Partially Ordered Permutation Complexity of Coupled Time Series

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Abstract

A new information-theoretic complexity measure of coupling among multiple time series is proposed based on ordinal patterns. We regard ordinal patterns extracted from time series as total orders and exploit their algebraic properties to introduce a multiplication among them. The result of the multiplication is in general a partial order. A Kullback-Leibler divergence for the partial orders gives rise to a complexity measure of coupling in the sense that it vanishes in two extreme cases: when time series are uncoupled and when they are identical. The performance of the proposed complexity measure is tested on two model multivariate dynamical systems and its advantage over the existing complexity measures is discussed.

Keywords: ordinal patterns; multivariate time series; partial order; Kullback-Leibler divergence

1 Introduction

When analyzing time series data, it is important to adopt a symbolization technique that is both simple and robust to get reliable results efficiently. Ordinal pattern analysis is a candidate of such technique in which a block of numerical values in time series is symbolized as a permutation called ordinal pattern that encodes the order relationship among numerical values in the block [6, 1]. Mapping the order relationship among numerical values in a given time series to ordinal patterns is a simple task and they are invariant under the presence of noise that does not change the order relationship. Various measures based on ordinal patterns have been effectively used to analyze real-world time series such as physiological and economic data [22, 3, 5, 10].

Ordinal pattern analysis has been extended to multivariate setting. For example, Hempel et al. [8]

proposed a permutation-based measure to detect coupling directionality from short time series called inner composition alignment. Given two time series, the first one is reordered in the increasing order and the second one is re-ordered according to the same order as the first one. The degree of monotonicity of the reordered second time series and its entropy have been used for coupling analysis [8, 19, 23].

Ordinal patterns have an extra advantage due to its intrinsic algebraic nature when analyzing multivariate time series. Since permutations of the same length can be regarded as elements of a symmetric group, a binary operation can be defined among them: the composition of permutations in the symmetric group [13]. Thus, we obtain an ordinal pattern from multiple ordinal patterns originating from multivariate time series by composing them as elements in the symmetric group. Amigo et al. [4] proposed the notion of transcript based on this viewpoint and applied it to defining complexity of coupling among multiple time series [4], measuring coupling directionality between two time series [14] and dimensional reduction to calculate multi-information [2].

Ordinal patterns can be regarded as carrying a different type of algebraic structure. They can be also seen as total orders on the set of time indices. In this paper, we propose a permutation-based entropy for multivariate time series based on this viewpoint. We introduce a multiplication among ordinal patterns by taking the intersection of them as total orders. The multiplication does not necessarily yield a total order but gives rise to a partial order in general. This multiplication enables us to define a new complexity measure of coupling among multiple time series.

This paper is organized as follows. In Sec. 2, we first review two existing complexity measures of coupling based on ordinal patterns and then introduce our new one. In Sec. 3, we compare performance of our new measure with existing two ones in two model multivariate dynamical systems. Finally, in Sec. 4, we summarize the results and give a conclusion.

2 Methods

In this section, we first recall the definition of permutation entropy. Second, we review coupling complexity based on transcript and statistical complexity. Finally, we introduce our new complexity measure of coupling among multiple time series called partially ordered permutation complexity.

2.1 Permutation entropy

Let $\{x_t\}_t = \{\dots, x_{t-1}, x_t, x_{t+1}, \dots\}$ be a realization of a sequence of real-valued random variables $\{X_t\}_t = \{\dots, X_{t-1}, X_t, X_{t+1}, \dots\}$. In practical applications, one has to choose an appropriate time delay before a symbolization scheme is applied but here we assume that it has already chosen and set it as the unit

of time steps. For a block of length L at time t denoted by $x_{t-L+1:t} := (x_{t-(L-1)}, x_{t-(L-2)}, \ldots, x_t)$, if

$$x_{s+\pi_0} \leq x_{s+\pi_1} \leq \cdots \leq x_{s+\pi_{L-1}}, \tag{1}$$

where $s = t-(L-1)$, then we say that $x_{t-L+1:t}$ is of type $\pi := \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle$. π is called an *ordinal L-pattern*. For ties in Eq. (1), we use a convention that determines a unique ordinal L-pattern of $x_{t-L+1:t}$. Here, we define $x_{s+i} \leq x_{s+j}$ when $x_{s+i} = x_{s+j}$ and $i < j$.

In the following, we assume that the underlying process $\{X_t\}_t$ is stationary. Let $p(\pi)$ be the probability distribution of the ordinal L-pattern π. The random variable of ordinal L-pattern is denoted by Π. The *permutation entropy* of order L is defined as [6]

$$H(\Pi) = -\sum_{\pi} p(\pi) \log_2 p(\pi), \tag{2}$$

where we omit the normalization constant.

2.2 Transcript and Coupling complexity

An ordinal L-pattern π is a permutation on the set $\{0, 1, \ldots, L-1\}$ and can be seen as an element of the symmetric group S_L of order L,

$$\pi = \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle = \begin{pmatrix} 0 & 1 & \ldots & L-1 \\ \pi_0 & \pi_1 & \ldots & \pi_{L-1} \end{pmatrix} \in S_L \tag{3}$$

The group operation in S_L is the composition of permutations:

$$\begin{aligned} \pi\sigma &= \begin{pmatrix} 0 & 1 & \ldots & L-1 \\ \pi_0 & \pi_1 & \ldots & \pi_{L-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & \ldots & L-1 \\ \sigma_0 & \sigma_1 & \ldots & \sigma_{L-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \ldots & L-1 \\ \sigma_{\pi_0} & \sigma_{\pi_1} & \ldots & \sigma_{\pi_{L-1}} \end{pmatrix} = \langle \sigma_{\pi_0}, \sigma_{\pi_1}, \ldots, \sigma_{\pi_{L-1}} \rangle \end{aligned} \tag{4}$$

for $\pi, \sigma \in S_L$, where the first factor of the composition is written on the left. Given $\alpha, \beta \in S_L$, the *L-transcript* from α to β [13] is a unique $\tau \in S_L$ such that

$$\tau\alpha = \beta. \tag{5}$$

Namely, $\tau = \beta\alpha^{-1}$ where α^{-1} is the inverse of α.

Let $\{x_t^{(n)}\}_t$ be a collection of N time series ($n = 1, 2, \ldots, N$). Let π_n be an ordinal L-pattern of

$x^{(n)}_{t-L+1:t}$ and τ_n the L-transcript from π_n to π_{n+1}. In the following, the subscript of an ordinal L-pattern signifies the index of time series. We have $\tau_n \pi_n = \pi_{n+1}$ for $n = 1, 2, \ldots, N-1$. The probability distribution of the L-transcripts $\tau_{1:N-1} := (\tau_1, \tau_2, \ldots, \tau_{N-1})$ is

$$p_T(\tau_{1:N-1}) = \sum_{\pi_{1:N} \in \Omega(\tau_{1:N-1})} p_J(\pi_{1:N}), \tag{6}$$

where $\pi_{1:N} := (\pi_1, \pi_2, \ldots, \pi_N)$, $p_J(\pi_{1:N})$ is the joint probability distribution of the ordinal L-patterns $\pi_1, \pi_2, \ldots, \pi_N$ and

$$\Omega(\tau_{1:N-1}) = \{\pi_{1:N} \in (S_L)^N \mid \tau_n \pi_n = \pi_{n+1} (1 \leq n \leq N-1)\}. \tag{7}$$

Let

$$H(\Pi_{1:N}) = -\sum_{\pi_{1:N}} p_J(\pi_{1:N}) \log_2 p_J(\pi_{1:N}) \tag{8}$$

be the Shannon entropy of the joint probability distribution p_J and

$$H(T_{1:N-1}) = -\sum_{\tau_{1:N-1}} p_T(\tau_{1:N-1}) \log_2 p_T(\tau_{1:N-1}) \tag{9}$$

the Shannon entropy of the transcript probability distribution p_T. $\Pi_{1:N}$ and $T_{1:N-1}$ denote random variables such that $\mathrm{Prob}\{\Pi_{1:N} = \pi_{1:N}\} = p_J(\pi_{1:N})$ and $\mathrm{Prob}\{T_{1:N-1} = \tau_{1:N-1}\} = p_T(\tau_{1:N-1})$, respectively. Note that $H(T_{1:N-1}) = 0$ if $\Pi_1 = \Pi_2 = \cdots = \Pi_N$.

Amigó et al. [4] introduced the *coupling complexity* C_t as a measure of complexity of coupling among multiple time series. It is defined as

$$C_t = \min_{1 \leq n \leq N} H(\Pi_n) - \left(H(\Pi_{1:N}) - H(T_{1:N-1})\right). \tag{10}$$

They showed that $C_t \geq 0$ and the following two properties hold [4]:

(a) If Π_n $(1 \leq n \leq N)$ are uniform and independent random variables, then $C_t = 0$.

(b) If $\Pi_1 = \Pi_2 = \cdots = \Pi_N$, then $C_t = 0$.

Thus, C_t vanishes in the two extreme cases: (a) one in which time series are uniformly random and independent and (b) the other in which they are completely synchronized. These properties justify that one calls C_t the coupling complexity, because C_t can take a maximum value between the two extreme cases.

2.3 Statistical complexity

The idea of statistical complexity is that a measure of complexity can be obtained by multiplying the Shannon entropy by a measure of 'disequilibrium' [11]. Let $p_n(\pi_n)$ be the marginal probability distribution of the ordinal L-pattern in n-th time series. We consider the product $p_I := \prod_{1\leq n\leq N} p_n$ of the marginal probability distributions as the 'equilibrium' distribution p_E. Following Rosso et al. [16], the degree of 'disequilibrium' is measured by the Jensen-Shannon divergence between p_J and p_E:

$$Q[p_J, p_E] = H[(p_J + p_E)/2] - H[p_J]/2 - H[p_E]/2, \tag{11}$$

where $H[p]$ denotes the Shannon entropy of a given probability distribution p. Note that we have $H[p_J] = H(\Pi_{1:N})$ etc. We use both notations interchangeably in the following. The *statistical complexity* C_s is defined as [16]

$$C_s = Q[p_J, p_I]H[p_J]. \tag{12}$$

Here, we omit the normalization constant [16] because we do not compare complexity of different systems.

When $\{x_t^{(n)}\}_t$ are independent, $C_s = 0$ holds since $p_J = p_I$ and thus $Q[p_J, p_I] = 0$. However, C_s does not necessarily vanish for completely synchronized time series.

2.4 Partially ordered permutation complexity

The ordinal L-patterns π_n $(1 \leq n \leq N)$ can be seen as a total order on the set $\{0, 1, \ldots, L-1\}$. Recall that a partial order on a set P is a binary relation $\leq$ on P satisfying the following three axioms: (i) $x \leq x$, (ii) if $x \leq y$ and $y \leq x$ then $x = y$, and (iii) if $x \leq y$ and $y \leq z$ then $x \leq z$, for all $x, y, z \in P$ [18]. A set P equipped with a partial order $\leq$ is called a partially ordered set and is denoted by $(P, \leq)$. Given a partial order $\leq$ on P, if $x \leq y$ or $y \leq x$ hold for all $x, y \in P$, then $\leq$ is called a total order.

Let μ be the function sending each N-tuple of total orders $\pi_{1:N}$ on $\{0, 1, \ldots, L-1\}$ to the partial order defined by the intersection

$$\bigcap_{1\leq n\leq N} \pi_n =: \mu(\pi_{1:N}), \tag{13}$$

where we take the intersection by regarding each π_n as a binary relation on $\{0, 1, \ldots, L-1\}$. We call $\mu(\pi_{1:N})$ the *partial ordinal L-pattern* induced by $\pi_{1:N}$. For example, let

$$\pi_1 = \langle 0, 2, 1\rangle = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\pi_2 = \langle 1, 0, 2 \rangle = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

where total orders as binary relations are represented as matrices. $\mu(\pi_1, \pi_2)$ can be obtained by the component-wise multiplication of the matrices representing π_1 and π_2:

$$\mu(\pi_1, \pi_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The partial order $\mu(\pi_1, \pi_2)$ includes only one non-equality relation between 0 and 2 and hence is not a total order.

Given a partial order $\leq$ on a set P, it is known that there exists a family of total orders on P $\{\leq_i\}_{i \in I}$ such that $\leq = \bigcap_{i \in I} \leq_i$. The family $\{\leq_i\}_{i \in I}$ is called a realization of $\leq$. The integer

$$k = \min\{|I| \mid \{\leq_i\}_{i \in I} \text{ is a realization of } \leq\} \tag{14}$$

is called the dimension of the partially ordered set $(P, \leq)$ and is denoted by $\dim(P, \leq)$ [18]. Hiraguchi's theorem [9] states that

$$\dim(P, \leq) \leq \lfloor |P|/2 \rfloor \tag{15}$$

for every finite partially ordered set $(P, \leq)$ with $|P| \geq 4$, where $\lfloor x \rfloor$ is the largest integer which does not exceed x. It is easy to see that if $|P| \leq 3$, then $\dim(P, \leq) \leq 2$. It is also known that there exists a partial order that satisfies the equality in Eq. (15) [18]. Thus, if $\max\{2, \lfloor L/2 \rfloor\} \leq N$, then every partial order on $\{0, 1, \ldots, L-1\}$ can arise as a partial ordinal L-pattern.

The partial ordinal L-pattern $\mu(\pi_{1:N})$ is an 'integrated' symbol for the collection of time series $\{x_t^{(n)}\}_t$ $(1 \leq n \leq N)$ in contrast to the 'atomic' symbol $\pi_{1:N}$ because it encodes the relationship between π_n's. Let $p_P(\xi)$ be the probability distribution of the partial ordinal L-pattern ξ, namely,

$$p_P(\xi) = \sum_{\pi_{1:N} \in \mu^{-1}(\xi)} p_J(\pi_{1:N}). \tag{16}$$

Let us consider the non-integrated part of $H(\Pi_{1:N})$ defined as the difference between the Shannon entropy of atomic symbol $H(\Pi_{1:N})$ and that of integrated symbol $H(\Xi)$:

$$H_p = H(\Pi_{1:N}) - H(\Xi), \tag{17}$$

where Ξ is the random variable of partial ordinal L-pattern. H_p can be written as the average entropy of atomic symbol per integrated symbol:

$$H_p = \sum_{\xi} p_P(\xi) \left(- \sum_{\pi_{1:N} \in \mu^{-1}(\xi)} \frac{p_J(\pi_{1:N})}{p_P(\xi)} \log_2 \frac{p_J(\pi_{1:N})}{p_P(\xi)} \right). \tag{18}$$

Indeed, H_p is the conditional entropy of $\Pi_{1:N}$ given Ξ because the joint probability of $\pi_{1:N}$ and ξ is $p_J(\pi_{1:N})$ if $\pi_{1:N} \in \mu^{-1}(\xi)$ and is otherwise 0. As in the case of $H(T_{1:N-1})$, H_p vanishes for completely synchronized time series.

The multi-information [12, 21, 17] of N-tuple of ordinal L-patterns is defined as

$$I(\Pi_{1:N}) = \sum_{\pi_{1:N}} p_J(\pi_{1:N}) \log_2 \frac{p_J(\pi_{1:N})}{p_I(\pi_{1:N})}. \tag{19}$$

$I(\Pi_{1:N})$ is the amount of information for atomic symbol. The amount of information for integrated symbol (partial ordinal L-pattern) can be quantified by the following Kullback-Leibler divergence [7]:

$$I(\Xi) := \sum_{\xi} p_P(\xi) \log_2 \frac{p_P(\xi)}{q(\xi)}, \tag{20}$$

where $q(\xi) = \sum_{\pi_{1:N} \in \mu^{-1}(\xi)} p_I(\pi_{1:N})$. In the same way as in H_p, we can introduce the non-integrated part of the amount of atomic information $I(\Pi_{1:N})$ as the difference $C_p := I(\Pi_{1:N}) - I(\Xi)$. It is the average amount of information of atomic symbol per integrated symbol:

$$C_p = \sum_{\xi} p_P(\xi) I_\xi, \tag{21}$$

where

$$I_\xi = \sum_{\pi_{1:N} \in \mu^{-1}(\xi)} \frac{p_J(\pi_{1:N})}{p_P(\xi)} \log_2 \frac{p_J(\pi_{1:N})/p_P(\xi)}{p_I(\pi_{1:N})/q(\xi)}. \tag{22}$$

C_p has the following properties:

(a') If Π_n $(1 \leq n \leq N)$ are independent random variables, then $C_p = 0$.

(b') If $\Pi_1 = \Pi_2 = \cdots = \Pi_N$, then $C_p = 0$.

Thus, C_p can be seen as a measure of complexity of coupling among multiple time series like C_t. We call C_p *partially ordered permutation complexity*. Note that the premise in (a') is weaker than that of (a). On the other hand, (a') holds for C_s. However, (b') does not hold for C_s in general. Thus, C_p combines the desirable aspects of both C_t and C_s.

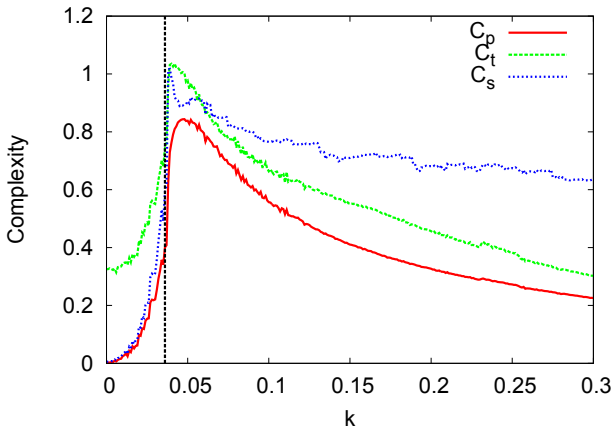

Figure 1: C_p, C_t and C_s for Eq. (23) are plotted against the coupling strength k. The vertical dotted line is $k = 0.036$.

3 Results

In this section, we calculate complexity measures of coupling among multiple time series described in Sec. 2 for two synthetic data and compare their performance. We take $L = 4$ in all the numerical experiments below. We checked that $L = 5$ also yields qualitatively similar results.

The first example is the following two non-identical coupled Rössler systems [15]:

$$\begin{aligned}
\dot{x}_{1,2} &= -w_{1,2}y_{1,2} - z_{1,2} + k(x_{2,1} - x_{1,2}), \\
\dot{y}_{1,2} &= w_{1,2}y_{1,2} + 0.165y_{1,2}, \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10),
\end{aligned} \tag{23}$$

where $w_1 = 0.99$ and $w_2 = 0.95$. Amigó et al. [4] investigated complexity of coupling in this system by C_t and a measure similar to C_s. The aim here to study this system is to reproduce their results and compare them with C_p. So, we numerically simulated Eq. (23) in the coupling strength range $0 \leq k \leq 0.3$ in the same setting. The fourth-order Runge-Kutta method with an increment $\delta t = 0.001$ was used to solve Eq. (23) from the initial condition: $x_1(0) = -0.4$, $y_1(0) = 0.6$, $z_1(0) = 5.8$, $x_2(0) = 0.8$, $y_2(0) = -2$ and $z_2(0) = -4$. For each k, we sampled data every 10 increments and generated a time series of length 2^{18}. Ordinal L-patterns were extracted from x_1 and x_2 time series with a time delay $\delta = 150$ [13]. In the uncoupled case ($k = 0$), the two systems show independent chaotic oscillations.

As the value of k increases, the system Eq. (23) exhibits two kinds of synchronization behaviors [15]. The first one is phase synchronization which starts to appear from $k \approx 0.036$. When the two Rössler systems are phase synchronized, the phases of them are locked while the amplitudes of them are still almost uncorrelated and change chaotically. The second one is a stronger form of synchronization known

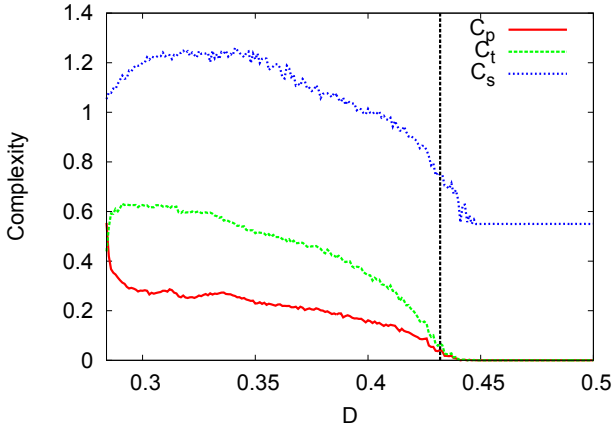

Figure 2: C_p, C_t and C_s for Eq. (24) are plotted against the coupling strength D in the range $0.284 < D < 0.5$. The vertical dotted line is $D = 0.432$.

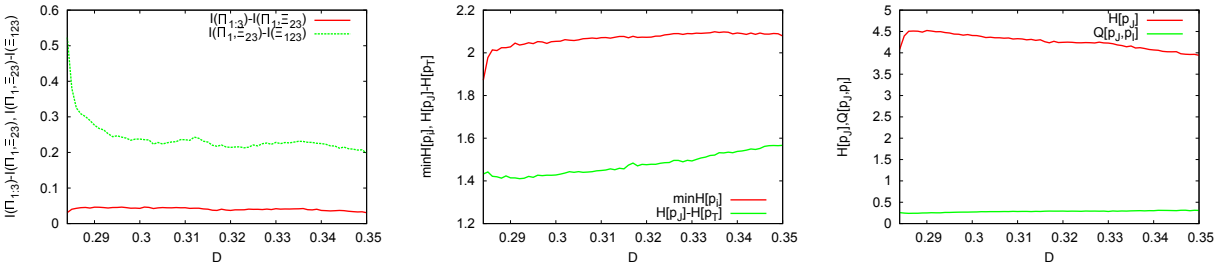

Figure 3: Decompositions of C_p (Left), C_t (Middle) and C_s (Right) for Eq. (24) into appropriate terms.

as lag-synchronization which starts at $k \approx 0.14$. In this regime, $x_1(t) = x_2(t+T)$ holds for some $T \geq 0$. Thus, once a synchronization behavior emerges, the form of synchronization becomes 'simpler' as the coupling strength increases [4].

Fig. 1 shows the three complexity measures C_p, C_t and C_s of coupling between x_1 and x_2. As k is increased, all of them gradually increase from $k = 0$, abruptly increase around the onset of phase synchronization $k \approx 0.036$ and gradually decrease after taking maximum values. Thus, our new measure C_p is as useful as C_t and C_s as a measure of complexity of coupling between x_1 and x_2 in this system. In addition to differences in their scales and the exact values of k at which they take maximum values, quantitative differences among the three measures are as follows: When k approaches 0, both C_p and C_s converge to 0 as we expect from their definitions. However, C_t does not. On the other hand, for large k, C_p and C_t decrease faster than C_s. Thus, it seems that C_p has both advantages of C_s and C_t: C_p converges to 0 as the correlation between time series is lost and it is sensitive enough to clearly discriminate the continuous change of the strength of coupling for large k in the range shown in Fig. 1.

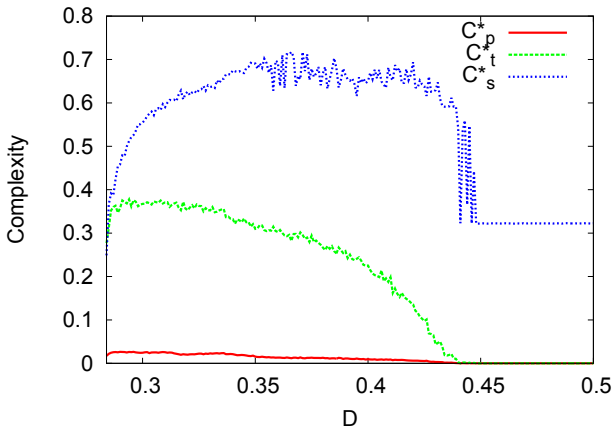

Figure 4: C_p, C_t and C_s for the coupling between x_1 and x_2 in Eq. (24) are plotted against the coupling strength D. Labels are denoted with $*$ to indicate that they are bivariate versions.

The second example is a system of three coupled identical chaotic maps defined by

$$\begin{aligned} x_1(t+1) &= (1-2J)f(x_1(t)) + J\left(f(x_2(t)) + f(x_3(t))\right), \\ x_2(t+1) &= (1-2J)f(x_2(t)) + J\left(f(x_3(t)) + f(x_1(t))\right), \\ x_3(t+1) &= (1-2J)f(x_3(t)) + J\left(f(x_1(t)) + f(x_2(t))\right), \end{aligned} \tag{24}$$

where $f(x) = 3.8x(1-x)$ and $J = (1-e^{-D})/3$ [20]. D is a parameter representing the strength of coupling among the three variables. We generated time series of length 2^{17} for each D by iteratively solving Eq. (24) with a small additive noise of amplitude 10^{-12} to avoid artificial numerical convergence and calculated the complexity measures with $N = 3$.

The system Eq. (24) exhibits two kinds of synchronization behavior [20]: one is partial synchronization in which two of the three components are synchronized and the remaining one is desynchronized ($D_c^2 < D < D_c^1$ where $D_c^2 \approx 0.201$ and $D_c^1 \approx 0.284$) and the other is complete synchronization in which all the three components are synchronized ($D_c^0 < D$ where $D_c^0 \approx 0.432$). These characteristic values of D are determined by the points where the transverse Lyapunov exponents for partial synchronization and complete synchronization change their sign. Here we focus on the range $D_c^1 < D < 0.5$ where break down of complete synchronization of the three chaotic oscillators occurs as D is decreased and a 'complex' behavior is observed as D approaches D_c^1 from above: intermittent switching behavior among three broken partial synchronization attractors with power law statistics. The parameter region $D_c^2 < D < D_c^1$ has a window structure where oscillation of each component is periodic. Since our primary interest is complexity of the relationship among chaotic time series, we exclude this region from the following analysis.

Fig. 2 shows C_p, C_t and C_s in the range $0.284 < D < 0.5$. They exhibit a similar behavior around

D_c^0. As D is decreased from above, all of them begin to increase from the constant values in the complete synchronization regime. In our numerical simulation, the onsets are slightly above D_c^0 due to the added noise. However, C_p behaves differently from C_t and C_s around D_c^1. We can see that when D approaches D_c^1 from above, C_p increases while the other two measures decrease. To see what causes the different behavior of C_p from C_t and C_s, we decompose each measure into appropriate terms (Fig. 3): C_p is the sum of two partial complexity terms $I(\Pi_{1,2,3}) - I(\Pi_1, \Xi_{2,3})$ and $I(\Pi_1, \Xi_{2,3}) - I(\Xi_{1,2,3})$, C_t is the difference between $\min_{i=1,2,3} H[p_i]$ and $H[p_J] - H[p_T]$, and C_s is the product of $Q[p_J, p_I]$ and $H[p_J]$. Here, $\Xi_{i_1,\ldots,i_k}$ is the random variable of partial ordinal L-patterns $\mu(\pi_{i_1}, \ldots, \pi_{i_k})$. Two other different decompositions can be made in the case of C_p but they yield almost the same result since the system Eq. (24) is symmetric with respect to exchanges of variables. From the middle and the right panels of Fig. 3, one can see that the behaviors of C_t and C_s are dominated by the entropy terms which decrease as D is decreased in the region close to D_c^1. On the other hand, as D is decreased, C_p increases due to the "upper level" complexity term $I(\Pi_1, \Xi_{2,3}) - I(\Xi_{1,2,3})$ which is the complexity of the relationship between Π_1 and $\Xi_{2,3}$ (Fig. 3, left). The involvement of all the three components seems to be essential for the behavior of C_p. Indeed, C_p for the relationship between Π_1 and Π_2 does not increase as D approaches D_c^1 from above (Fig. 4). C_t and C_s behave similarly. This suggests that the behavior of C_p reflects three component relationships that cannot be captured by either C_t or C_s.

4 Conclusion

In this paper, we proposed a new measure of complexity for coupling among multiple time series called partially ordered permutation complexity C_p. It is based on the intersection of ordinal L-patterns regarded as total orders which yields an integrated symbol for the whole system (a partial ordinal L-pattern) that is in general a partial order.

We numerically studied the behavior of C_p in a bivariate and a trivariate dynamical systems by comparing it with two existing complexity measures of coupling C_t and C_s. In the first case, the performance of C_p is qualitatively comparable with C_t and C_s. In the second case, C_p can reflect a 'complex' behavior unique in the trivariate system that cannot be detected by C_t and C_s.

Our results demonstrated the advantage of C_p in analyzing complexity of coupling among a small number of time series. However, as N increases, the number of non-trivial partial ordinal L-patterns would typically tend to decrease for a fixed L. Hence, further theoretical development would be necessary for practical measurement of complexity of coupling among a large number of time series. This is out of the scope of this paper and is left as future work.

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References

[1] J. M. Amigó. *Permutation Complexity in Dynamical Systems.* Springer-Verlag Berlin Heidelberg, 2010.

[2] J. M. Amigó, T. Aschenbrenner, W. Bunk, and R. Monetti. Dimensional reduction of conditional algebraic multi-information via transcripts. *Information Sciences*, 278:298–310, 2014.

[3] J. M. Amigó, K. Keller, and V. A. Unakafova. Ordinal symbolic analysis and its application to biomedical recordings. *Phil. Trans. R. Soc. A*, 373:20140091, 2015.

[4] J. M. Amigó, R. Monetti, T. Aschenbrenner, and W. Bunk. Transcripts: An algebraic approach to coupled time series. *CHAOS*, 22:013105, 2012.

[5] C. Bandt. A new kind of permutation entropy used to classify sleep stages from invisible EEG microstructure. *Entropy*, 19:197, 2017.

[6] C. Bandt and B. Pompe. Permutation entropy: a natural complexity measure for time series. *Phys. Rev. Lett.*, 88:174102, 2002.

[7] T. M. Cover and J. A. Thomas. *Elements of Information Theory.* John Wiley & Sons, Inc., 1991.

[8] S. Hempel, A. Koseska, J. Kurths, and Z. Nikoloski. Inner composition alignment for inferring directed networks from short time series. *Phys. Rev. Lett.*, 107:054101, 2011.

[9] T. Hiraguchi. On the dimension of partially ordered sets. *Sci. Rep. Kanazawa Univ.*, 1:77–94, 1951.

[10] K. Keller, T. Mangold, I. Stolz, and J. Werner. Permutation entropy: New ideas and challenges. *Entropy*, 19:134, 2017.

[11] R. López-Ruiz, H. L. Mancini, and X. Calbet. A statistical measure of complexity. *Phys. Lett. A*, 209:321–326, 1995.

[12] W. J. McGill. Multivariate information transmission. *Psychometrika*, 19:97–116, 1954.

[13] R. Monetti, W. Bunk, T. Aschenbrenner, and F. Jamitzky. Characterizing synchronization in time series using information measures extracted from symbolic representations. *Phys. Rev. E*, 79:046207, 2009.

[14] R. Monetti, W. Bunk, T. Aschenbrenner, S. Springer, and J. M. Amigó. Information directionality in coupled time series using transcripts. *Phys. Rev. E*, 88:022911, 2013.

[15] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths. From phase to lag synchronization in coupled chaotic oscillators. *Phys. Rev. Lett.*, 78:4193–4196, 1997.

[16] O. A. Rosso, H. A. Larrondo, M. T. Martin, A. Plastino, and M. A. Fuentes. Distinguishing noise from chaos. *Phys. Rev. Lett.*, 99:154102, 2007.

[17] E. Schneidman, S. Still, M. J. Berry II, and W. Bialek. Network information and connected correlations. *Phys. Rev. Lett.*, 91:238701, 2003.

[18] B. S. W. Schröder. *Ordered Sets: An Introduction*. Springer Science+Business Media, New York, 2003.

[19] W. Shi, P. Shang, and A. Lin. The coupling analysis of stock market indices based on cross-permutation entropy. *Nonlinear Dyn.*, 79:2439–2447, 2015.

[20] N. Tsukamoto, S. Miyazaki, and H. Fujisaka. Synchronization and intermittency in three-coupled chaotic oscillators. *Phys. Rev. E*, 67:016212, 2003.

[21] S. Watanabe. Information theoretical analysis of multivariate correlation. *IBM J. Res. Dev.*, 4:66–82, 1960.

[22] M. Zanin, L. Zunino, O. Rosso, and D. Papo. Permutation entropy and its main biomedical and econophysics applications: A review. *Entropy*, 14:1553–1577, 2012.

[23] X. Zhao and P. Shang. Measuring the uncertainty of coupling. *Euro. Phys. Lett.*, 110:60007, 2015.