Classification of spatial complete graphs on four vertices up to $C_5$-moves

Ryo Nikkuni (Tokyo Woman’s Christian University)
(joint work with Akira Yasuhara (Tokyo Gakugei University))

Dedicated to Professor Kunio Murasugi for his 80th birthday

1. $C_k$-moves on spatial graphs and finite type invariants

Let $S^3$ be the unit 3-sphere in $\mathbb{R}^4$ centered at the origin and $S^2$ the unit 2-sphere in $S^3$. Let $f$ be an embedding of a finite graph $G$ into $S^3$. Then $f$ is called a spatial embedding of $G$ or simply a spatial graph. Two spatial embeddings $f$ and $g$ of $G$ are said to be ambient isotopic if there exists an orientation-preserving self homeomorphism $\Phi$ on $S^3$ such that $\Phi \circ f = g$. A graph $G$ is said to be planar if there exists an embedding of $G$ into $S^2$. A spatial embedding $f$ of a planar graph $G$ is said to be trivial if there exists an embedding $h$ of $G$ into $S^2$ such that $f$ and $h$ are ambient isotopic.

A $C_1$-move is a crossing change and a $C_k$-move is a local move on spatial graphs as illustrated in Fig. 1.1 for $k \geq 2$ [4], [2]. Note that a $C_2$-move is equal to a delta move [8], [12], and a $C_3$-move is equal to a clasp-pass move [3]. Two spatial embeddings of a graph are said to be $C_k$-equivalent if they are transformed into each other by $C_k$-moves and ambient isotopies. By the definition of a $C_k$-move, it is easy to see that $C_k$-equivalence implies $C_{k-1}$-equivalence.

A $C_k$-move is closely related to finite type invariants of knots, links and spatial graphs. For a graph $G$, we give an orientation to each of the edges of $G$. A singular spatial embedding of $G$ is an immersion of $G$ into $S^3$ whose multipoints are only transversal double points away from vertices. Let $v$ be an ambient isotopy invariant of spatial graphs taking values in an additive group. We extend $v$ to
singular spatial embeddings of $G$ by $v(K_\times) = v(K_+) - v(K_-)$, where $K_\times, K_+$ and $K_-$ are singular spatial embeddings of $G$ which are identical except inside the depicted regions as illustrated in Fig. 1.2. Then $v$ is called a finite type invariant of order $\leq n$ if $v$ vanishes on every singular spatial embedding of $G$ with at least $n + 1$ double points [18], [1], [14]. If $v$ is of order $\leq n$ but not of order $\leq n - 1$, then $v$ is called a finite type invariant of order $n$.

We say that two spatial embeddings $f$ and $g$ of $G$ are $FT_n$-equivalent if $v(f) = v(g)$ for any finite type invariant $v$ of order $\leq n$. In particular for oriented knots, Goussarov and Habiro showed independently the following.

**Theorem 1.1.** ([2], [4]) Two oriented knots $J$ and $K$ are $C_k$-equivalent if and only if they are $FT_{k-1}$-equivalent.

The ‘only if’ part of Theorem 1.1 is also true for oriented links [2], [4] and spatial graphs [17], but the ‘if’ part does not always hold. For example, the Whitehead link and the trivial 2-component link are $FT_2$-equivalent but not $C_3$-equivalent [16]. By finding a basis for the space of finite type invariants for knots, we also have the following.

**Theorem 1.2.** Let $J$ and $K$ be two oriented knots. Then we have the following.

1. ([12], [8]) $J$ and $K$ are $C_2$-equivalent.
2. ([3]) $J$ and $K$ are $C_3$-equivalent if and only if $a_2(J) = a_2(K)$.
3. $J$ and $K$ are $C_4$-equivalent if and only if they are $C_3$-equivalent and $P^{(3)}_0(J; 1) = P^{(3)}_0(K; 1)$.
4. $J$ and $K$ are $C_5$-equivalent if and only if they are $C_4$-equivalent, $a_4(J) = a_4(K)$ and $P^{(4)}_0(J; 1) = P^{(4)}_0(K; 1)$.

Here, $a_n(\cdot)$ denotes the $n$th coefficient of the Conway polynomial and $P^{(n)}_m(\cdot; 1)$ denotes the $n$th derivative at 1 of the $m$th coefficient polynomial $P_m(\cdot; t)$. For spatial embeddings of a graph which may not be homeomorphic to the circle, a $C_k$-classification of them has been completed with the comparatively small $k$. The following table shows the present status of the completion of these classifications.
Let $\Theta$ be the theta curve and $K_4$ the complete graph on four vertices as illustrated in Fig. 1.3. Note that each of $\Theta$ and $K_4$ is planar and does not contain a pair of disjoint cycles. The following are $C_2$ and $C_3$-classifications of spatial embeddings of such graphs.

**Theorem 1.3.** Let $G$ be a planar graph which does not contain a pair of disjoint cycles and $f$ and $g$ two spatial embeddings of $G$. Then we have the following.

(1) ([15], [11]) $f$ and $g$ are $C_2$-equivalent.

(2) ([16]) $f$ and $g$ are $C_3$-equivalent if and only if $a_2(f(\gamma)) = a_2(g(\gamma))$ for any subgraph $\gamma$ of $G$ which is homeomorphic to the circle.

![Figure 1.3](image-url)
compact and orientable surface in $S^3$ such that $f(G)$ is a deformation retract of $S_f$ contained in the interior of $S_f$ [7]. In particular, if $G = \Theta$ or $K_4$, then the disk/band surface of $f(G)$ with zero Seifert linking form is unique with respect to $f$ under ambient isotopy [7], see Fig. 1.4.

Let $e_1, e_2, \ldots, e_l$ be all edges of $G$. Let $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ ($\varepsilon_i = 0, \pm 1, \infty$) be a surface in $S^3$ obtained from $S_f$ as illustrated in Fig. 1.5. Note that $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ depends only on $S_f$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$. Thus in the case of $\Theta$ and $K_4$, $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ is also the unique surface for $f$ if $S_f$ has zero Seifert linking form. To classify spatial theta curves and spatial complete graphs on four vertices under $C_4$ and $C_5$-equivalences, we use some $a_n(\cdot)$ and $P_n^{(m)}(\cdot; 1)$ for knots which appear as the boundary component of the surfaces above. For a knot $J$, recall that $P(J; t, z)$ does not depend on the orientation of $J$. Therefore $a_m(J) = P_m(J; 1)$ and $P_m^{(n)}(J; 1)$ also do not depend on the orientation of $J$.

2. Classification of spatial theta curves and spatial complete graphs on four vertices

First we state complete classifications of spatial theta curves under $C_4$ and $C_5$-equivalences. Let $f$ be a spatial theta curve and $S_f$ the disk/band surface of $f(\Theta)$
with zero Seifert linking form. We put
\[ J_1(f) = f(e_2 \cup e_3), \quad J_2(f) = f(e_1 \cup e_3), \quad J_3(f) = f(e_1 \cup e_2). \]

Let \( J_4(f) \) (resp. \( J_5(f), J_6(f) \)) be the component of \( \partial S_f(0, 0, -1) \) (resp. \( \partial S_f(-1, 0, 0), \partial S_f(0, 1, 0) \)) which is not corresponding to \( J_3(f) \) (resp. \( J_1(f), J_2(f) \)). Now we state classification theorems for spatial theta curves under \( C_4 \) and \( C_5 \)-equivalence.

**Theorem 2.1.** Two spatial theta curves \( f \) and \( g \) are \( C_4 \)-equivalent if and only if the following conditions hold:

1. \( f \) and \( g \) are \( C_3 \)-equivalent,
2. \( P_0^{(3)}(J_i(f); 1) = P_0^{(3)}(J_i(g); 1) \) (\( i = 1, 2, 3, 4 \)).

**Theorem 2.2.** Two spatial theta curves \( f \) and \( g \) are \( C_5 \)-equivalent if and only if the following conditions hold:

1. \( f \) and \( g \) are \( C_4 \)-equivalent,
2. \( a_4(J_i(f)) = a_4(J_i(g)) \) (\( i = 1, 2, 3, 5 \)),
3. \( P_0^{(4)}(J_i(f); 1) = P_0^{(4)}(J_i(g); 1) \) (\( i = 1, 2, 3, 5, 6 \)).

**Example 2.3.** There exists a spatial theta curve \( f \) such that \( J_i(f) \) is trivial for \( i = 1, 2, 3 \) but \( f \) is not \( C_4 \)-equivalent to the trivial spatial theta curve \( h \). For example, let \( f \) be Kinoshita’s theta curve as illustrated in Fig. 1.4. It is clear that \( J_i(f) \) is trivial for \( i = 1, 2, 3 \). But we have \( P_0^{(3)}(J_4(f); 1) = 48 \neq 0 \). Thus \( f \) and \( h \) are not \( C_4 \)-equivalent by Theorem 2.1. Note that \( f \) and \( h \) are \( C_3 \)-equivalent by Theorem 1.3.

Next we give complete classifications of spatial complete graphs on four vertices under \( C_4 \) and \( C_5 \)-equivalences. Let \( f \) be a spatial complete graph on four vertices
and $S_f$ the disk/band surface of $f(K_4)$ with zero Seifert linking form. We put
\[
J_1(f) = f(e_1 \cup e_2 \cup e_5 \cup e_6), \quad J_2(f) = f(e_2 \cup e_3 \cup e_4 \cup e_6),
\]
\[
J_3(f) = f(e_1 \cup e_3 \cup e_4 \cup e_5), \quad J_4(f) = f(e_2 \cup e_4 \cup e_5),
\]
\[
J_5(f) = f(e_3 \cup e_5 \cup e_6), \quad J_6(f) = f(e_1 \cup e_4 \cup e_6), \quad J_7(f) = f(e_1 \cup e_2 \cup e_3).
\]

Let $J_8(f)$ (resp. $J_9(f), J_{10}(f)$) be the component of $\partial S_f((\infty,0,0,0,0,0,0))$ which is not corresponding to $J_8(f)$ (resp. $J_9(f), J_{10}(f)$). Let $J_{11}(f)$ (resp. $J_{12}(f), J_{13}(f)$) be the component of $\partial S_f(0,1,1,1,1,1) \cup \partial S_f(0,0,0,0,0,0)$ which is not corresponding to $J_{11}(f)$. Let $J_{14}(f)$ (resp. $J_{15}(f)$) be the component of $\partial S_f(\infty,0,0,0,0,0)$ which is not corresponding to $J_{14}(f)$. Let $J_{16}(f)$ (resp. $J_{17}(f)$) be the component of $\partial S_f(0,0,0,0,0,0)$ which is not corresponding to $J_{16}(f)$. Let $J_{18}(f)$ (resp. $J_{19}(f)$) be the component of $\partial S_f(0,1,0,0,0)$ which is not corresponding to $J_{18}(f)$. Let $J_{20}(f)$ (resp. $J_{21}(f), J_{22}(f)$) be the component of $\partial S_f(0,0,0,0,0,0)$ which is not corresponding to $J_{20}(f)$. Let $J_{23}(f)$ (resp. $J_{24}(f)$) be the component of $\partial S_f(1,0,0,0,0,0)$ which is not corresponding to $J_{23}(f)$.

Now we state classification theorems for spatial complete graphs on four vertices under $C_4$ and $C_5$-equivalence.

**Theorem 2.4.** Two spatial complete graphs on four vertices $f$ and $g$ are $C_4$-equivalent if and only if the following conditions hold:

1. $f$ and $g$ are $C_5$-equivalent,
2. $P_0^{(3)}(J_i(f); 1) = P_0^{(3)}(J_i(g); 1)$ ($i = 1, 2, \ldots, 13$).

Let $\Theta_i$ be the subgraph of $\Theta$ which is obtained from $\Theta$ by deleting the edge $e_i$ ($i = 1, 2, \ldots, 6$). Note that $\Theta_i$ is homeomorphic to $\Theta$. Then, by Theorems 2.1 and 2.4, we have the following.

**Corollary 2.5.** Two spatial complete graphs on four vertices $f$ and $g$ are $C_4$-equivalent if and only if $f|_{\Theta_i}$ and $g|_{\Theta_i}$ are $C_4$-equivalent ($i = 1, 2, \ldots, 6$).

**Example 2.6.** Let $f$ and $g$ be two spatial complete graphs on four vertices as illustrated in Fig. 2.1. Since $J_i(f)$ is trivial for $i = 1, 2, \ldots, 7$, by Theorem 1.3 it follows that $f$ is $C_5$-equivalent to the trivial spatial complete graph on four vertices $h$. But we can see that $f|_{\Theta_i}$ is the Kinoshita's theta curve. Thus by Example 2.3 and Corollary 2.5, $f$ and $h$ are not $C_4$-equivalent. On the other hand, we can see that $g|_{\Theta_i}$ is trivial for $i = 1, 2, \ldots, 6$. Thus by Corollary 2.5, it follows that $g$ and $h$ are $C_5$-equivalent. Note that $g$ is not trivial under ambient isotopy.
Theorem 2.7. Two spatial complete graphs on four vertices $f$ and $g$ are $C_5$-equivalent if and only if the following conditions hold:

1. $f$ and $g$ are $C_4$-equivalent,
2. $a_4(J_i(f)) = a_4(J_i(g))$ ($i = 1, 2, \ldots, 16$),
3. $P_0^{(4)}(J_i(f); 1) = P_0^{(4)}(J_i(g); 1)$ ($i = 1, 2, \ldots, 24$).

Example 2.8. There exists a spatial complete graph on four vertices $f$ such that $f|_{\Theta_i}$ is $C_5$-equivalent to the trivial spatial embedding of $\Theta_i$ for $i = 1, 2, \ldots, 6$ but $f$ is not $C_5$-equivalent to the trivial spatial complete graph on four vertices $h$. For example, let $f$ be the spatial complete graph on four vertices as illustrated in Fig. 2.2. We can see that $f|_{\Theta_i}$ is trivial for $i = 1, 2, \ldots, 5$. Though $f|_{\Theta_6}$ is not trivial, by checking the conditions in Theorem 2.2, we can see that $f|_{\Theta_6}$ is $C_5$-equivalent to the trivial spatial embedding of $\Theta_6$. But we have $P_0^{(4)}(J_{20}(f); 1) = 384 \neq 0$. Thus $f$ and $h$ are not $C_5$-equivalent by Theorem 2.7.

Question 2.9. Does there exist a spatial complete graph on four vertices $f$ such that $f|_{\Theta_i}$ is trivial for $i = 1, 2, \ldots, 6$ but $f$ is not $C_5$-equivalent to the trivial spatial complete graph on four vertices?
Let $f$ be a spatial theta curve (resp. spatial complete graph on four vertices) and $S_f$ the disk/band surface of $f(\Theta)$ (resp. $f(K_4)$) with zero Seifert linking form. Then it is known that a finite type invariant of order $\leq n$ of $\partial S_f(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ (resp. $\partial S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6)$) is also a finite type invariant of order $\leq n$ of $f$ [13], [19]. On the other hand, $a_n(\cdot)$ is a finite type invariant of order $\leq n$ [1], and $P_0^{(n)}(\cdot; 1)$ is a finite type invariant of order $\leq n$ [6]. Therefore, by Theorems 1.3, 2.1, 2.2, 2.4 and 2.7, we have the following.

**Corollary 2.10.** Let $G = \Theta$ or $K_4$. For $k \leq 5$, two spatial embeddings $f$ and $g$ of $G$ are $C_k$-equivalent if and only if they are $FT_{k-1}$-equivalent.

**Remark 2.11.**

1. Proofs of Theorems 2.1, 2.2, 2.4 and 2.7 are done by showing a slightly modified version of Meilhan and the second author’s $C_4$ and $C_5$-classifications of string links [10].

2. Let $H$ be the handcuff graph, which is constructed by connecting two loops by a single edge. Then, there exists a spatial handcuff graph $f$ such that $f$ is $FT_{k-1}$-equivalent to the trivial spatial handcuff graph $h$ but not $C_k$-equivalent to $h$ for $k = 3, 4, 5$. Classification of spatial handcuff graphs under $C_k$-equivalence for $k = 3, 4, 5$ are due to be mentioned in [5].

**References**


Department of Mathematics, School of Arts and Sciences
Tokyo Woman’s Christian University
2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585, Japan
nick@lab.twcu.ac.jp