A remark on the identifiable projections of planar graphs

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Abstract

A generic immersion of a graph into the 2-space is said to be *identifiable* if any two embeddings of the graph into the 3-space obtained by lifting the immersion with respect to the natural projection from the 3-space to the 2-space are ambient isotopic. We show that a generic immersion of a planar graph is identifiable if and only if any of the spatial embeddings obtained by lifting the immersion is trivial. Besides we characterize almost identifiable projections for a class of planar graphs.

Keywords: Spatial graph; Identifiable projection; Strongly almost trivial embedding
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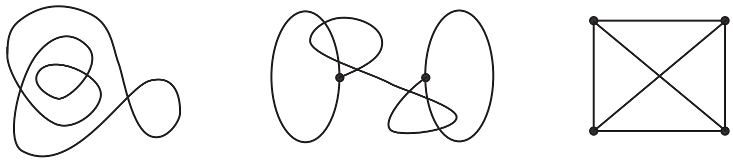
1. Introduction

Throughout this paper we work in the piecewise linear category. Let G be a finite graph. We refer the reader to [1] for fundamental terminology of the graph theory. We consider G as a topological space in the usual way. An embedding of G into $\mathbf{R}^3$ is called a *spatial embedding* of G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G into $\mathbf{R}^2$. A spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding of the graph into $\mathbf{R}^2 \subset \mathbf{R}^3$. Such an embedding is unique up to ambient isotopy in $\mathbf{R}^3$ [8].

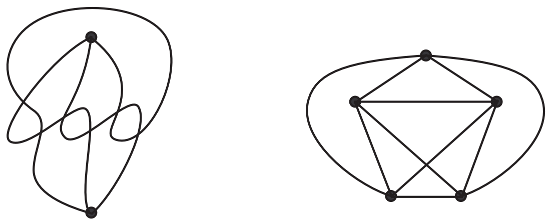
A *regular projection* of G is a continuous map $G \to \mathbf{R}^2$ whose multiple points are only finitely many transversal double points away from vertices. For a regular projection $\hat{f}$ of G with p double points, we can obtain 2^p regular diagrams of the spatial embeddings of G from $\hat{f}$ by giving over/under information to each double point. Then we say that a spatial embedding f of G is *obtained from* $\hat{f}$ if f is ambient isotopic to a spatial embedding of G which is represented by one of these 2^p regular diagrams. Then we also call $\hat{f}$ a *regular projection of* f.

A regular projection $\hat{f}$ of a graph is said to be *identifiable* [6] if any two spatial embeddings of the graph obtained from $\hat{f}$ are ambient isotopic. For example, each of the regular projections as illustrated in Fig. 1.1 (1), (2) and (3) is identifiable. It is known that only planar graphs have identifiable projections [6, Proposition 1.1]. A regular projection $\hat{f}$ of a graph G is said to be *almost identifiable* if $\hat{f}|_H$ is identifiable for any proper subgraph H of G. It is clear that if $\hat{f}$ is identifiable then it is almost identifiable. But the converse is not true. For example, let us consider the regular projections as illustrated in Fig. 1.2 (1) and (2). We can see that each of the projections is almost identifiable. But (1) is not identifiable because Kinoshita's theta curve can be obtained from (1), and (2) is also not identifiable because the graph is the *complete graph* on five vertices K_5, namely non-planar. A regular projection $\hat{f}$ of a graph G is said to be *minimally unidentifiable* if $\hat{f}$ is almost identifiable but not identifiable.



(1) (2) (3)

Fig. 1.1.



(1) (2)

Fig. 1.2.

If a graph G is homeomorphic to the disjoint union of 1-spheres, then the following is clear because a trivial link is always obtained from any regular projection of G.

Proposition 1.1. *Let G be a graph which is homeomorphic to the disjoint union of 1-spheres. Then, a regular projection of G is identifiable if and only if any of the spatial embeddings obtained from the projection is a trivial link.* $\square$

In this paper we generalize Proposition 1.1 to arbitrary planar graphs as follows.

Theorem 1.2. *A regular projection of a planar graph is identifiable if and only if any of the spatial embeddings obtained from the projection is trivial.*

Note that there exist infinitely many planar graphs which have a *knotted* regular projection [14], [12], [9], namely any of the spatial embeddings obtained from the projection is non-trivial. So the proof of Proposition 1.1 does not work for the proof of Theorem 1.2. We prove Theorem 1.2 in the next section.

A non-trivial spatial embedding f of a planar graph G is said to be *strongly almost trivial* [3] if there exists a regular projection $\hat{f}$ of f such that any of the spatial embeddings of H obtained from $\hat{f}|_H$ is trivial for any proper subgraph H of G. For a strongly almost trivial spatial embedding f of a planar graph G, it is clear that there exists a regular projection $\hat{f}$ of f such that $\hat{f}$ is minimally unidentifiable. Conversely, we have the following as a direct consequence of Theorem 1.2.

Corollary 1.3. *Let $\hat{f}$ be a regular projection of a planar graph. If $\hat{f}$ is minimally unidentifiable then there exists a spatial embedding f of the graph obtained from $\hat{f}$ such that f is strongly almost trivial.* $\square$

By Corollary 1.3, we can characterize almost identifiable projections for a class of planar graphs. In the next section we show the following.

Theorem 1.4. *Suppose that a planar graph G has no strongly almost trivial spatial embeddings. Then a regular projection of G is almost identifiable if and only if it is identifiable.*

We also give some examples in the next section.

2. Proofs and examples

A spatial embedding f of a graph G is said to be *free* if $\pi_1(\mathbf{R}^3 - f(G))$ is a free group. The following is Scharlemann-Thompson's famous criterion.

Theorem 2.1. ([11]) *For a planar graph G, a spatial embedding f of G is trivial if and only if $f|_H$ is free for any subgraph H of G.* $\square$

On the other hand, the following is known.

Lemma 2.2. ([9, Proposition 3.2]) *Let $\hat{f}$ be a regular projection of a graph. Then there exists a free spatial embedding of the graph obtained from $\hat{f}$.* $\square$

Proof of Theorem 1.2. We have the 'if' part by the uniqueness of the trivial spatial embeddings of a planar graph. Next we show the 'only if' part. Let $\hat{f}$ be an identifiable projection of a planar graph G and f the spatial embedding of G obtained from $\hat{f}$. We note that $\hat{f}|_H$ is also identifiable for any subgraph H of G. Then by Lemma 2.2 the spatial embedding g of H obtained from $\hat{f}|_H$ is free. Since $g = f|_H$, we have that $f|_H$ is free for any subgraph H of G. Therefore by Theorem 2.1 we have that f is trivial. This completes the proof. $\square$

Proof of Theorem 1.4. We have the 'if' part immediately. Next we show the 'only if' part. Let $\hat{f}$ be an almost identifiable projection of G. Suppose that $\hat{f}$ is not identifiable. Then $\hat{f}$ is minimally unidentifiable and therefore by Corollary 1.3 there exists a strongly almost trivial spatial embedding of G obtained from $\hat{f}$. This is a contradiction. Thus we have that $\hat{f}$ is identifiable. $\square$

Example 2.3. Let us consider the following two conditions of graphs:

(C1) If e_1, e_2 and e_3 are three edges of G such that $e_1 \cup e_2 \cup e_3$ is a path of G then there exists a cycle γ of G such that $e_1 \cup e_2 \cup e_3 \subset \gamma$.

(C2) If e_1 and e_2 are disjoint edges of G then there exist two disjoint cycles γ_1 and γ_2 of G such that $e_i \subset \gamma_i$ $(i = 1, 2)$.

It is known that every 3-connected graph satisfies the condition (C1), and every 4-connected planar graph satisfies the condition (C2) [6, Proposition 2.1]. Hence, for example, a 4-connected planar graph satisfies the conditions (1) and (2). Huh and Oh showed that if a simple 2-connected planar graph G satisfies the conditions (C1) and (C2) then G has no strongly almost trivial spatial embeddings [5, Theorem 1.1]. Thus by Theorem 1.4 we have that a regular projection of a simple 2-connected planar graph G satisfying the conditions (C1) and (C2) is almost identifiable if and only if it is identifiable.

Example 2.4. Though the complete graph on four vertices K_4 does not satisfy the condition (C2), Huh and Oh showed that K_4 has no strongly almost trivial spatial embeddings [5, Example 2]. Thus by Theorem 1.4 we have that a regular projection of K_4 is almost identifiable if and only if it is identifiable.

A regular projection of a graph is said to be *reduced* if the image of the projection has no local parts as illustrated in Fig. 2.1 (a) and (b). For example, each of the regular projections as illustrated in Fig. 1.1 (2) and (3) is reduced. Huh and Taniyama showed that any reduced identifiable projection of a simple 2-connected planar graph G satisfying the conditions (C1) and (C2) is an embedding from G to $\mathbf{R}^2$ [6, Theorem 1.2]. Thus by Example 2.3 we have the following.

Corollary 2.5. *Let G be a simple 2-connected planar graph satisfying the conditions (C1) and (C2). If a regular projection of G is reduced and almost* *identifiable then it is an embedding from G to $\mathbf{R}^2$.* $\square$

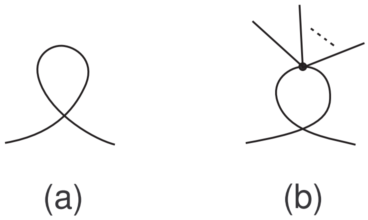


Fig. 2.1.

On the other hand, a reduced (almost) identifiable projection of K_4 is not always an embedding from K_4 to $\mathbf{R}^2$, see Fig. 1.1 (3).

Remark 2.6. As we saw in Corollary 2.5, there exist infinitely many planar graphs whose reduced identifiable projections are only embeddings from the graph to $\mathbf{R}^2$ (see also [2], [10], [13], [7], [15], [4]). Theorem 1.2 indicates that the spatial embedding of a planar graph obtained from an identifiable projection of the graph is always trivial even if the projection is reduced and not an embedding into $\mathbf{R}^2$.

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