On generalizations of the Conway-Gordon theorems

Ryo Nikkuni
Department of Mathematics, Tokyo Woman's Christian University

1 Introduction

An embedding f of a finite graph G into $\mathbb{R}^3$ is called a *spatial embedding* of G, and the image $f(G)$ is called a *spatial graph* of G. A subgraph γ of G homeomorphic to the circle is called a *cycle* of G. A cycle of G is also called a *k-cycle* if it contains exactly k vertices, and a *Hamiltonian cycle* if it contains all of the vertices of G. We denote the set of all k-cycles of G by $\Gamma_k(G)$, and the set of all pairs of two disjoint cycles of G consisting of a k-cycle and an l-cycle by $\Gamma_{k,l}(G)$. For a cycle γ (resp. a pair of disjoint cycles λ) and a spatial embedding f of G, $f(\gamma)$ (resp. $f(\lambda)$) is none other than a knot (resp. a 2-component link) in $f(G)$. For a Hamiltonian cycle γ of G, we also call $f(\gamma)$ a *Hamiltonian knot* in $f(G)$.

Let K_n be the *complete graph* on n vertices, that is the graph consisting of n vertices such that each pair of its distinct vertices is connected by exactly one edge. Then let us recall the following Conway-Gordon theorems, which are very famous theorems in spatial graph theory.

Theorem 1.1 (Conway-Gordon [6])

(1) *For any spatial embedding f of K_6, we have*

$$\sum_{\lambda \in \Gamma_{3,3}(K_6)} \operatorname{lk}\left(f(\lambda)\right) \equiv 1 \pmod 2,$$

where lk *denotes the linking number in $\mathbb{R}^3$.*

(2) *For any spatial embedding f of K_7, we have*

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2\left(f(\gamma)\right) \equiv 1 \pmod 2,$$

where a_2 denotes the second coefficient of the Conway polynomial.

Theorem 1.1 implies that K_6 is *intrinsically linked*, that is, every spatial graph of K_6 contains a nonsplittable 2-component link, and K_7 is *intrinsically knotted*, that is, every spatial graph of K_7 contains a nontrivial knot. Characterizing all intrinsically linked graphs and intrinsically knotted graphs is an important research theme in spatial graph theory, see for example [8, §§2-6]. However, our interest in this article is in another direction; it is to generalize the Conway-Gordon theorems to K_n with arbitrary $n \geq 6$. As far as the author knows, there have been little results about a generalization of the Conway-Gordon type congruences on K_n with $n \geq 8$. It is all with the following results.

Theorem 1.2 (1) (Foisy [10], Hirano [12])[1] *For any spatial embedding f of K_8, we have*

$$\sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \equiv 3 \pmod{6}.$$

(2) (Hirano [12]) *Let $n \geq 9$ be an integer. For any spatial embedding f of K_n, we have*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv 0 \pmod{2}.$$

(3) (Kazakov-Korablev [17]) *Let $n \geq 7$ be an integer. For any spatial embedding f of K_n, we have*

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda)) \equiv 0 \pmod{2}.$$

On the other hand, let us also recall integral lifts of the Conway-Gordon theorems for K_6 and K_7 proven by the author as follows.

Theorem 1.3 (Nikkuni [21])

(1) *For any spatial embedding f of K_6, we have*

$$\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \frac{1}{2} \sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}(f(\lambda))^2 - \frac{1}{2}. \tag{1.1}$$

(2) *For any spatial embedding f of K_7, we have*

$$\begin{aligned} &\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma)) \\ =\ & \frac{1}{7}\left(2 \sum_{\lambda \in \Gamma_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2 + 3 \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2\right) - 6. \end{aligned} \tag{1.2}$$

Actually, Theorem 1.1 (1) can be recovered from Theorem 1.3 (1) by multiplying by 2 and taking the mod 2 reduction, and Theorem 1.1 (2) can also be recovered from Theorem 1.3 (2) by multiplying by 7 and taking the mod 2 reduction (note that $\sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2$ is odd as we will see later). Our purposes are to generalize the integral Conway-Gordon theorems to K_n with arbitrary $n \geq 6$ and to investigate the behavior of the nontrivial Hamiltonian knots and links in a spatial graph of K_n. This is a joint work with H. Morishita.

[1] For any spatial embedding f of K_8, Foisy showed that $\sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \equiv 0 \pmod{3}$ [10], and Hirano showed that $\sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \equiv 1 \pmod{2}$ [12].

2 Generalizations of the Conway-Gordon theorems

First, we generalize Theorem 1.3 to K_n with arbitrary $n \geq 6$ as follows.

Theorem 2.1 (Morishita-Nikkuni [18]) *Let $n \geq 6$ be an integer. For any spatial embedding f of K_n, we have*

$$\begin{aligned}&\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)!\sum_{\gamma\in\Gamma_5(K_n)} a_2(f(\gamma))\\ &= \frac{(n-5)!}{2}\left(\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 - \binom{n-1}{5}\right).\end{aligned}$$

Example 2.2 (1) In the case of $n = 6$ in Theorem 2.1, we have (1.1).

(2) In the case of $n = 7$ in Theorem 2.1, we have

$$\sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(K_7)} a_2(f(\gamma)) = \sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2 - 6. \tag{2.1}$$

Though (1.2) and (2.1) are slightly different, since $\sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2$ is twice $\sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2$ (see Example 2.8 (1)), these equations are equivalent.

Note that any pair of two disjoint 3-cycles λ of K_n is not shared by any two different subgraphs of K_n isomorphic to K_6. Then Theorem 1.1 (1) implies that $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2$ is greater than or equal to the number of subgraphs of K_n isomorphic to K_6, that is equal to $\binom{n}{6}$. Thus by Theorem 2.1, we have the following.

Corollary 2.3 *Let $n \geq 6$ be an integer. For any spatial embedding f of K_n, we have*

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)!\sum_{\gamma\in\Gamma_5(K_n)} a_2(f(\gamma)) \geq \frac{(n-5)(n-6)(n-1)!}{2\cdot 6!}.$$

Remark 2.4 Endo-Otsuki introduced a certain special spatial embedding f_{b} of K_n, a *canonical book presentation* of K_n [7], and Otsuki also showed that $f_{\mathrm{b}}(K_n)$ contains exactly $\binom{n}{6}$ Hopf links as all of the nonsplittable triangle-triangle links [22]. Thus the lower bound of Corollary 2.3 is sharp. Furthermore, every 5-cycle knot in $f_{\mathrm{b}}(K_n)$ is trivial. Thus for an integer $n \geq 6$, we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f_{\mathrm{b}}(\gamma)) = \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right) = \frac{(n-5)(n-6)(n-1)!}{2\cdot 6!}. \tag{2.2}$$

Moreover, for $n \geq 7$, let f and g be two spatial embeddings of K_n. Let us consider the difference between $\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma))$ and $\sum_{\gamma\in\Gamma_n(K_n)} a_2(g(\gamma))$ modulo $(n-5)!$. By Theorem 2.1, we have

$$\begin{aligned}&\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_n(K_n)} a_2(g(\gamma))\\ &\equiv \frac{(n-5)!}{2}\left(\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 - \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(g(\lambda))^2\right) \pmod{(n-5)!}.\end{aligned} \tag{2.3}$$

Note that both $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2$ and $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(g(\lambda))^2$ have the same parity, that is equal to the parity of $\binom{n}{6}$. Thus by (2.3), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) \equiv \sum_{\gamma\in\Gamma_n(K_n)} a_2(g(\gamma)) \pmod{(n-5)!}. \tag{2.4}$$

This shows that $\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma))$ does not depend on an embedding f of K_n. Now let us select a canonical book presentation f_{b} of K_n as g. Then by (2.2) and (2.4), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) \equiv \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right) \pmod{(n-5)!}. \tag{2.5}$$

Then we also note that $\binom{n}{6}$ is odd if and only if $n \equiv 6, 7 \pmod 8$, and $\binom{n-1}{5}$ is odd if and only if $n \equiv 0, 6 \pmod 8$. Thus by (2.5), we have the following congruence, that contains Theorem 1.1 (2) and Theorem 1.2 (1) as the cases of $n = 7, 8$ and also generalizes Theorem 1.2 (2) remarkably.

Corollary 2.5 *Let $n \geq 7$ be an integer. For any spatial embedding f of K_n, we have the following congruence modulo $(n-5)!$:*

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} -\dfrac{(n-5)!}{2}\dbinom{n-1}{5} & (n \equiv 0 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8) \\ \dfrac{(n-5)!}{2}\dbinom{n}{6} & (n \equiv 7 \pmod 8). \end{cases}$$

Next, we also generalize Theorem 1.2 (3) from a viewpoint of the linking numbers of 2-component "Hamiltonian" links as follows.

Theorem 2.6 (Morishita-Nikkuni [19]) *Let $n \geq 6$ be an integer. Let p and q be two positive integers satisfying $n = p + q$, where $p, q \geq 3$. For any spatial embedding f of K_n, we have*

$$\sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = \begin{cases} (n-6)! \displaystyle\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p = q) \\ 2(n-6)! \displaystyle\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p \neq q). \end{cases}$$

In particular, we also have

$$\sum_{p+q=n} \sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = (n-5)! \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2.$$

Theorem 2.6 also implies that for any spatial embedding f of K_n ($n \geq 6$), the sum of lk^2 over all of the Hamiltonian 2-component links is congruent to 0 modulo $(n-5)!$.

As we have already seen, $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2$ is greater than or equal to $\binom{n}{6}$. Thus by Theorem 2.6, we have the following.

Corollary 2.7 *Let $n \geq 6$ be an integer. Let p and q be two positive integers satisfying $n = p + q$, where $p, q \geq 3$. For any spatial embedding f of K_n, we have*

$$\sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq \begin{cases} \dfrac{n!}{6!} & (p = q) \\ 2 \cdot \dfrac{n!}{6!} & (p \neq q). \end{cases}$$

In particular,

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq (n-5) \cdot \frac{n!}{6!}.$$

Example 2.8 (1) In the case of $n = 7$ in Corollary 2.7, we have

$$\sum_{\lambda \in \Gamma_{3,4}(K_n)} \mathrm{lk}(f(\lambda))^2 = 2 \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq 14.$$

It has been shown in Fleming-Mellor [9] that every spatial graph of K_7 contains at least 14 triangle-square links with odd linking number.

(2) In the case of $n = 8$ in Corollary 2.7, we have

$$\sum_{\lambda \in \Gamma_{3,5}(K_n)} \mathrm{lk}(f(\lambda))^2 = 2 \sum_{\lambda \in \Gamma_{4,4}(K_n)} \mathrm{lk}(f(\lambda))^2 = 4 \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq 112. \quad (2.6)$$

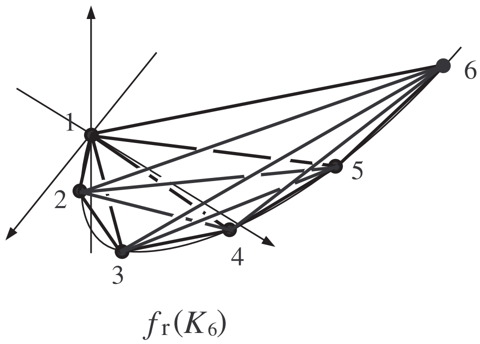
It has also been shown in [9] that every spatial graph of K_8 contains at least 42 nonsplittable triangle-pentagon links and 35 nonsplittable square-square links. Based on (2.6), we expect that every spatial graph of K_8 contains many more nonsplittable triangle-pentagon links and square-square links.

3 Applications to rectilinear spatial complete graphs

A spatial embedding f_{r} of a simple graph G is said to be *rectilinear* if for any edge e of G, $f_{\mathrm{r}}(e)$ is a straight line segment in $\mathbb{R}^3$. We can construct a special rectilinear spatial embedding of K_n by taking n vertices of K_n on the moment curve (t, t^2, t^3) in $\mathbb{R}^3$ and connecting every pair of two distinct vertices by a straight line segment, see Fig. 3.1 for $n = 6, 7$.[2] We say such a rectilinear spatial embedding of K_n to be *standard*.

A rectilinear spatial graph appears in polymer chemistry as a mathematical model for chemical compounds (see [1, §7], for example), and the range of rectilinear spatial graph types is much narrower than the general spatial graphs. So we are interested in the behavior of the nontrivial Hamiltonian knots and links in a rectilinear spatial graph of K_n. Note that knot and link types appearing as constituent knots and links in a rectilinear spatial graph of K_n are limited because they have the *stick number* $\leq n$, where the stick number of a link L, denoted by $s(L)$, is the minimum number of edges in a polygon

[2] In fact, the standard rectilinear spatial graph of K_7 is equivalent to the Conway-Gordon's spatial graph of K_7 in [6] which contains exactly one trefoil knot as the nontrivial Hamiltonian knots.



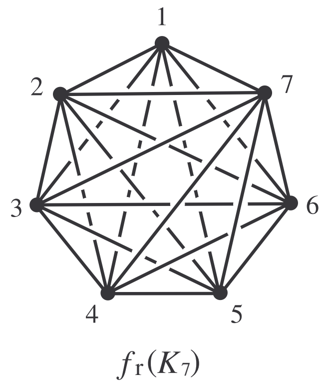


Figure 3.1: Standard rectilinear spatial embedding f_{r} of K_n $(n = 6, 7)$

representing L. We recall the following results on stick numbers (see [1], [20], [2], [5]), where we denote each of knots and links appearing in the statement by using its label in Rolfsen's table.

Proposition 3.1 *Let L be a link. Then the following statements hold.*

(1) *If L is a nontrivial knot, then $s(L) \geq 6$.*

(2) *$s(L) = 6$ if and only if L is equivalent to 3_1, 0_1^2 or 2_1^2.*

(3) *$s(L) = 7$ if and only if L is equivalent to 4_1 or 4_1^2.*

(4) *$s(L) = 8$ if and only if L is equivalent to 5_1, 5_2, 6_1, 6_2, 6_3, the granny knot $3_1\#3_1$, the square knot $3_1\#3_1^*$, 8_{19}, 8_{20} or 5_1^2.*

Proposition 3.1 (1) says that every polygonal knot with five sticks is trivial. Thus for rectilinear spatial graph of K_n, by Theorem 2.1 we have the following immediately.

Theorem 3.2 *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding f_{r} of K_n, we have*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2 - \binom{n-1}{5}\right).$$

Also note that a 2-component link with exactly six sticks is either a trivial link or a Hopf link by Proposition 3.1 (2). Thus for any rectilinear spatial embedding f_{r} of K_n, $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2$ is equal to the number of triangle-triangle Hopf links in $f_{\mathrm{r}}(K_n)$. Namely, Theorem 3.2 says that for every rectilinear spatial graph of K_n $(n \geq 6)$, the sum of a_2 over all of the Hamiltonian knots is determined explicitly in terms of the number of triangle-triangle Hopf links.

In the same way as Corollary 2.3, we can obtain a lower bound of $\sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right)$ for all rectilinear spatial embeddings f_{r} of K_n. On the other hand, since the number of triangle-triangle Hopf links in a rectilinear spatial graph of K_6 is strongly limited, we can also obtain an upper bound of $\sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right)$. Actually, it is known the following.

Proposition 3.3 (Hughes [15], Huh-Jeon [14], Nikkuni [21]) *Every rectilinear spatial graph of K_6 contains at most three Hopf links.*

This implies that $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2$ is less than or equal to $3\binom{n}{6}$. Thus by Theorem 3.2, we have the following.

Corollary 3.4 *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding f_{r} of K_n, we have*

$$\frac{(n-5)(n-6)(n-1)!}{2\cdot 6!} \leq \sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq \frac{3(n-2)(n-5)(n-1)!}{2\cdot 6!}.$$

Example 3.5 (1) In the case of $n=6$ in Corollary 3.4, we have

$$0 \leq \sum_{\gamma\in\Gamma_6(K_6)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq 1.$$

By Proposition 3.1 (2), $f_{\mathrm{r}}(\gamma)$ is either a trivial knot or a trefoil knot. Since $a_2(3_1)=1$, we have that every rectilinear spatial graph of K_6 contains at most one trefoil knot, which was also observed in [14] in combinatorial way.

(2) In the case of $n=7$, by Corollary 3.4 and Theorem 1.1 (2), we have

$$1 \leq \sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq 15, \quad \sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \equiv 1 \pmod 2.$$

By Proposition 3.1 (2) and (3), if $f_{\mathrm{r}}(\gamma)$ is a nontrivial knot then it is either a trefoil knot or a figure eight knot. Since $a_2(4_1)=-1$, we have that every rectilinear spatial graph of K_7 contains a trefoil knot with 7 sticks, which was originally proven by Brown [4] and Ramírez Alfonsín [24] in combinatorial and computational way.

(3) In the case of $n=8$, by Corollary 3.4 and Theorem 1.2 (1), we have

$$21 \leq \sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq 189, \quad \sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \equiv 3 \pmod 6.$$

By Proposition 3.1, all of the polygonal knots with 8 sticks are classified completely, and, in addition, only 3_1, 5_1, 5_2, 6_3, $3_1\#3_1$, $3_1\#3_1^*$, 8_{19} and 8_{20} have a_2 with a positive value. Thus every rectilinear spatial graph of K_8 contains one of these knots as a Hamiltonian knot. Since the maximum value of a_2 for them is 5 ($a_2(8_{19})=5$), we have that the minimum number of nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_8 is at least $\lceil 21/5\rceil = 5$. However this is not yet the sharp lower bound, see Remark 4.2. In Section 4, we will discuss the minimum number of nontrivial Hamiltonian knots in a rectilinear spatial graph of K_n with arbitrary $n \geq 7$.

Remark 3.6 The lower bound in Corollary 3.4 is also sharp, actually the standard rectilinear spatial embedding of K_n realizes the lower bound. On the other hand, in the case of $n = 7$, our upper bound is 15. However, according to a computer search in [16] (by using the oriented matroid theory), there seems to be no rectilinear embedding f_{r} of K_7 such that $\sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = 13, 15$, or equivalently by Theorem 3.2, $\sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}(f_{\mathrm{r}}(\gamma))^2 = 19, 21$. Thus the author does not expect that the upper bound in Corollary 3.4 is sharp if $n \geq 7$.

Problem 3.7 Determine the sharp upper bound of $\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right)$ for all rectilinear spatial embeddings f_{r} of K_n for each $n \geq 7$, or equivalently by Theorem 3.2, determine the maximum number of triangle-triangle Hopf links in $f_{\mathrm{r}}(K_n)$ for each $n \geq 7$.

Example 3.8 Let us consider two spatial embeddings of K_8 as illustrated in Fig. 3.2. The left one was given in [3], and the right one is the standard rectilinear spatial embedding of K_8. It is known that, they are not equivalent[3] but each of them contains exactly 21 trefoil knots as all of the nontrivial Hamiltonian knots [3], [25]. We did not understand the meaning of this number "21" over ten years. In our current research, we have that for any spatial embedding f of K_8, if every 5-cycle knot in $f(K_8)$ is trivial then $\sum_{\gamma\in\Gamma_8(K_8)} a_2(f(\gamma)) \geq 21$. This also implies that if every nontrivial Hamiltonian knot is a trefoil knot, then there must exist at least 21 trefoil knots as Hamiltonian knots.

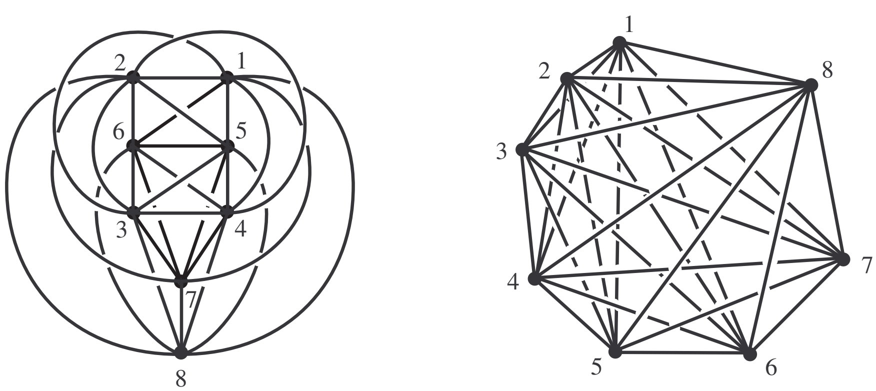


Figure 3.2: Two spatial embeddings of K_8

4 Further applications

In the rest of this article, we shall mention further two applications. First, let us consider the minimum number of nontrivial Hamiltonian knots in a rectilinear spatial graph of K_n. Note that we can obtain an estimate of a_2 over Hamiltonian knots in a rectilinear spatial graph of K_n from above as follows. The *crossing number* of a link L is the minimum number of crossings in a regular diagram of L on the plane, denoted by $c(L)$. In particular

[3]The left spatial graph of K_8 contains a triangle-pentagon link with nonzero even lk (actually $[257] \cup [13846]$), but the right spatial graph of K_8 does not contain such a triangle-pentagon link.

for a knot K, it has been shown that

$$c(K) \leq \frac{(s(K)-3)\,(s(K)-4)}{2} \tag{4.1}$$

by Calvo [5], and also has been shown that

$$a_2(K) \leq \frac{c(K)^2}{8} \tag{4.2}$$

by Polyak-Viro [23]. By combining (4.1) and (4.2), for a polygonal knot K with $\leq n$ sticks, we have

$$a_2(K) \leq \left\lfloor \frac{(n-3)^2(n-4)^2}{32} \right\rfloor, \tag{4.3}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Then by the lower bound in Corollary 3.4 and (4.3), we have the following estimate of the minimum number of nontrivial Hamiltonian knots in every rectilinear spatial graph of K_n from below.

Corollary 4.1 *Let $n \geq 7$ be an integer. The minimum number of nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_n is at least*

$$r_n = \left\lceil \frac{(n-5)(n-6)(n-1)!/(2\cdot 6!)}{\lfloor (n-3)^2(n-4)^2/32 \rfloor} \right\rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

The concrete values of r_n for $7 \leq n \leq 16$ are given in the following table.

n	7	8	9	10	11	12	13	14	15	16	$\cdots$
r_n	1	2	12	92	772	7187	73628	823680	10015889	131436569	$\cdots$

Remark 4.2 For every spatial embedding f of K_n (which does not need to be rectilinear), Hirano showed that there exist at least three nontrivial Hamiltonian knots with an odd value of a_2 in $f(K_8)$ [13], and Foisy showed that there exist at least $(n-1)(n-2)\cdots 9\cdot 8$ nontrivial Hamiltonian knots with an odd value of a_2 in $f(K_n)$ if $n \geq 9$ [3]. We see that r_n is greater than Foisy's lower bound of the minimum number of nontrivial Hamiltonian knots with an odd value of a_2 if $n = 9, 10, 11$. On the other hand, in the case of $n = 8$, as we have already seen in Example 3.5 (3), we can obtain an estimate "5" from below better than r_8. Moreover, it is known that every rectilinear spatial graph of the complete four-partite graph $K_{3,3,1,1}$ contains at least one nontrivial Hamiltonian knot with a positive value of a_2 (Hashimoto-Nikkuni [11]). Since there are 280 subgraphs of K_8 isomorphic to $K_{3,3,1,1}$ and for any 8-cycle γ of K_8 there exist 36 subgraphs of K_8 isomorphic to $K_{3,3,1,1}$ containing γ, we have that there are at least $\lceil 280/36 \rceil = 8$ nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_8.

Problem 4.3 Determine the minimum number of nontrivial Hamiltonian knots (with a positive value of a_2) in every rectilinear spatial graph of K_n for each $n \geq 8$.

Next, we can also obtain the result about the maximum value of a_2 for all of the Hamiltonian knots in a rectilinear spatial graph of K_n as follows.

Theorem 4.4 (Morishita-Nikkuni [19]) *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding $f_{\rm r}$ of K_n, we have*

$$\max_{\gamma\in\Gamma_n(K_n)}\{a_2(f_{\rm r}(\gamma))\} \geq \frac{(n-5)(n-6)}{6!}.$$

Actually, by Corollary 3.4 we have

$$\max_{\gamma\in\Gamma_n(K_n)}\{a_2(f_{\rm r}(\gamma))\}\cdot \sharp\Gamma_n(K_n) \geq \sum_{\gamma\in\Gamma_n(K_n)} a_2(f_{\rm r}(\gamma)) \geq \frac{(n-5)(n-6)(n-1)!}{2\cdot 6!},$$

and since $\sharp\Gamma_n(K_n) = (n-1)!/2$, we have the result.[4] Theorem 4.4 implies that if n is sufficiently large then every rectilinear spatial graph of K_n contains a Hamiltonian knot with arbitrary large value of a_2. By Theorem 4.4, we also have the following corollary.

Corollary 4.5 *Let m be a positive integer. If $n > \left(11+\sqrt{2880m-2879}\right)/2$, then for any rectilinear spatial embedding $f_{\rm r}$ of K_n, there exists a Hamiltonian cycle $\gamma \in \Gamma_n(K_n)$ such that $a_2\left(f_{\rm r}(\gamma)\right) \geq m$.*

Remark 4.6 It has been shown in Shirai-Taniyama [26] that

(1) For any spatial embedding f of $K_{48\cdot 2^k}$, there exists a cycle γ of $K_{48\cdot 2^k}$ such that $|a_2(f(\gamma))| \geq 2^{2k}$,

(2) Let m be a positive integer. If $n \geq 96\sqrt{m}$, then for any spatial embedding f of K_n, there exists a cycle γ of K_n such that $|a_2(f(\gamma))| \geq m$.

If we restrict ourselves to rectilinear spatial graphs of K_n, Corollary 4.5 is better than Shirai-Taniyama's result, see the following table.

m	1	2	3	4	5	6	7	8	9	10
n (Shirai-Taniyama [26])	48	136	167	96	215	236	254	272	288	304
n (Morishita-Nikkuni [19])	7	33	44	52	60	66	72	77	82	87

For any knot (resp. link) type L, Negami proved in [20] that there exists a positive integer n such that every rectilinear spatial graph of K_n contains a knot (resp. link) equivalent to L. The minimum value of such n is called the *Ramsey number* of L, denoted by $R(L)$. For a positive integer m, let us define $R(m)$ by the minimum value of n such that every rectilinear spatial graph of K_n contains a knot with $a_2 \geq m$. For example, since the standard rectilinear spatial graph of K_6 (Fig. 3.1) does not contain a trefoil knot, we have $R(1) = 7$. Note that for a knot type K with $a_2(K) > 0$, $R(K)$ is evaluated by $R(a_2(K))$ from below.

Problem 4.7 Determine $R(m)$ for each $m \geq 2$.

[4] Though we can also obtain a similar result about the maximum value of lk^2 for all of the Hamiltonian links, we omit it.

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Department of Mathematics, School of Arts and Sciences
Tokyo Woman's Christian University
2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585
JAPAN
E-mail address: `nick@lab.twcu.ac.jp`

東京女子大学 現代教養学部　新國 亮