{i,2}$= .

It is easy to see that each of the two diagrams on the lower part is ambient isotopic to the diagram on the upper part. On the other hand, the two diagrams on the lower part are transformed into each other by an obvious adjacent C_k-move and ambient isotopies, namely this C_k-move is realized by a band sum of a $(k+1)$-component Milnor link. Then we can produce a $(k+1)$-component Milnor link with the arbitrary pattern of the half twists of the fusion bands with the link up to C_{k+1}-moves and ambient isotopies, see the case of $k=5$ as illustrated in Fig. 2.8. This implies that an adjacent C_k-move can be realized by C_{k+1}-moves and ambient isotopies. $\square$

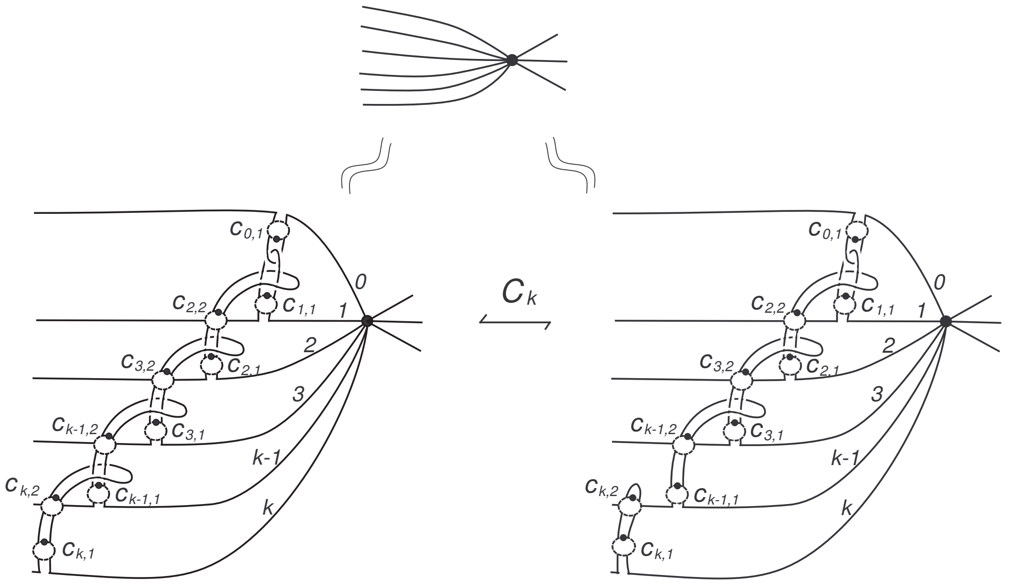


Fig. 2.7.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. (1) $\Rightarrow$ (2): It is known that a delta move is realized by sharp moves on the strings in the delta move and ambient isotopies [14]. Thus the result is clear.

(2) $\Rightarrow$ (3): We note that if two spatial embeddings f and g of G are sharp edge-homotopic then $f|_H$ and $g|_H$ are sharp edge-homotopic for any subgraph H of G. It is known that a spatial theta curve as illustrated in Fig. 3.1 is non-trivial up to sharp edge-homotopy [17, Example 3.6]. Besides we have that a Hopf link is non-trivial up to sharp edge-homotopy because the *linking number* is a sharp-edge homotopy invariant of an oriented link. So we have the result.

(3) $\Rightarrow$ (4): Let G be a graph which does not contain a subgraph which is homeomorphic to Θ or the disjoint union of two 1-spheres. Let K_5 and $K_{3,3}$ be the graphs as illustrated in Fig. 3.2. It is well known that a graph is planar if and only if the graph does not contain a subgraph which is homeomorphic to K_5 or $K_{3,3}$ [5]. Since each of K_5 and $K_{3,3}$ contains a subgraph which is homeomorphic to Θ, we have that G is a planar graph which does not contain mutually disjoint cycles. Thus by [23, Theorem C] we have that G is homeomorphic to a *double trident*, a *multi-spoke wheel* or a generalized bouquet. Here a double trident and a multi-spoke wheel are graphs as illustrated in Fig. 3.3 (1) and (2), respectively, where a gray edge is allowed to have arbitrary multiplicity. Since each of double tridents and multi-spoke wheels contains a subgraph which is homeomorphic to Θ, the

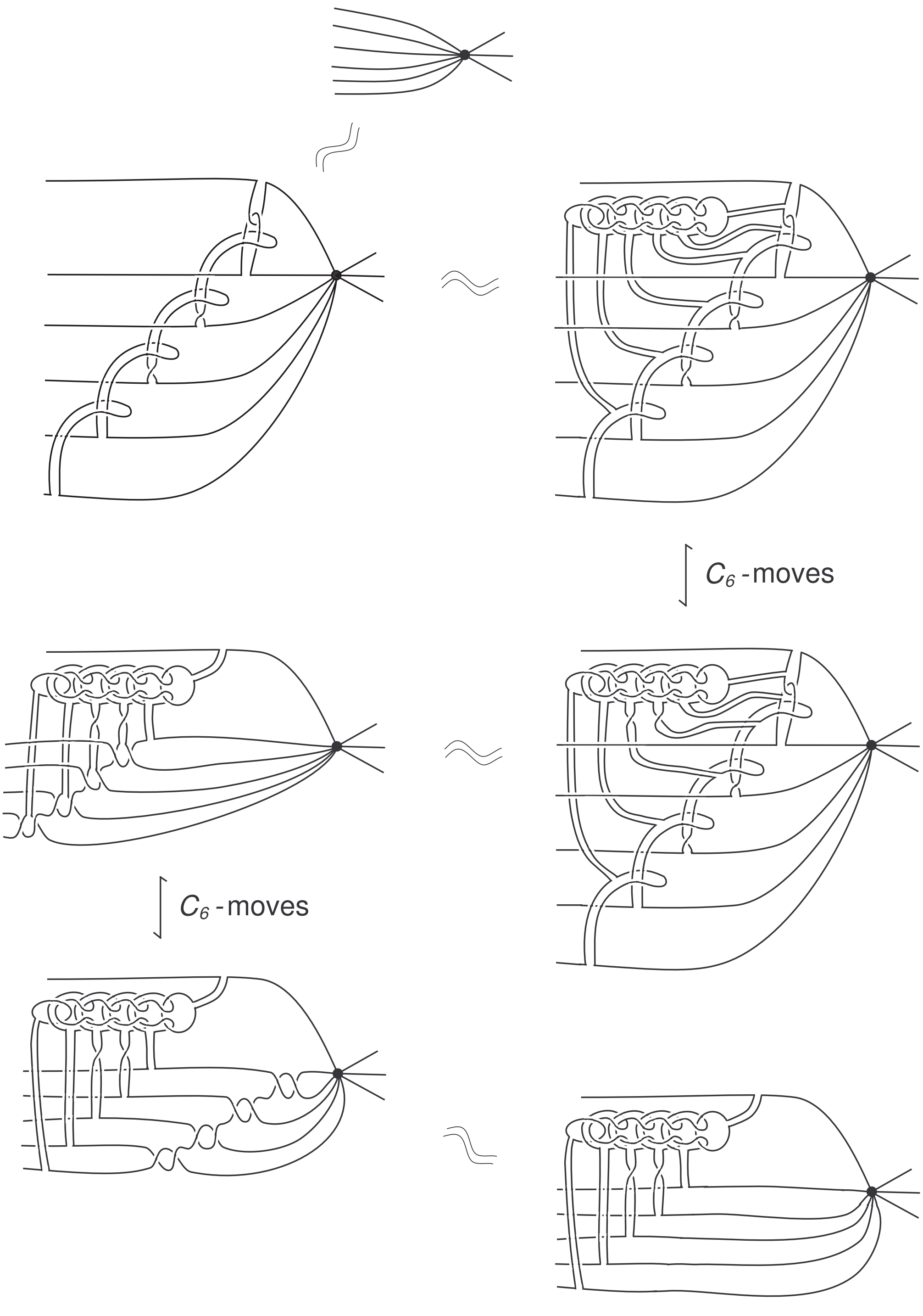


Fig. 2.8.

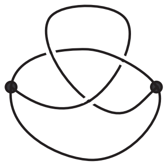


Fig. 3.1.

graph G must be a generalized bouquet. Moreover G must be a bouquet because G does not contain a subgraph which is homeomorphic to Θ. So we have the result.

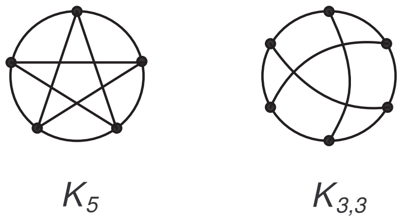


Fig. 3.2.

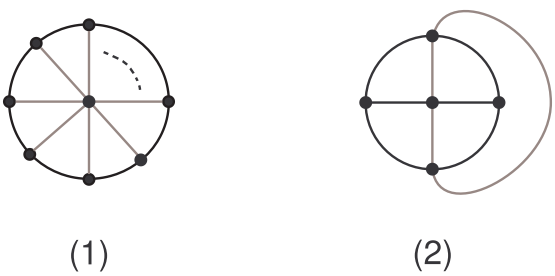


Fig. 3.3.

(4) $\Rightarrow$ (1): We show that any spatial embedding f of B_m is trivial up to delta edge-homotopy. Let h be the trivial spatial embedding of B_m. It is clear that f and h are transformed into each other by C_1-moves and ambient isotopies. Then we can regard each of the C_1-moves as an adjacent C_1-move. Thus by Lemma 2.3 we have that f and h are transformed into each other by C_2-moves and ambient isotopies. For each of these C_2-moves, if all of the three strings in the C_2-move belong to the same knot in $f(B_m)$, it is realized by self delta moves and ambient isotopies by Lemma 2.1.

Otherwise we can regard this C_2-move as an adjacent C_2-move. Thus by Lemma 2.3 we have that f and h are transformed into each other by C_3-moves, self delta moves and ambient isotopies. For each of these C_3-moves, if at least three of the four strings in the C_3-move belong to the same knot in $f(B_m)$, it is realized by self delta moves and ambient isotopies by Lemma 2.1. Otherwise we can regard this C_3-move as an adjacent C_3-move. Thus by Lemma 2.3 we have that f and h are transformed into each other by C_4-moves, self delta moves and ambient isotopies.

By following the procedure above repeatedly, we have that f and h are transformed into each other by C_{2m}-moves, self delta moves and ambient isotopies. Then we can see that for each of the C_{2m}-moves there exists a knot in $f(B_m)$ such that at least three of the $(2m+1)$ strings in the C_{2m}-move belong to it. So it is realized by self delta moves and ambient isotopies by Lemma 2.1. Therefore we have that f and h are transformed into each other by self delta moves and ambient isotopies, namely f is delta edge-homotopic to h. This completes the proof. $\square$

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