Homotopy on Spatial Graphs and Generalized Sato-Levine Invariants

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1. Introduction

An embedding f of a finite graph G into the 3-sphere $\mathbb{S}^3$ is called a *spatial embedding of* G or simply a *spatial graph*. We call the image of f restricted on a cycle (resp. mutually disjoint cycles) in G a *constituent knot* (resp. *constituent link*) of f, where a *cycle* is a graph homeomorphic to a circle. A spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding of the graph into a 2-sphere in $\mathbb{S}^3$. A spatial embedding f of G is said to be *split* if there exists a 2-sphere S in $\mathbb{S}^3$ such that $S \cap f(G) = \emptyset$ and each connected component of $\mathbb{S}^3 - S$ has intersection with $f(G)$, and otherwise f is said to be *non-splittable*.

Two spatial embeddings of G are said to be *edge-homotopic* if they are transformed into each other by *self crossing changes* and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge, and *vertex-homotopic* if they are transformed into each other by crossing changes on two adjacent spatial edges and ambient isotopies. These equivalence relations were introduced by Taniyama [12] as generalizations of Milnor's *link-homotopy* on oriented links [7], namely if G is a mutually disjoint union of cycles then these are none other than link-homotopy. It is known that edge (resp. vertex)-homotopy on spatial graphs behaves quite differently than link-homotopy on oriented links. Taniyama introduced the α*-invariant* of spatial graphs by taking a weighted sum of the second coefficient of the Conway polynomial of the constituent knots [11]. By applying the α-invariant, it is shown that the spatial embedding of K_4 as illustrated in Fig. 1.1 (1) is not trivial up to edge-homotopy, and two spatial embeddings of $K_{3,3}$ as illustrated in Fig. 1.1 (2) and (3) are not vertex-homotopic. Note that each of these spatial graphs does not have a constituent link. On the other hand, some invariants of spatial graphs defined by taking a weighted sum of the third coefficient of the Conway polynomial of the constituent 2-component links were introduced by Taniyama as $\mathbb{Z}_2$-valued invariants if the linking numbers are even [13], and by Fleming and the author as integer-valued invariants if the linking numbers vanish [3]. By applying these invariants, it is shown that each of the spatial graphs as illustrated in Fig. 1.2 (1) and (2) is non-splittable up to edge-homotopy, and the spatial graph as illustrated in Fig. 1.2 (3) is non-splittable up to vertex-homotopy. Note that each of these spatial graphs does not contain a constituent link which is not trivial up to link-homotopy.

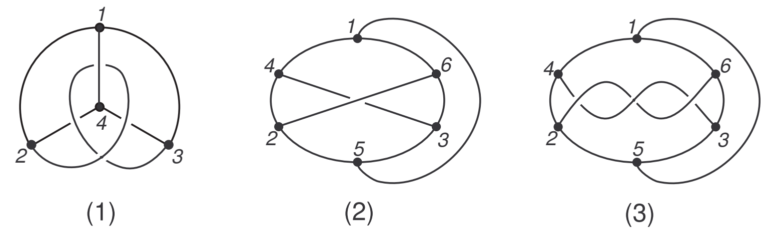

Fig. 1.1.

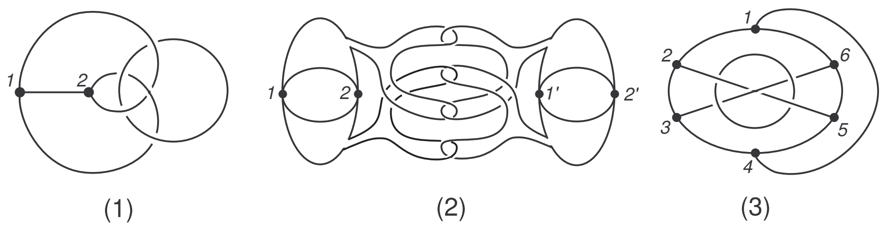

Fig. 1.2.

In this report, we construct some new edge (resp. vertex)-homotopy invariants of spatial graphs without any restriction of linking numbers of the constituent 2-component links by applying a weighted sum of the *generalized Sato-Levine invariant*. Here the generalized Sato-Levine invariant $\tilde{\beta}(L) = \tilde{\beta}(K_1, K_2)$ is an ambient isotopy invariant of an oriented 2-component link $L = K_1 \cup K_2$ which appears in various ways independently [1], [2], [5], [6], [4], [8] and can be calculated by

$$\tilde{\beta}(L) = a_3(L) - \mathrm{lk}(L)\left\{a_2(K_1) + a_2(K_2)\right\},$$

where a_i denotes the i-th coefficient of the Conway polynomial and $\mathrm{lk}(L) = \mathrm{lk}(K_1, K_2)$ denotes the linking number of L. It is known that if $\mathrm{lk}(L) = 0$ then $\tilde{\beta}(L)$ coincides with the original *Sato-Levine invariant* $\beta(L)$ defined in [10]. As a consequence, our invariants are generalizations of Fleming and the author's homotopy invariants of spatial graphs defined in [3].

In this report, we narrow our results down to edge-homotopy invariants and omit to give a concrete proof for some propositions. Please see [9] for the details.

2. Some formulas about the generalized Sato-Levine invariant

We need the following two formulas, 'self crossing change formula' and 'orientation-inverting formula', for the generalized Sato-Levine invariant of oriented 2-component links.

Lemma 2.1. *Let* $L_+ = J_+ \cup K$ *and* $L_- = J_- \cup K$ *be two oriented* 2*-component links and* $L_0 = J_1 \cup J_2 \cup K$ *an oriented* 3*-component link which are identical except inside the depicted regions as illustrated in Fig. 2.1. Suppose that* $\operatorname{lk}(L_+) = \operatorname{lk}(L_-) = m$. *Then it holds that*

$$\tilde{\beta}(L_+) - \tilde{\beta}(L_-) = \operatorname{lk}(K, J_i)\,\{m - \operatorname{lk}(K, J_i)\} \ \ (i = 1, 2).$$

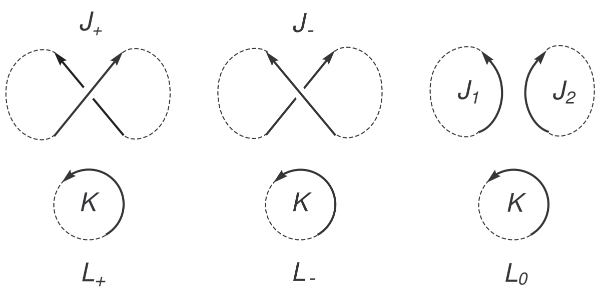

Fig. 2.1.

Theorem 2.2. *Let* $L = J_1 \cup J_2$ *be an oriented* 2*-component link with* $\operatorname{lk}(L) = m$. *Let* $L' = (-J_1) \cup J_2$ *be the oriented* 2*-component link obtained from* L *by inverting the orientation of* J_1. *Then it holds that*

$$\tilde{\beta}(L) - \tilde{\beta}(L') = \frac{1}{6}(m^3 - m).$$

Remark 2.3. Let f be a spatial embedding of a graph G and $\gamma,\ \gamma'$ two disjoint cycles of G. By Theorem 2.2, if $\operatorname{lk}(f(\gamma), f(\gamma')) = 0,\ \pm 1$ then the value of $\tilde{\beta}(f(\gamma), f(\gamma'))$ does not depend on the orientation of $f(\gamma)$ and $f(\gamma')$, namely it is well-defined. But if $\operatorname{lk}(f(\gamma), f(\gamma')) \neq 0$, then Theorem 2.2 implies that the value of $\tilde{\beta}(f(\gamma), f(\gamma'))$ have the indeterminacy arisen from a choice of the orientations of $f(\gamma)$ and $f(\gamma')$.

3. Definitions of invariants

From now onward, we assume that a graph G is *oriented*, namely an orientation is given for each edge of G. For a subgraph H of G, we denote the set of all cycles of H by $\Gamma(H)$. For an edge e of H, we denote the set of all oriented cycles of H which contain the edge e and have the orientation induced by the orientation of e by $\Gamma_e(H)$. We set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ for a positive integer n and $\mathbb{Z}_0 = \mathbb{Z}$. We call a map $\omega : \Gamma(H) \to \mathbb{Z}_n$ a *weight on* $\Gamma(H)$ *over* $\mathbb{Z}_n$. Then we say that a weight ω on $\Gamma(H)$ over $\mathbb{Z}_n$ is *weakly balanced on an edge* e if

$$\sum_{\gamma \in \Gamma_e(H)} \omega(\gamma) \equiv 0 \pmod{n}.$$

Let $G = G_1 \cup G_2$ be a disjoint union of two graphs, ω_i a weight on $\Gamma(G_i)$ over $\mathbb{Z}_n$ $(i = 1, 2)$ and f a spatial embedding of G. Then we say that a weight ω_i is *null-homologous on an edge* e *of* G_i *with respect to* f *and* ω_j $(i \neq j)$ if

$$\mathrm{lk}\left(\sum_{\gamma \in \Gamma_e(G_i)} \omega_i(\gamma) f(\gamma), f(\gamma') \right) \equiv 0 \pmod{n}$$

for any $\gamma' \in \Gamma(G_j)$ with $\omega_j(\gamma') \neq 0$.

Example 3.1. Let $G = G_1 \cup G_2$ is the graph as illustrated in Fig. 3.1. We denote the cycle $e_i \cup e_j$ of G_1 by γ_{ij}. Let ω_1 be the weight on $\Gamma(G_1)$ over $\mathbb{Z}$ defined by

$$\omega_1(\gamma) = \begin{cases} 1 & (\gamma = \gamma_{12},\ \gamma_{34}) \\ -1 & (\gamma = \gamma_{23},\ \gamma_{14}) \\ 0 & (\text{otherwise}), \end{cases}$$

and ω_2 the weight on $\Gamma(G_2)$ over $\mathbb{Z}$ defined by $\omega_2(\gamma') = 1$. Let f be the spatial embedding of G as illustrated in Fig. 3.1. Note that

$$\Gamma_{e_1}(G_1) = \{\gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{e_1 + e_2,\ e_1 - e_3,\ e_1 + e_4\}$$

and

$$\sum_{\gamma \in \Gamma_{e_1}(G_1)} \omega_1(\gamma)\gamma = (e_1 + e_2) - (e_1 + e_4) = e_2 - e_4.$$

Then we have that

$$\mathrm{lk}\left(\sum_{\gamma \in \Gamma_{e_1}(G_1)} \omega_1(\gamma) f(\gamma), f(\gamma') \right) = \mathrm{lk}\,(f(e_2 - e_4), f(\gamma')) = 0.$$

Therefore ω_1 is null-homologous on e_1 with respect to f and ω_2.

Now let $G = G_1 \cup G_2$ be a disjoint union of graphs, ω_i a weight on $\Gamma(G_i)$ over $\mathbb{Z}_n$ $(i = 1, 2)$ and f a spatial embedding of G. For $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$, we put

$$\eta(f(\gamma), f(\gamma')) = \frac{1}{6}(m^3 - m)$$

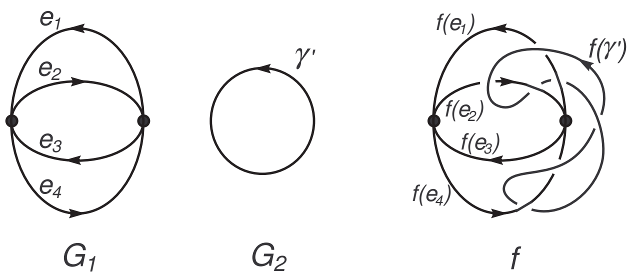

Fig. 3.1.

where $m = \mathrm{lk}(f(\gamma), f(\gamma'))$ under arbitrary orientations of γ and γ'. Then we put

$$\tilde{\eta}_{\omega_1,\omega_2}(f) = \gcd\{\eta(f(\gamma), f(\gamma')) \mid \gamma \in \Gamma(G_1),\ \gamma' \in \Gamma(G_2),\ \omega_1(\gamma)\omega_2(\gamma') \not\equiv 0 \pmod n\},$$

where gcd means the greatest common divisor. Note that $\tilde{\eta}_{\omega_1,\omega_2}(f)$ is a well-defined non-negative integer which does not depends on the choice of orientations of each pair of disjoint cycles. Then we define $\tilde{\beta}_{\omega_1,\omega_2}(f) \in \mathbb{Z}_n$ by

$$\tilde{\beta}_{\omega_1,\omega_2}(f) \equiv \sum_{\substack{\gamma\in\Gamma(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\tilde{\beta}(f(\gamma), f(\gamma')) \pmod{\gcd\{n, \tilde{\eta}_{\omega_1,\omega_2}(f)\}}.$$

Here we may calculate $\tilde{\beta}(f(\gamma), f(\gamma'))$ under arbitrary orientations of γ and γ'.

Remark 3.2. (1) For an oriented 2-component link L, $\tilde{\beta}(L)$ is not a link-homotopy invariant of L. Thus $\tilde{\beta}_{\omega_1,\omega_2}(f)$ may be not an edge (resp. vertex)-homotopy invariant of f as it is. See also Remark 4.4.
(2) By Theorem 2.2, the value of $\tilde{\beta}(f(\gamma), f(\gamma'))$ is well-defined modulo $\eta(f(\gamma), f(\gamma'))$. This is the reason why we consider the modulo $\tilde{\eta}_{\omega_1,\omega_2}(f)$ reduction.

Then, let us state the invariance of $\tilde{\beta}_{\omega_1,\omega_2}$ up to edge-homotopy under some conditions on graphs and its spatial embeddings.

Theorem 3.3. *If ω_i is weakly balanced on any edge of G_i and null-homologous on any edge of G_i with respect to f and ω_j ($i = 1, 2,\ i \neq j$), then $\tilde{\beta}_{\omega_1,\omega_2}(f)$ is an edge-homotopy invariant of f.*

Proof. Let f and g be two spatial embeddings of G such that g is edge-homotopic to f. Then it holds that

$$\tilde{\eta}_{\omega_1,\omega_2}(f) = \tilde{\eta}_{\omega_1,\omega_2}(g) \tag{3.1}$$

because the linking number of a constituent 2-component link of a spatial graph is an edge-homotopy invariant. First we show that if f is transformed into g by self crossing changes on $f(G_1)$ and ambient isotopies then $\tilde{\beta}_{\omega_1,\omega_2}(f) = \tilde{\beta}_{\omega_1,\omega_2}(g)$. It is clear that any link invariant of a constituent link of a spatial graph is also an ambient isotopy invariant of the spatial graph. Thus we may assume that g is obtained from f by a single crossing change on $f(e)$ for an edge e of G_1 as illustrated in Fig. 3.2. Moreover, by smoothing on this crossing point we can obtain the spatial embedding h of G and the knot J_h as illustrated in Fig. 3.2. Then by (3.1), Lemma 2.1 and the assumptions for ω_1, we have that

$$\begin{aligned}
\tilde{\beta}_{\omega_1,\omega_2}(f) - \tilde{\beta}_{\omega_1,\omega_2}(g) &\equiv \sum_{\substack{\gamma\in\Gamma_e(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\left\{\tilde{\beta}(f(\gamma),f(\gamma')) - \tilde{\beta}(g(\gamma),g(\gamma'))\right\} \\
&= \sum_{\substack{\gamma\in\Gamma_e(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(h(\gamma'),J_h)\left\{\mathrm{lk}(f(\gamma),f(\gamma')) - \mathrm{lk}(h(\gamma'),J_h)\right\} \\
&= \sum_{\gamma'\in\Gamma(G_2)} \omega_2(\gamma')\Bigg\{\mathrm{lk}(h(\gamma'),J_h)\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\mathrm{lk}(f(\gamma),f(\gamma')) \\
&\qquad - \sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\mathrm{lk}(h(\gamma'),J_h)^2\Bigg\} \\
&= \sum_{\gamma'\in\Gamma(G_2)} \omega_2(\gamma')\Bigg\{\mathrm{lk}(h(\gamma'),J_h)\mathrm{lk}\left(\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)f(\gamma),f(\gamma')\right) \\
&\qquad -\mathrm{lk}(h(\gamma'),J_h)^2\left(\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\right)\Bigg\} \\
&\equiv 0 \pmod{\gcd\{n,\tilde{\eta}_{\omega_1,\omega_2}(f)\}}.
\end{aligned}$$

Therefore we have that $\tilde{\beta}_{\omega_1,\omega_2}(f) = \tilde{\beta}_{\omega_1,\omega_2}(g)$. In the same way we can show that if f is transformed into g by self crossing changes on $f(G_2)$ and ambient isotopies then $\tilde{\beta}_{\omega_1,\omega_2}(f) = \tilde{\beta}_{\omega_1,\omega_2}(g)$. Thus $\tilde{\beta}_{\omega_1,\omega_2}(f)$ is an edge-homotopy invariant of f. □

Remark 3.4. In particular, if it holds that

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma),f(\gamma')) = 0$$

for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$, then $\tilde{\beta}_{\omega_1,\omega_2}(f)$ coincides with Fleming and the author's invariant $\beta_{\omega_1,\omega_2}(f)$ defined in [3].

4. Examples

Let G be a planar graph which is not a cycle. An embedding $p : G \to \mathbb{S}^2$ is said to be *cellular* if the closure of each of the connected components of $\mathbb{S}^2 - p(G)$ on $\mathbb{S}^2$

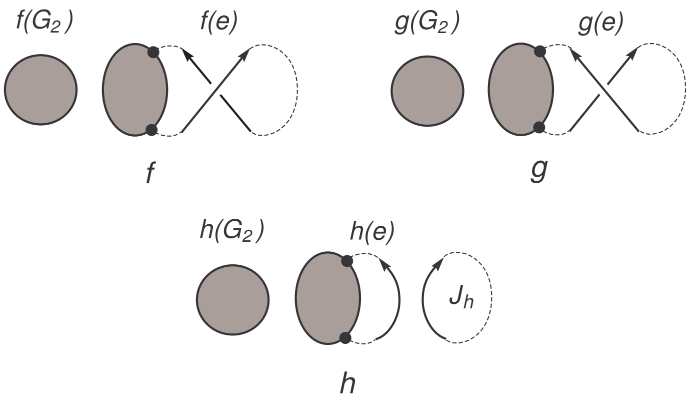

Fig. 3.2.

is homeomorphic to the disk. Then we regard the set of the boundaries of all of the connected components of $\mathbb{S}^2 - p(G)$ as a subset of $\Gamma(G)$ and denote it by $\Gamma_p(G)$. We say that G *admits a checkerboard coloring on* $\mathbb{S}^2$ if there exists a cellular embedding $p : G \to \mathbb{S}^2$ such that we can color all of the connected components of $\mathbb{S}^2 - p(G)$ by two colors (black and white) so that any of the two components which are adjacent by an edge have distinct colors. We denote the subset of $\Gamma_p(G)$ which corresponds to the black (resp. white) colored components by $\Gamma_p^b(G)$ (resp. $\Gamma_p^w(G)$).

Proposition 4.1. *Let G be a planar graph which is not a cycle and admits a checkerboard coloring on $\mathbb{S}^2$ with respect to a cellular embedding $p : G \to \mathbb{S}^2$. Let ω_p be the weight on $\Gamma(G)$ over $\mathbb{Z}_n$ defined by*

$$\omega_p(\gamma) = \begin{cases} 1 & (\gamma \in \Gamma_p^b(G)) \\ n-1 & (\gamma \in \Gamma_p^w(G)) \\ 0 & (\gamma \in \Gamma(G) - \Gamma_p(G)). \end{cases}$$

Then ω_p is weakly balanced on any edge of G.

We call the weight ω_p in Proposition 4.1 a *checkerboard weight*. Moreover, by giving the counter clockwise orientation to each $p(\gamma)$ for $\gamma \in \Gamma_p^b(G)$ and the clockwise orientation to each $p(\gamma)$ for $\gamma \in \Gamma_p^w(G)$ with respect to the orientation of $\mathbb{S}^2$, an orientation is given for each edge of G naturally. We call this orientation of G a *checkerboard orientation* over the checkerboard coloring. Since the orientation of each edge e is coherent with the orientation of each cycle $\gamma \in \Gamma_p(G)$ which contains e, by Theorem 3.3 we have the following.

Theorem 4.2. *Let $G = G_1 \cup G_2$ be a disjoint union of two planar graphs such that G_i is not a cycle and admits a checkerboard coloring on $\mathbb{S}^2$ with respect to a* *cellular embedding $p_i : G \to \mathbb{S}^2$ $(i = 1, 2)$. Let ω_{p_i} be the checkerboard weight on $\Gamma(G_i)$ over $\mathbb{Z}_n$ $(i = 1, 2)$. We orient G by the checkerboard orientation of G_i over the checkerboard coloring $(i = 1, 2)$. Then, for a spatial embedding f of G, if ω_i is null-homologous on any edge of G_i with respect to f and ω_j $(i = 1, 2,\ i \neq j)$, then $\tilde{\beta}_{\omega_1,\omega_2}(f) \pmod n$ is an edge-homotopy invariant of f.*

Example 4.3. Let $G = G_1 \cup G_2$ be a disjoint union of two planar graphs as in Theorem 4.2 and f a spatial embedding of G. Let $\omega_{p_i} : \Gamma(G_i) \to \mathbb{Z}_n$ be the checkerboard weight $(i = 1, 2)$, where

$$n = \gcd \left\{ \mathrm{lk}(f(\gamma), f(\gamma')) \mid \gamma \in \Gamma_{p_1}(G_1),\ \gamma' \in \Gamma_{p_2}(G_2) \right\}.$$

Then, for any edge e of G_i and any $\gamma' \in \Gamma_{p_j}(G_j)$ $(i \neq j)$, we have that

$$\mathrm{lk}\left(\sum_{\gamma \in \Gamma_e(G_i)} \omega_i(\gamma) f(\gamma), f(\gamma') \right) = \sum_{\gamma \in \Gamma_e(G_i)} \omega_i(\gamma) \mathrm{lk}\left(f(\gamma), f(\gamma')\right) \equiv 0 \pmod n.$$

Thus we have that ω_i is null-homologous on any edge of G_i with respect to f and ω_j $(i = 1, 2,\ i \neq j)$. Therefore we have that $\tilde{\beta}_{\omega_{p_1},\omega_{p_2}}(f) \pmod n$ is an edge-homotopy invariant of f.

For example, let Θ_4 be the graph with two vertices u and v and 4 edges e_1, e_2, e_3, e_4 each of which joins u and v. We denote the cycle of Θ_4 consists of two edges e_i and e_j by γ_{ij}. Let $p : \Theta_4 \to \mathbb{S}^2$ be the cellular embedding as illustrated in the left-hand side of Fig. 4.1. It is clear that Θ_4 admits the checkerboard coloring on $\mathbb{S}^2$ with respect to p as illustrated in the center of Fig. 4.1. The right-hand side of Fig. 4.1 shows the checkerboard orientation of Θ_4 over the checkerboard coloring.

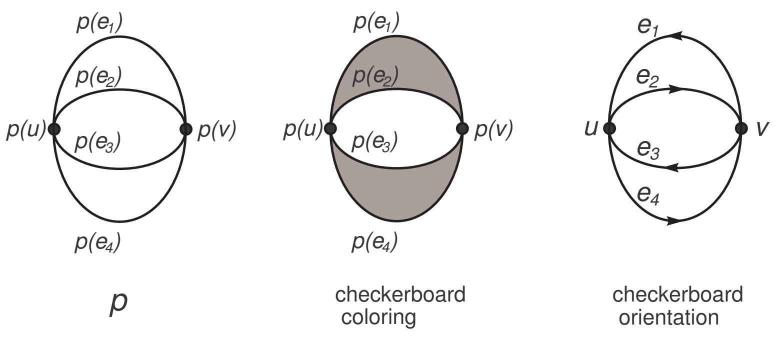

Fig. 4.1.

Let $G = \Theta_4^1 \cup \Theta_4^2$ be a disjoint union of two copies of Θ_4. For a non-negative integer m, let f_m and g_m be two spatial embeddings of G as illustrated in Fig. 4.2. Note that

$$\mathrm{lk}(f_m(\gamma), f_m(\gamma')) = \mathrm{lk}(g_m(\gamma), g_m(\gamma')) = 0 \text{ or } m$$

for any $\gamma \in \Gamma(\Theta_4^1)$ and $\gamma' \in \Gamma(\Theta_4^2)$. So we have that $n = m$. Let $\omega_i : \Gamma(\Theta_4^i) \to \mathbb{Z}_m$ be the checkerboard weight ($i = 1, 2$). Then, by a direct calculation we can see that the constituent 2-component link of f_m which has a non-zero generalized Sato-Levine invariant is only $L = f_m(\gamma_{14} \cup \gamma'_{14})$ and $\tilde{\beta}(L) = 2$. Thus we have that $\tilde{\beta}_{\omega_1,\omega_2}(f_m) \equiv 2 \pmod{m}$. On the other hand, we can see that each constituent 2-component link $g_m(\gamma \cup \gamma')$ for $\gamma \in \Gamma_p(\Theta_4^1)$ and $\gamma' \in \Gamma_p(\Theta_4^2)$ is a trivial 2-component link or T'_m. Thus we have that $\tilde{\beta}_{\omega_1,\omega_2}(g_m) \equiv 0 \pmod{m}$. Therefore we have that f_m and g_m are not edge-homotopic if $m \neq 1, 2$. We remark here that the case of $m = 0$ has already shown by Fleming and the author in [3, Example 4.3].

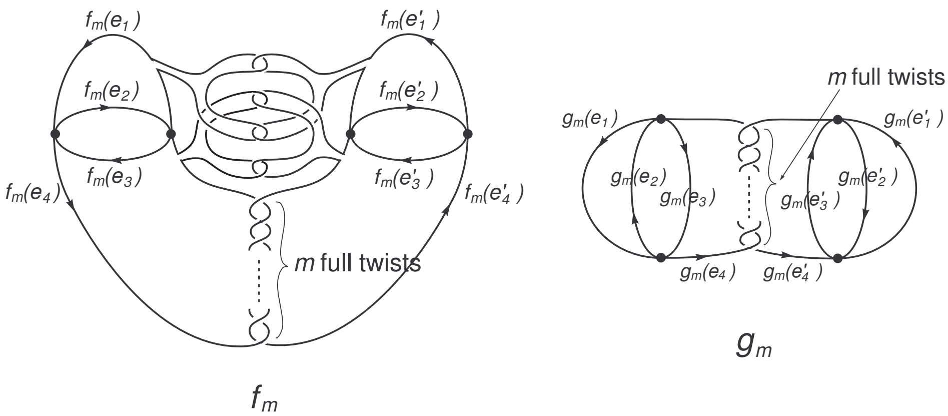

Fig. 4.2.

Example 4.4. Let $G = \Theta_4^1 \cup \Theta_4^2$ be a disjoint union of two copies of Θ_4 oriented in the same way as Example 4.3 and $\omega_i : \Gamma(\Theta_4^i) \to \mathbb{Z}$ the checkerboard weight ($i = 1, 2$). For Θ_4^1, we have that

$$\sum_{\gamma \in \Gamma_{e_1}(\Theta_4^1)} \omega_1(\gamma)\gamma = e_2 - e_4, \quad \sum_{\gamma \in \Gamma_{e_2}(\Theta_4^1)} \omega_1(\gamma)\gamma = e_1 - e_3,$$
$$\sum_{\gamma \in \Gamma_{e_3}(\Theta_4^1)} \omega_1(\gamma)\gamma = e_4 - e_2, \quad \sum_{\gamma \in \Gamma_{e_4}(\Theta_4^1)} \omega_1(\gamma)\gamma = e_3 - e_1.$$

This implies that ω_1 is null-homologous on any edge of G_1 with respect to a spatial embedding f of G and ω_2 if and only if

$$\mathrm{lk}\,(f(\gamma_{13}), f(\gamma'))) = \mathrm{lk}\,(f(\gamma_{24}), f(\gamma'))) = 0 \tag{4.1}$$

for any $\gamma' \in \Gamma_p(\Theta_4^2)$. The same condition can be said of ω_2. For an integer m, let f_m be the spatial embedding of G as illustrated in Fig. 4.3. Note that

$$\mathrm{lk}(f_k(\gamma), f_k(\gamma')) = \mathrm{lk}(f_l(\gamma), f_l(\gamma')) = 0 \text{ or } 1 \ (k \neq l)$$

for any $\gamma \in \Gamma(\Theta_4^1)$ and $\gamma' \in \Gamma(\Theta_4^2)$. Since we can see that ω_i satisfies (4.1), we have that ω_i is null-homologous on any edge of G_i with respect to f_m and ω_j ($i = 1,2,\ i \neq j$). Namely $\tilde{\beta}_{\omega_1,\omega_2}(f_m)$ is an integer-valued edge-homotopy invariant of f_m. Then, by a direct calculation we can see that the constituent 2-component link of f_m which has a non-zero generalized Sato-Levine invariant is only $L = f_m(\gamma_{14} \cup \gamma'_{14})$ and $\tilde{\beta}(L) = 2m$. Thus we have that $\tilde{\beta}_{\omega_1,\omega_2}(f_m) = 2m$. Therefore we have that f_k and f_l are not edge-homotopic for $k \neq l$.

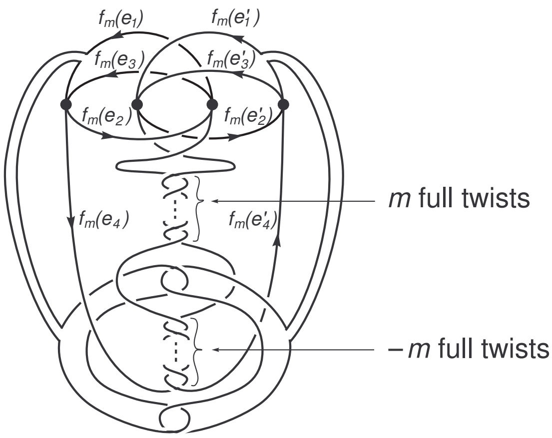

Fig. 4.3.

Remark 4.5. In Theorems 3.3 and 4.2, the condition "ω_i is null-homologous on any edge of G_i with respect to f and ω_j ($i = 1,2,\ i \neq j$)" is essential. Let $G = \Theta_4^1 \cup \Theta_4^2$ be a disjoint union of two copies of Θ_4 oriented in the same way as Example 4.4 and $\omega_i : \Gamma(\Theta_4^i) \to \mathbb{Z}$ the checkerboard weight ($i = 1,2$). Let f and g be two spatial embeddings of G as illustrated in Fig. 4.4. Note that f and g are edge-homotopic. But by a direct calculation we have that $\tilde{\beta}_{\omega_1,\omega_2}(f) = -1$ and $\tilde{\beta}_{\omega_1,\omega_2}(g) = 0$, namely $\tilde{\beta}_{\omega_1,\omega_2}(f)$ is not an edge-homotopy invariant of f. Actually ω_1 is not null-homologous on e_4 with respect to f and ω_2.

References

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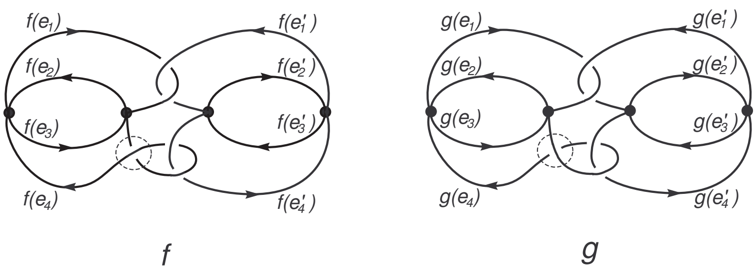

Fig. 4.4.

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