

Theory of Interface: Category Theory, Directed Networks and Evolution of Biological Networks

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Abstract

Biological networks have two modes. The first mode is static: a network is a passage on which something flows. The second mode is dynamic: a network is a pattern constructed by gluing functions of entities constituting the network. In this paper, first we discuss that these two modes can be associated with the category theoretic duality (adjunction) and derive a natural network structure (a path notion) for each mode by appealing to the category theoretic universality. The path notion corresponding to the static mode is just the usual directed path. The path notion for the dynamic mode is called lateral path which is the alternating path considered on the set of arcs. Their general functionalities in a network are transport and coherence, respectively. Second, we introduce a betweenness centrality of arcs for each mode and see how the two modes are embedded in various real biological network data. We find that there is a trade-off relationship between the two centralities: if the value of one is large then the value of the other is small. This can be seen as a kind of division of labor in a network into transport on the network and coherence of the network. Finally, we propose an optimization model of networks based on a quality function involving intensities of the two modes in order to see how networks with the above trade-off relationship can emerge through evolution. We show that the trade-off relationship can be observed in the evolved networks only when the dynamic mode is dominant in the quality function by numerical simulations. We also show that the evolved networks have features qualitatively similar to real biological networks by standard complex network analysis.

1 Introduction

In this decade, large interaction network data on biological, social and technological systems have become available. Science of complex networks has attempted to reveal structures and functions underlying these network data by proposing various mathematical indices and models (Albert and Barabási, 2002;

Newman, 2003; Boccaletti et al., 2006). For example, the notions of small-world property, scale-free property and modular organization have become important tools to understand complex systems. On the other hand, we are aware of the criticism for purely graph-theoretic analysis forgetting meanings of networks (Arita, 2004). However, it makes mathematical analysis difficult if we stick to a meaning of each individual network too much. We might need a mathematical language that makes discussion on meaning of networks in a large sense possible. This paper is an attempt to discuss a comprehensive meaning of nodes and arcs in directed biological networks by appealing to category theory (MacLane, 1998).

Applications of category theory to biology originates from papers by R. Rosen published in the late 1950s (Rosen, 1958a,b). In the beginning, Rosen proposed a model of the maintenance mechanism of metabolic networks in terms of category theory. However, it seems that the viewpoint of ‘network’ became implicit as his theory of the metabolism-repair system developed. There are several other attempts to describe functions of biological systems by category theory (for example, (Ehresmann and Vanbreemersch, 1987; Wolkenhauer and Hofmeyr, 2007)). Since these studies propose highly abstract models of biological systems, it is not so clear how they can be applied to real world data. In this paper, we theoretically extend the research direction which we have sought in recent years (Haruna and Gunji, 2007; Haruna, 2008; Haruna and Gunji, 2009; Haruna, 2010, 2011) and try to make a bridge to real world data analysis.

Our starting point is two modes of biological networks. One is static and the other is dynamic. In the static mode, a network is a passage on which something flows. In the dynamic mode, a network is a pattern constructed by gluing functions of entities constituting the network. For example, let us consider a neuronal network: nodes are neurons and arcs are synaptic connections between neurons. In the static mode, the neuronal network is a passage on which electric or chemical signals flow. On the other hand, in the dynamic mode, each node (a neuron) is an information processing entity that receives signals from other nodes, modifies them and sends its response signals to other nodes. Similar pictures can hold for other biological networks such as ecological flow networks and gene regulation networks. Note that our dynamic mode does not consider change along the time parameter directly. Hence, it is different from both dynamics on networks such as percolation (Dorogovtsev et al., 2008), synchronization (Arenas et al., 2008) and games (Szabó and Fáth, 2007) and dynamic structural change in network structure as in temporal networks (Holme and Saramäki, 2012), although there may be conceptual links with them.

How can we extract information about the two mode from a given directed networks? In this paper, we argue that there is a natural pair of network transformations capturing features of the two modes. One can obtain information on the static mode by applying one network transformation to the given network and that on the dynamic mode by applying the other. For each network transformation, a natural network path notion is associated. For the static mode, it is simple and intuitive. If we assume that something flows along the direction of arcs, then the notion of directed path may be a natural network

structure corresponding to the static mode. It arises from the network transformation transforming the given network to its line graph as shown below. For the dynamic mode, the corresponding network transformation and the path notion are derived via the category theoretic universality with respect to the idea that functions of entities constituting the network are glued on the interface among them. Once the network transformation for the dynamic mode is derived, that for the static mode is obtained as the unique dual transformation for it. Thus, the dynamic mode is primary in terms of the category theoretic universality.

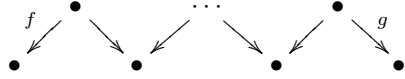
Before closing this section, we sketch the story without using category theoretic terminology. The directed path corresponding to the static mode is generated by the network transformation R ¹ defined below which sends arcs to nodes in the following sense: let $G = (A, N, \partial_0, \partial_1)$ be a directed network, where A is the set of arcs, N is the set of nodes and ∂_0, ∂_1 are maps from A to N sending each arc to its source or target, respectively. A directed network $R(G) = (A^*, N^*, \partial_0^*, \partial_1^*)$ is defined by putting $A^* = \{(f, g) \in A \times A \mid \partial_1(f) = \partial_0(g)\}$, $N^* = A$ and $\partial_0^*(f, g) = f$, $\partial_1^*(f, g) = g$ for $(f, g) \in A^*$. The set of arcs for $R(G)$ is the set of directed paths of length 2 in G . If we apply R to G twice, then we obtain the set of directed paths of length 3 in G as the set of arcs for $R^2(G)$. In general, the set of arcs for $R^n(G)$ is the set of directed paths of length $n + 1$ in G for any $n \geq 0$.

For the network transformation R , we have the dual network transformation L (in category theoretic terminology, both R and L can be extended to endofunctors on the category of directed networks and L is the left adjoint functor to R). The network transformation L sends each node to an arc: for any directed network $G = (A, N, \partial_0, \partial_1)$, $L(G) = (A_*, N_*, \partial_{0*}, \partial_{1*})$ is defined by putting $A_* = N$, $N_* = N \times \{0, 1\} / \sim$ and $\partial_{0*}(x) = [(x, 0)]$, $\partial_{1*}(x) = [(x, 1)]$ for $x \in A_*$. Here, \sim is the equivalence relation on the set $N \times \{0, 1\}$ generated by the relation r defined by $(x, 1)r(y, 0) :\Leftrightarrow$ there exists $f \in A$ such that $\partial_0(f) = x$ and $\partial_1(f) = y$. $[(x, i)]$ is the equivalence class containing (x, i) . As we will discuss in Section 2, the network transformation L can be associated with the dynamic mode. From this, we can derive a path notion called lateral path that can be seen as the one dual to the directed path.

Let us consider the meaning of the network transformation L . When we apply L to a directed network G , each node is mapped to an arc. We regard this arc as representing *function* of the node, namely, the arc in $L(G)$ to which a node in G is mapped is thought of representing a process occurring on the node. On the other hand, we can regard each arc f in G as being sent to the node $[(\partial_0(f), 1)] (= [(\partial_1(f), 0)])$ connecting two arcs $\partial_0(f)$ and $\partial_1(f)$ in $L(G)$. Namely, *interaction is interface between functions*. This idea is materialized as a mathematical entity by the map $\varphi : A \rightarrow N_*$ defined by $f \mapsto [(\partial_0(f), 1)]$. For arcs f, g in G , a necessary and sufficient condition for the equality $\varphi(f) = \varphi(g)$ is the existence of an

¹For a directed network G , $R(G)$ is so-called **line graph** of G . We use the two terms *network* and *graph* as synonymous words.

alternating sequence of arcs in G connecting f and g such as



Note that there are $2 \times 2 = 4$ possibilities for the situation at the two ends of the sequence. One of them is drawn above. We call such an alternating sequence of arcs **lateral path**². Since the lateral path is associated with gluing of functions, we introduce the term *coherence* for its general functionality. On the other hand, the functionality of the directed path is considered as *transport* on a network here.

We claim that *the notion of lateral path is a natural network structure associated with the dynamic mode*. This claim is precisely formulated and proved in Section 2. In this rough sketch, it may be enough to give the following explanation: in the network transformation L , function of a node is represented by a single arc. However, we can represent function by a more complicated graph. It does not even need to be a graph. We can show that the representation of function corresponding to L occupies a special position among *all* representations of function of a node (in the language of category theory, it satisfies a certain universality): any representation of function gives rise to a map on the set of arcs that materializes the idea “interaction as interface between functions”. This map in turn induces an equivalence relation on the set of arcs by its fibers. The claim is that the equivalence relation for the above map φ induced by L gives the finest partition of the set of arcs among all such equivalence relations.

This paper is organized as follows. In Section 2, we describe a mathematically precise formulation of the story sketched above. In Section 3, we introduce betweenness centralities of arcs for the static and dynamic modes based on directed path and lateral path, respectively. We see how these two modes are embedded in various real biological network data. As a result, we find that there is a trade-off relationship between the two centralities: if the value of one is large then the value of the other is small. In Section 4, we propose an optimization model of networks based on a quality function involving intensities of the static and dynamic modes. By numerical simulation, we see how networks with the above trade-off relationship can emerge. We show that the trade-off relationship emerges after the evolution only when the dynamic mode is dominant in the quality function. We also show that the evolved networks have features qualitatively similar to real biological networks by standard complex network analysis. In Section 5, we give conclusions and indicate some future directions. In Appendix, we present a generalization of the theory for directed networks described in Section 2. The aim of Appendix is twofold. One is to extend the theory in Section 2 to more general data structures. The other is to understand what features of directed networks are relevant to the main results in Section 2. We show that *self-duality* and *acyclicity*

²Similar notion called **alternating walk** is considered in (Crofts et al., 2010). However, it is defined on the set of nodes. In (Crofts et al., 2010), it is used as an auxiliary means to obtain a bipartition of directed networks. On the other hand, lateral path in this paper is a central stuff associated with the dynamic mode of biological networks.

possessed by the type category for the category of directed networks are enough to reproduce the main results in Section 2 in more generalized situation.

2 Theory of interface in directed networks

In this section, we develop category theoretic formulation of the story sketched in Introduction. For necessary category theoretic background, readers are referred to (MacLane, 1998) or (Borceux, 1994).

Although this section is the main part of this paper, the readers who do not want to go into category theory can safely skip this section and jump to Section 3 because the main concern in this section is a mathematical justification of the connection between the dynamic mode of directed networks and the notion of lateral path and how the relationship between the static and the dynamic modes is associated with an adjunction. The subject of Section 3 and 4 in turn is applications of the lateral path and the directed path.

2.1 Left Kan extension along the Yoneda embedding functor

In this subsection, we review the left Kan extension along the Yoneda embedding functor which is a key material for our theory.

Let \mathcal{C} be a small category, \mathcal{D} a cocomplete category and $M : \mathcal{C} \rightarrow \mathcal{D}$ a functor. Recall that a **presheaf** on \mathcal{C} is a contravariant functor from \mathcal{C} to **Set**. The category of presheaves on \mathcal{C} is a functor category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$. We denote it by $\hat{\mathcal{C}}$. The **Yoneda embedding functor** $\mathbf{y} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ is defined by sending each object $c \in \mathcal{C}$ to the presheaf $\mathbf{y}(c) = \mathcal{C}(-, c)$ and each morphism $f : c \rightarrow c'$ in \mathcal{C} to the natural transformation $\mathbf{y}(f) = f \circ (-) : \mathbf{y}(c) \rightarrow \mathbf{y}(c')$.

The **left Kan extension** of functor M along the Yoneda embedding functor \mathbf{y} is the functor $\text{Lan}_{\mathbf{y}}M : \hat{\mathcal{C}} \rightarrow \mathcal{D}$ defined as follows: for any object $G \in \hat{\mathcal{C}}$, $\text{Lan}_{\mathbf{y}}M(G)$ is given by the colimit

$$\text{Lan}_{\mathbf{y}}M(G) = \text{colim} \left(\text{Elts}(G) \xrightarrow{\pi_G} \mathcal{C} \xrightarrow{M} \mathcal{D} \right).$$

Here, $\text{Elts}(G)$ is the category of elements of G whose objects are pairs (c, x) of object c in \mathcal{C} and $x \in G(c)$ and morphisms from (c, x) to (c', x') are morphisms $f : c \rightarrow c'$ in \mathcal{C} such that $G(f)(x') = x$. $\pi_G : \text{Elts}(G) \rightarrow \mathcal{C}$ is the projection functor to \mathcal{C} .

As we will review below, the functor $\text{Lan}_{\mathbf{y}}M$ is the left adjoint functor to the Hom functor. As usual, we denote $\text{Lan}_{\mathbf{y}}M(G)$ by $G \otimes M$.

For any morphism $\theta : G \rightarrow H$ in $\hat{\mathcal{C}}$, morphism $\theta \otimes M : G \otimes M \rightarrow H \otimes M$ in \mathcal{D} is defined as follows: let $\{\mu_{(c,x)}^{M,G} : M \circ \pi_G(c, x) = M(c) \rightarrow G \otimes M\}_{(c,x) \in \text{Elts}(G)}$ be the colimit on $M \circ \pi_G$ defining $G \otimes M$.

For any $x \in G(c)$ and $c \in \mathcal{C}$, we have $\theta_c(x) \in H(c)$. Hence, $\{\mu_{(c,\theta_c(x))}^{M,H} : M \circ \pi_H(c, \theta_c(x)) = M(c) \rightarrow H \otimes M\}_{(c,x) \in \text{Elts}(G)}$ is a cocone on $M \circ \pi_G$, where $\mu^{M,H}$ is the colimit on $M \circ \pi_H$ defining $H \otimes M$. By the universality of $G \otimes M$, there is a unique morphism $\xi : G \otimes M \rightarrow H \otimes M$ such that $\mu_{(c,\theta_c(x))}^{M,H} = \xi \circ \mu_{(c,x)}^{M,G}$ for all $(c, x) \in \text{Elts}(G)$. We define $\theta \otimes M := \xi$.

The following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{M} & \mathcal{D} \\ & \searrow \mathbf{y} & \uparrow \text{Lan}_{\mathbf{y}} M \\ & & \hat{\mathcal{C}}. \end{array}$$

Namely, $\mathbf{y} \otimes M \cong M$.

The functor $(-) \otimes M : \hat{\mathcal{C}} \rightarrow \mathcal{D}$ is the left adjoint functor to the functor $\mathcal{D}(M(-), -) : \mathcal{D} \rightarrow \hat{\mathcal{C}}$. Namely, we have a natural bijection

$$\mathcal{D}(G \otimes M, F) \cong \hat{\mathcal{C}}(G, \mathcal{D}(M(-), F))$$

for any $G \in \hat{\mathcal{C}}$ and $F \in \mathcal{D}$. Indeed, if we define the natural transformation $\eta^{M,G} : G \rightarrow \mathcal{D}(M(-), G \otimes M)$ by $\eta_c^{M,G} : G(c) \rightarrow \mathcal{D}(M(c), G \otimes M) : x \mapsto \mu_{(c,x)}^{M,G}$ for each object $c \in \mathcal{C}$, then for any morphism $\theta : G \rightarrow \mathcal{D}(M(-), F)$ there exists a unique morphism $\hat{\theta} : G \otimes M \rightarrow F$ such that $\theta = \mathcal{D}(M(-), \hat{\theta}) \circ \eta^{M,G}$: since $\{\theta_c(x) : M \circ \pi_G(c, x) = M(c) \rightarrow F\}_{(c,x) \in \text{Elts}(G)}$ is a cocone on $M \circ \pi_G$, there exists a unique morphism $\hat{\theta} : G \otimes M \rightarrow F$ such that $\theta_c(x) = \hat{\theta} \circ \mu_{(c,x)}^{M,G}$ for all $(c, x) \in \text{Elts}(G)$. The condition that $\theta_c(x) = \hat{\theta} \circ \mu_{(c,x)}^{M,G}$ for any $(c, x) \in \text{Elts}(G)$ is equivalent to the equality $\theta = \mathcal{D}(M(-), \hat{\theta}) \circ \eta^{M,G}$.

$$\begin{array}{ccc} G & \xrightarrow{\eta^{M,G}} & \mathcal{D}(M(-), G \otimes M) \\ & \searrow \theta & \downarrow \mathcal{D}(M(-), \hat{\theta}) \\ & & \mathcal{D}(M(-), F) \end{array} \quad \begin{array}{ccc} G \otimes M & & \\ \downarrow \hat{\theta} & & \\ F & & \end{array}$$

2.2 Interface of a representation

Let $\uparrow\uparrow$ be the category freely generated by the following directed graph:

$$0 \xrightarrow{m_0} 1.$$

We can identify the category of directed networks consisting of directed networks and homomorphisms between directed networks with $\uparrow\uparrow$, the category of presheaves on $\uparrow\uparrow$. Indeed, for any object $G \in \uparrow\uparrow$, the quadruplet $(G(1), G(0), G(m_0), G(m_1))$ defines a directed network. On the other hand, any directed network $(A, N, \partial_0, \partial_1)$ gives rise to a presheaf G on $\uparrow\uparrow$ by $G(0) = N, G(1) = A, G(m_0) = \partial_0$ and $G(m_1) =$

∂_1 . It is obvious that directed network homomorphisms between directed networks are translated into natural transformations between corresponding presheaves, and vice versa.

We can regard $\uparrow\uparrow$ as a kind of data type corresponding to directed networks and any functor $M : \uparrow\uparrow \rightarrow \mathcal{D}$ is a representation of it. Hence, hereafter, for any small category \mathcal{C} and category \mathcal{D} , we call any functor $M : \mathcal{C} \rightarrow \mathcal{D}$ a **representation** of \mathcal{C} . In the following, we frequently assume that \mathcal{D} is bicomplete for technical reasons. Even under this constraint, since any presheaf category is bicomplete, we still have a large freedom for representations.

Let us define the representation $M_0 : \uparrow\uparrow \rightarrow \hat{\uparrow\uparrow}$ as follows:

$$M_0(0) = p_0 \xrightarrow{a} p_1, \quad M_0(1) = q_0 \xrightarrow{b_0} q_1 \xrightarrow{b_1} q_2,$$

$m_0 \cdot a = b_0$ and $m_1 \cdot a = b_1$, where we write $m_i \cdot a := M_0(m_i)(1)(a)$ for $i = 0, 1$. $M_0(0)$ is the representation of a node and $M_0(1)$ is the representation of interaction between nodes (arc). In the terminology of Introduction, we can regard function of a node as being represented by the arc a in M_0 and an arc between two nodes (interaction) in a directed network is represented by the node q_1 connecting the two arcs (representations of function of a node) b_0 and b_1 (“interaction as interface between functions”).

In general, we regard any representation $M : \uparrow\uparrow \rightarrow \mathcal{D}$ as a representation of function of the unit of directed network, namely an arc together with its source and target nodes. For a directed network G , $G \otimes M$ can be seen as a representation of the dynamic mode – *a network is a pattern constructed by gluing functions of entities constituting the network* – because $G \otimes M$ is constructed by gluing copies of M .

One can see that the network transformation L introduced in Introduction is isomorphic to the left Kan extension of M_0 along the Yoneda embedding functor and the network transformation R is isomorphic to its right adjoint. Namely, $L \cong (-) \otimes M_0$ and $R \cong \hat{\uparrow\uparrow}(M_0(-), -)$. Thus, we obtain the functor associated with the static mode as the right adjoint functor to the functor associated with the dynamic mode.

The node q_1 in $M_0(1)$ is a pullback of $M_0(m_0)$ and $M_0(m_1)$. Motivated by this fact, we define the notion of interface as follows:

Definition 2.1 *Let \mathcal{C} be a small category and \mathcal{D} a complete category. For any representation $M : \mathcal{C} \rightarrow \mathcal{D}$ of \mathcal{C} , its **interface** is the functor $\text{Int}_M : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ defined as follows: for any object $c \in \mathcal{C}$, we define*

$$\text{Int}_M(c) = \lim \left(\text{Elts}(\mathbf{y}(c)) \xrightarrow{\pi_c} \mathcal{C} \xrightarrow{M} \mathcal{D} \right).$$

Let $f : c' \rightarrow c$ be a morphism in \mathcal{C} and $\{\nu_{(c'',g)}^{M,c'} : \text{Int}_M(c') \rightarrow M \circ \pi_{c'}(c'',g) = M(c'')\}_{(c'',g) \in \text{Elts}(\mathbf{y}(c'))}$ the limit on $M \circ \pi_{c'}$ defining $\text{Int}_M(c')$. Since $\{\nu_{(c'',f \circ g)}^{M,c} : \text{Int}_M(c) \rightarrow M \circ \pi_c(c'',f \circ g) = M(c'')\}_{(c'',g) \in \text{Elts}(\mathbf{y}(c'))}$

is a cone on $M \circ \pi_{c'}$, there exists a unique morphism $\xi : \text{Int}_M(c) \rightarrow \text{Int}_M(c')$ such that $\nu_{(c'',g)}^{M,c'} \circ \xi = \nu_{(c'',f \circ g)}^{M,c}$ for all $(c'', g) \in \text{Elts}(\mathbf{y}(c'))$. We define $\text{Int}_M(f) = \xi$.

2.3 Interface transformation

In this subsection, we introduce what we call interface transformation which generalizes the map φ defined in Introduction.

Definition 2.2 Let \mathcal{C} be a small category, \mathcal{D} a bicomplete category and $M : \mathcal{C} \rightarrow \mathcal{D}$ a representation of \mathcal{C} . Fix an object $G \in \hat{\mathcal{C}}$. For any $x \in G(c)$ and $c \in \mathcal{C}$, we put $\varphi_c^{M,G}(x) := \mu_{(c,x)}^{M,G} \circ \nu_{(c,\text{id}_c)}^{M,c}$. As we will see below in Proposition 2.3, $\varphi^{M,G}$ is a natural transformation from the contravariant functor G on \mathcal{C} to the covariant functor $\mathcal{D}(\text{Int}_M(-), G \otimes M)$ ³. We call $\varphi^{M,G}$ **interface transformation** of G with respect to M .

$$\begin{array}{ccc} \text{Int}_M(c) & \xrightarrow{\nu_{(c,\text{id}_c)}^{M,c}} & M(c) \xrightarrow{\mu_{(c,x)}^{M,G}} G \otimes M. \\ & \searrow \varphi_c^{M,G}(x) & \nearrow \end{array}$$

Let us check that interface transformations defined in Definition 2.2 are actually natural transformations from contravariant functors to covariant functors. Hereafter, we write $x \cdot f := G(f)(x)$ for any $x \in G(c)$ and $f : c' \rightarrow c$ in \mathcal{C} .

Proposition 2.3 Let \mathcal{C} be a small category and \mathcal{D} a bicomplete category. For any representation $M : \mathcal{C} \rightarrow \mathcal{D}$ of \mathcal{C} and presheaf $G \in \hat{\mathcal{C}}$, any morphism $f : c' \rightarrow c$ in \mathcal{C} makes the following diagram commute:

$$\begin{array}{ccc} G(c) & \xrightarrow{\varphi_c^{M,G}} & \mathcal{D}(\text{Int}_M(c), G \otimes M) \\ G(f) \downarrow & & \uparrow (-) \circ \text{Int}_M(f) \\ G(c') & \xrightarrow{\varphi_{c'}^{M,G}} & \mathcal{D}(\text{Int}_M(c'), G \otimes M). \end{array}$$

³Natural transformations from a contravariant functor to a covariant functor can be seen as dinatural transformations (MacLane, 1998). Let \tilde{G} be the composition $\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\pi_1} \mathcal{C}^{\text{op}} \xrightarrow{G} \mathbf{Set}$ and \tilde{H} the composition $\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{H} \mathbf{Set}$, where $H = \mathcal{D}(\text{Int}_M(-), G \otimes M)$. Then, $\varphi^{M,G}$ is a dinatural transformation from \tilde{G} to \tilde{H} .

Proof. Let us show $\varphi_c^{M,G}(x) = \varphi_{c'}^{M,G}(x \cdot f) \circ \text{Int}_M(f)$ for any $x \in G(c)$. However, the equality follows from a diagram chasing over the following commutative diagram:

$$\begin{array}{ccccc}
\text{Int}_M(c) & \xrightarrow{\nu_{(c, \text{id}_c)}^{M,c}} & M(c) & & \\
\text{Int}_M(f) \downarrow & \searrow \nu_{(c', f)}^{M,c} & \uparrow M(f) & \searrow \mu_{(c, x)}^{M,G} & \\
\text{Int}_M(c') & \xrightarrow{\nu_{(c', \text{id}_{c'})}^{M,c'}} & M(c') & \xrightarrow{\mu_{(c', x \cdot f)}^{M,G}} & G \otimes M.
\end{array}$$

□

2.4 Category \mathcal{IT}_G

Now, we define a category where interface transformations live.

Definition 2.4 Let \mathcal{C} be a small category. For any $G \in \hat{\mathcal{C}}$, we define a category \mathcal{IT}_G as follows: objects in \mathcal{IT}_G are pairs (H, φ) consisting of a functor $H : \mathcal{C} \rightarrow \mathbf{Set}$ and a natural transformation $\varphi : G \rightarrow H$. Here, the natural transformation φ from the contravariant functor G to the covariant functor H is defined as a family of maps $\{\varphi_c : G(c) \rightarrow H(c)\}_{c \in \mathcal{C}}$ that makes the following diagram commute for any morphism $f : c' \rightarrow c \in \mathcal{C}$:

$$\begin{array}{ccc}
G(c) & \xrightarrow{\varphi_c} & H(c) \\
G(f) \downarrow & & \uparrow H(f) \\
G(c') & \xrightarrow{\varphi_{c'}} & H(c').
\end{array}$$

Morphisms from (H, φ) to (H', φ') are natural transformations $n : H \rightarrow H'$ such that $\varphi' = n \circ \varphi$:

$$\begin{array}{ccc}
H(c) & \xrightarrow{n_c} & H'(c) \\
\swarrow \varphi_c & & \searrow \varphi'_c \\
& G(c) &
\end{array}$$

From Proposition 2.3, we have the following fact:

Proposition 2.5 Let \mathcal{C} be a small category and \mathcal{D} a bicomplete category. For any representation $M : \mathcal{C} \rightarrow \mathcal{D}$ of \mathcal{C} and presheaf $G \in \hat{\mathcal{C}}$, the interface transformation $\varphi^{M,G}$ of G with respect to M gives rise to the object $(\mathcal{D}(\text{Int}_M(-), G \otimes M), \varphi^{M,G})$ in \mathcal{IT}_G .

2.5 Initiality of \mathcal{IT}_G

Theorem 2.6 *For any directed network $G \in \hat{\uparrow}$, \mathcal{IT}_G has an initial object. In particular, the object $(\hat{\uparrow}(\text{Int}_{M_0}(-), G \otimes M_0), \varphi^{M_0, G})$ induced by the interface transformation $\varphi^{M_0, G}$ of G with respect to the representation $M_0 : \hat{\uparrow} \rightarrow \hat{\uparrow}$ given in subsection 2.2 is an initial object in \mathcal{IT}_G .*

Proof. We show that for any object (H, ψ) in \mathcal{IT}_G , there exists a unique morphism $\iota : \hat{\uparrow}(\text{Int}_{M_0}(-), G \otimes M_0) \rightarrow H$ such that $\psi = \iota \circ \varphi^{M_0, G}$.

By $\text{Int}_{M_0}(0) = M_0(0) \cong \mathbf{y}(1)$, we have $\hat{\uparrow}(\text{Int}_{M_0}(0), G \otimes M_0) \cong G \otimes M_0(1) \cong G(0)$. If we identify the leftmost-hand side with the rightmost-hand side of this sequence of isomorphisms, then one can see that $\varphi_0^{M_0, G}$ becomes the identity $\text{id}_{G(0)}$ by chasing how elements move by each isomorphism. Hence, we should take ι_0 as $\iota_0 := \psi_0$.

Let us construct ι_1 . In general, for any representation $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ of a small category \mathcal{C} , we can take

$$G \otimes M(c) = \left(\sum_{c' \in \mathcal{C}} G(c') \times M(c')(c) \right) / \sim$$

for any $G \in \hat{\mathcal{C}}$ and $c \in \mathcal{C}$. Here, \sim is the equivalence relation generated by the following binary relation r : $(x, f \cdot p)r(x \cdot f, p)$ for $x \in G(c')$, $p \in M(c')(c)$ and $f : c' \rightarrow c'' \in \mathcal{C}$. Components of the colimit $\mu^{M, G}$ are given by $\mu_{(c, x)}^{M, G}(c') : M(c)(c') \rightarrow G \otimes M(c') : p \mapsto [(x, p)]$, where $[(x, p)]$ is the equivalence class containing (x, p) .

Now, we have $\text{Int}_{M_0}(1) = \{(p_1, p_0)\} \cong \mathbf{y}(0)$ and $\hat{\uparrow}(\text{Int}_{M_0}(1), G \otimes M_0) \cong G \otimes M_0(0)$. We also have $\nu_{(1, \text{id}_1)}^{M_0, 1}(0)(p_1, p_0) = q_1 \in M_0(1)(0)$, $\nu_{(0, m_i)}^{M_0, 1}(0)(p_1, p_0) = p_{i+1} \in M_0(0)(0)$ for $i = 0, 1$. Since $\text{Int}_{M_0}(0) = M_0(0)$, $\text{Int}_{M_0}(m_i) = \nu_{(0, m_i)}^{M_0, 1}$ for $i = 0, 1$. Here, $i + 1$ is considered modulo 2. Hence, we have $\varphi_1^{M_0, G}(x) = [(x, q_1)]$ for any $x \in G(1)$. $(-) \circ \text{Int}_{M_0}(m_i) : \hat{\uparrow}(\text{Int}_{M_0}(0), G \otimes M_0) = G(0) \rightarrow \hat{\uparrow}(\text{Int}_{M_0}(1), G \otimes M_0) = G \otimes M_0(0)$ is given by $x \mapsto [(x, p_{i+1})]$.

For $i = 0, 1$, ι_1 should make the following diagram commute:

$$\begin{array}{ccc} G \otimes M_0(0) & \xrightarrow{\iota_1} & H(1) \\ \uparrow [(-, p_{i+1})] & & \uparrow H(m_i) \\ G(0) & \xrightarrow{\iota_0} & H(0). \end{array}$$

Namely, for any $x \in G(0)$, we should have $\iota_1([(x, p_{i+1})]) = H(m_i) \circ \iota_0(x)$, $i = 0, 1$. Let us see that ι_1 is uniquely determined by this condition.

First, any element of $G \otimes M_0(0) = [G(0) \times \{p_0, p_1\} + G(1) \times \{q_0, q_1, q_2\}] / \sim$ has a representative of the form (x, p) , $x \in G(0)$, $p \in \{p_0, p_1\}$. Indeed, for any $y \in G(1)$, we have $(y, q_0) = (y, m_0 \cdot p_0) \sim (y \cdot m_0, p_0)$, $(y, q_1) = (y, m_0 \cdot p_1) \sim (y \cdot m_0, p_1)$ and $(y, q_2) = (y, m_1 \cdot p_1) \sim (y \cdot m_1, p_1)$.

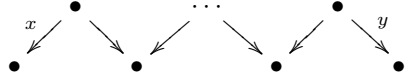
Second, ι_1 is well-defined. It is enough to show that $H(m_0) \circ \iota_0(x) = H(m_1) \circ \iota_0(y)$ for any $x, y \in G(0)$ such that $x = z \cdot m_0$, $y = z \cdot m_1$ for some $z \in G(1)$. However, since ψ is a natural transformation from G to H , we obtain $H(m_0) \circ \iota_0(x) = H(m_0) \circ \psi_0(x) = H(m_0) \circ \psi_0(z \cdot m_0) = \psi_1(z) = H(m_1) \circ \psi_0(z \cdot m_1) = H(m_1) \circ \psi_0(y) = H(m_1) \circ \iota_0(y)$.

Finally, $\psi_1 = \iota_1 \circ \varphi_1^{M_0, G}$ holds because $\iota_1 \circ \varphi_1^{M_0, G}(x) = \iota_1([(x, q_1)]) = \iota_1([(x \cdot m_0, p_1)]) = H(m_0) \circ \iota_0(x \cdot m_0) = H(m_0) \circ \psi_0(x \cdot m_0) = \psi_1(x)$ for any $x \in G(1)$.

□

Corollary 2.7 *Let G be a directed network. For any representation $M : \uparrow\uparrow \rightarrow \mathcal{D}$ of $\uparrow\uparrow$ in any bicomplete category \mathcal{D} , let \sim_M be the equivalence relation on $G(1)$ induced by the fibers of the map $\varphi_1^{M, G} : G(1) \rightarrow \mathcal{D}(\text{Int}_M(1), G \otimes M)$, the 1-component of the interface transformation of G with respect to M . Namely, we define $x \sim_M y : \Leftrightarrow \varphi_1^{M, G}(x) = \varphi_1^{M, G}(y)$. Then, $\sim_{M_0} \subseteq \sim_M$ for any M .*

The equivalence relation \sim_{M_0} on $G(1)$ in Corollary 2.7 can be given explicitly as follows: for $x, y \in G(1)$, since $\varphi_1^{M_0, G}(x) = \varphi_1^{M_0, G}(y) \Leftrightarrow [(x, q_1)] = [(y, q_1)]$, we have $x \sim_{M_0} y$ if and only if there exists an alternating sequence of arcs between x and y in G such as



Note that there are four cases for the situation at both ends of the sequence. For $x, y \in G(1)$, we say that x and y are **laterally connected** if $\varphi_1^{M_0, G}(x) = \varphi_1^{M_0, G}(y)$. The alternating sequence of arcs connecting x and y is called **lateral path**.

2.6 Examples

Example 2.8 Let us consider the representation $\mathbf{y} : \uparrow\uparrow \rightarrow \hat{\uparrow\uparrow}$. For any directed network $G \in \hat{\uparrow\uparrow}$, $\varphi_0^{\mathbf{y}, G}$ is an isomorphism by $\text{Int}_{\mathbf{y}}(0) = \mathbf{y}(0)$. $\varphi_1^{\mathbf{y}, G}$ is a map to a singleton set because $\text{Int}_{\mathbf{y}}(1) = \emptyset$. Consequently, the equivalence relation $\sim_{\mathbf{y}}$ on $G(1)$ induced by the map $\varphi_1^{\mathbf{y}, G}$ is $G(1) \times G(1)$. Namely, any $x, y \in G(1)$ are equivalent.

Example 2.9 Let us define the representation $M_u : \uparrow\uparrow \rightarrow \hat{\uparrow\uparrow}$ as follows:

$$\begin{aligned} M_u(0) &= p_0 \xrightarrow{a} p_1, \\ M_u(1) &= q_0 \xrightarrow{b_0} q_1 \xleftarrow{b_1} q_2, \end{aligned}$$

$m_0 \cdot a = b_0$ and $m_1 \cdot a = b_1$. We have $\text{Int}_{M_u}(0) = M_u(0)$ and $\text{Int}_{M_u}(1) = \{(p_1, p_1)\}$.

For any directed network $G \in \hat{\uparrow}$, $\varphi_0^{M_u, G}$ is an isomorphism. The equivalence relation \sim_{M_u} on $G(1)$ defined by the fibers of $\varphi_1^{M_u, G}$ is generated by the equations $x \cdot m_0 = y \cdot m_0$, $x \cdot m_0 = y \cdot m_1$, $x \cdot m_1 = y \cdot m_0$ and $x \cdot m_1 = y \cdot m_1$ because $\varphi_1^{M_u, G}(x) = \varphi_1^{M_u, G}(y) \Leftrightarrow [(x, q_1)] = [(y, q_1)]$ for any $x, y \in G(1)$ and $q_1 = m_0 \cdot p_1 = m_1 \cdot p_1$. Thus, for any $x, y \in G(1)$, $x \sim_{M_u} y$ if and only if there exists a sequence of arcs connecting x and y in G without regard to the direction of arcs. We call this kind of sequence of arcs **undirected path**.

Example 2.10 In general, we can calculate the interface transformation $\varphi^{M, G}$ of a directed network G with respect to a representation $M : \hat{\uparrow} \rightarrow \hat{\uparrow}$ as follows: for any $x \in G(0)$, the 0-component

$$\varphi_0^{M, G}(x) : \text{Int}_M(0) = M(0) \xrightarrow{\nu_{(0, \text{id}_0)}^{M, 0} = \text{id}_{M(0)}} M(0) \xrightarrow{\mu_{(0, x)}^{M, G}} G \otimes M$$

is given by $\varphi_0^{M, G}(x)(0)(p) = [(x, p)]$ for any $p \in M(0)(0)$ and $\varphi_0^{M, G}(x)(1)(a) = [(x, a)]$ for any $a \in M(0)(1)$.

Next we consider the 1-component $\varphi_1^{M, G}$. We have $\text{Int}_M(1)(0) = \{(p, p') \in M(0)(0) \times M(0)(0) \mid m_0 \cdot p = m_1 \cdot p'\}$ and $\text{Int}_M(1)(1) = \{(a, a') \in M(0)(1) \times M(0)(1) \mid m_0 \cdot a = m_1 \cdot a'\}$. In the commutative diagram

$$\begin{array}{ccc} \text{Int}_M(1) & \xrightarrow{\nu_{(0, m_1)}^{M, 1}} & M(0) \\ \nu_{(0, m_0)}^{M, 1} \downarrow & \searrow \nu_{(1, \text{id}_1)}^{M, 1} & \downarrow M(m_1) \\ M(0) & \xrightarrow{M(m_0)} & M(1), \end{array}$$

morphisms $\nu_{(0, m_0)}^{M, 1}$ and $\nu_{(0, m_1)}^{M, 1}$ consists of projection maps to the first component and the second component, respectively. Consequently, for any $x \in G(1)$,

$$\varphi_1^{M, G}(x) : \text{Int}_M(1) \xrightarrow{\nu_{(1, \text{id}_1)}^{M, 1}} M(1) \xrightarrow{\mu_{(1, x)}^{M, G}} G \otimes M$$

is given by $\varphi_1^{M, G}(x)(0)(p, p') = [(x, q)]$ for any $(p, p') \in \text{Int}_M(1)(0)$, where $q := m_0 \cdot p = m_1 \cdot p'$, and $\varphi_1^{M, G}(x)(1)(a, a') = [(x, b)]$ for any $(a, a') \in \text{Int}_M(1)(1)$, where $b := m_0 \cdot a = m_1 \cdot a'$.

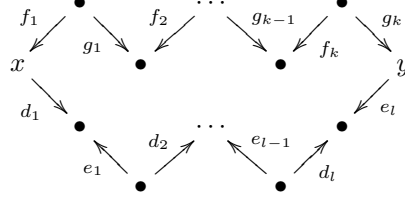
Example 2.11 We define the representation $M_1 : \hat{\uparrow} \rightarrow \hat{\uparrow}$ by

$$\begin{aligned} M_1(0) &= p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} p_2, \\ M_1(1) &= q_0 \xrightarrow{b_0} q_1 \xrightarrow{b_1} q_2 \xrightarrow{b_2} q_3, \end{aligned}$$

$m_0 \cdot a_0 = b_0$, $m_0 \cdot a_1 = b_1$, $m_1 \cdot a_0 = b_1$ and $m_1 \cdot a_1 = b_2$. We have $\text{Int}_{M_1}(1)(0) = \{(p_1, p_0), (p_2, p_1)\}$ and

$$\text{Int}_{M_1}(1)(1) = \{(a_1, a_0)\}.$$

Let G be a directed network. Since $m_0 \cdot a_1 = b_1$ and $m_1 \cdot a_0 = b_1$, $\varphi_0^{M_1, G}(x) = \varphi_0^{M_1, G}(y) \Leftrightarrow$ there exist $f_1, g_1, \dots, f_k, g_k \in G(1)$ ($k \geq 0$) such that $x = f_1 \cdot m_1, f_i \cdot m_0 = g_i \cdot m_0, g_i \cdot m_1 = f_{i+1} \cdot m_1$ ($i = 1, 2, \dots, k-1$), $y = g_k \cdot m_1$ and there exist $d_1, e_1, \dots, d_l, e_l \in G(1)$ ($l \geq 0$) such that $x = d_1 \cdot m_0, d_i \cdot m_1 = e_i \cdot m_1, e_i \cdot m_0 = d_{i+1} \cdot m_0$ ($i = 1, 2, \dots, l-1$), $y = e_l \cdot m_0$ for any $x, y \in G(0)$.



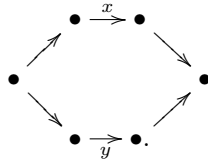
For any $x, y \in G(1)$, we have $\varphi_1^{M_1, G}(x) = \varphi_1^{M_1, G}(y) \Leftrightarrow [(x, b_1)] = [(y, b_1)]$. One can see that $[(x, b_1)] = [(y, b_1)]$ is equivalent to the existence of a lateral path between x and y .

Example 2.12 The representation $M_2 : \uparrow\uparrow \rightarrow \hat{\uparrow}\hat{\uparrow}$ is defined by

$$\begin{aligned} M_2(0) &= p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} p_2 \xrightarrow{a_2} p_3, \\ M_2(1) &= q_0 \xrightarrow{b_0} q_1 \xrightarrow{b_1} q_2 \xrightarrow{b_2} q_3 \xrightarrow{b_3} q_4 \end{aligned}$$

and $m_0 \cdot a_i = b_i$ and $m_1 \cdot a_i = b_{i+1}$ for $i = 0, 1, 2$.

Let G be a directed network. For any $x, y \in G(1)$, one can see that if x and y are connected by a lateral path in G , then $\varphi_1^{M_2, G}(x) = \varphi_1^{M_2, G}(y)$, and if $\varphi_1^{M_2, G}(x) = \varphi_1^{M_2, G}(y)$, then there exists an undirected path between x and y in G . However, the converses do not necessarily hold for both implications. For the first one, if G is given by the following directed network, then we have $\varphi_1^{M_2, G}(x) = \varphi_1^{M_2, G}(y)$:



For the second one, if G is the following directed network, then we have $\varphi_1^{M_2, G}(x) \neq \varphi_1^{M_2, G}(y)$:

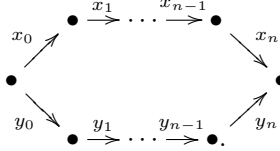


Example 2.13 In general, we define the representation $M_n : \uparrow \rightarrow \hat{\uparrow}$ by

$$\begin{aligned} M_n(0) &= p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} \cdots \xrightarrow{a_n} p_{n+1} , \\ M_n(1) &= q_0 \xrightarrow{b_0} q_1 \xrightarrow{b_1} \cdots \xrightarrow{b_{n+1}} q_{n+2} \end{aligned}$$

and $m_0 \cdot a_i = b_i$ and $m_1 \cdot a_i = b_{i+1}$ for $i = 0, 1, \dots, n$.

Let G_n be the following directed network:

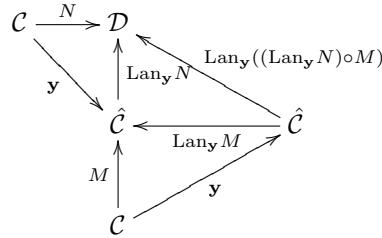


For $n \geq 1$, we have $\varphi_1^{M_{2n}, G_{2n}}(x_n) = \varphi_1^{M_{2n}, G_{2n}}(y_n)$ and $\varphi_1^{M_{2(n-1)}, G_{2n}}(x_n) \neq \varphi_1^{M_{2(n-1)}, G_{2n}}(y_n)$.

M_n can be seen as a $n+1$ -fold tensor product of M_0 . In the next subsection, we discuss tensor product of representations.

2.7 Tensor product of representations and morphisms in \mathcal{IT}_G

Let \mathcal{C} be a small category and \mathcal{D} a cocomplete category. The tensor product of representations $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ and $N : \mathcal{C} \rightarrow \mathcal{D}$ is defined by $M \otimes N := (\text{Lan}_{\mathbf{y}} N) \circ M$.



The above diagram commutes up to isomorphism. Indeed, since left adjoints preserve colimits, we have a canonical isomorphism

$$\alpha_G : \text{Lan}_{\mathbf{y}} N (\text{Lan}_{\mathbf{y}} M(G)) \xrightarrow{\cong} \text{Lan}_{\mathbf{y}} ((\text{Lan}_{\mathbf{y}} N) \circ M) (G)$$

for any presheaf $G \in \hat{\mathcal{C}}$. If we use the symbol \otimes , then this becomes

$$\alpha_G : (G \otimes M) \otimes N \xrightarrow{\cong} G \otimes (M \otimes N) .$$

Note that α_G is a unique morphism such that $\alpha_G^{-1} \circ \mu_{(c,x)}^{M \otimes N, G} = \mu_{(c,x)}^{M, G} \otimes N$ for all $(c, x) \in \text{Elts}(G)$.

Example 2.14 For any representation $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ of a small category \mathcal{C} , we define $M^{\otimes n} := \underbrace{M \otimes \cdots \otimes M}_{n \text{ times}}$ up to canonical isomorphisms. We have $M_0^{\otimes n+1} \cong M_n$, where M_n is the representation introduced in Example 2.13.

Let \mathcal{C} be a small category, \mathcal{D} a bicomplete category and G a presheaf on \mathcal{C} . From Proposition 2.5, we can obtain an object of \mathcal{IT}_G from any representation $M : \mathcal{C} \rightarrow \mathcal{D}$. Now, we describe a way to obtain a morphism of \mathcal{IT}_G .

Let $\{\nu_{(c',f)}^{M,c} : \text{Int}_M(c) \rightarrow M \circ \pi_c(c',f) = M(c')\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))}$ be the limit on $M \circ \pi_c$ defining $\text{Int}_M(c)$ and $\{\nu_{(c',f)}^{M \otimes N, c} : \text{Int}_{M \otimes N}(c) \rightarrow (M \otimes N) \circ \pi_c(c',f) = M(c') \otimes N\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))}$ the limit on $(M \otimes N) \circ \pi_c$ defining $\text{Int}_{M \otimes N}(c)$. Since $\{\nu_{(c',f)}^{M,c} \otimes N : \text{Int}_M(c) \otimes N \rightarrow M(c') \otimes N\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))}$ is a cone on $(M \otimes N) \circ \pi_c$, there exists a unique morphism $\beta_c : \text{Int}_M(c) \otimes N \rightarrow \text{Int}_{M \otimes N}(c)$ such that $\nu_{(c',f)}^{M,c} \otimes N = \nu_{(c',f)}^{M \otimes N, c} \circ \beta_c$ for all $(c',f) \in \text{Elts}(\mathbf{y}(c))$ by the universality of $\text{Int}_{M \otimes N}(c)$. β_c is natural with respect to c . Indeed, we have the following commutative diagram for any morphisms $c'' \xrightarrow{f'} c' \xrightarrow{f} c$ in \mathcal{C} :

$$\begin{array}{ccc}
\text{Int}_M(c) \otimes N & \xrightarrow{\beta_c} & \text{Int}_{M \otimes N}(c) \\
\downarrow \text{Int}_M(f) \otimes N & \searrow \nu_{(c'',f \circ f')}^{M,c} \otimes N & \swarrow \nu_{(c'',f \circ f')}^{M \otimes N, c} \\
& M \otimes N(c'') & \\
& \swarrow \nu_{(c'',f')}^{M,c'} \otimes N & \searrow \nu_{(c'',f')}^{M \otimes N, c'} \\
\text{Int}_M(c') \otimes N & \xrightarrow{\beta_{c'}} & \text{Int}_{M \otimes N}(c')
\end{array}$$

Lemma 2.15 Let \mathcal{A} and \mathcal{B} be categories and $F : \mathcal{A} \rightarrow \mathcal{B}$ a functor. Assume that F has a limit $\{\nu_A : L \rightarrow F(A)\}_{A \in \mathcal{A}}$. Given two morphisms $u, v : B \rightarrow L$ in \mathcal{B} , if $\nu_A \circ u = \nu_A \circ v$ for all $A \in \mathcal{A}$, then $u = v$.

Proof. Put $\delta_A := \nu_A \circ u = \nu_A \circ v$. Since $\{\delta_A : B \rightarrow F(A)\}_{A \in \mathcal{A}}$ is a cone on F , there exists a unique morphism η in \mathcal{B} such that $\nu_A \circ \eta = \delta_A$ for all $A \in \mathcal{A}$. However, both u and v satisfy this condition. Consequently, $u = \eta = v$. □

By applying Lemma 2.15 with $u = \text{Int}_{M \otimes N}(f) \circ \beta_c$ and $v = \beta_{c'} \circ \text{Int}_M(f) \otimes N$, we get the equality $\text{Int}_{M \otimes N}(f) \circ \beta_c = \beta_{c'} \circ \text{Int}_M(f) \otimes N$. Consequently, we obtain a natural transformation $\beta : \text{Int}_M(-) \otimes N \rightarrow \text{Int}_{M \otimes N}$.

Proposition 2.16 Let \mathcal{C} be a small category and \mathcal{D} a bicomplete category. For representations $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ and $N : \mathcal{C} \rightarrow \mathcal{D}$, assume that the natural transformation $\beta : \text{Int}_M(-) \otimes N \rightarrow \text{Int}_{M \otimes N}$ is an isomorphism. Then, for any presheaf $G \in \hat{\mathcal{C}}$, a morphism $j : (\hat{\mathcal{C}}(\text{Int}_M(-), G \otimes M), \varphi^{M,G}) \rightarrow (\mathcal{D}(\text{Int}_{M \otimes N}(-), G \otimes (M \otimes$

$N)), \varphi^{M \otimes N, G})$ in \mathcal{IT}_G is obtained by defining $j_c(\theta)$ as the composition

$$\text{Int}_{M \otimes N}(c) \xrightarrow{\beta_c^{-1}} \text{Int}_M(c) \otimes N \xrightarrow{\theta \otimes N} (G \otimes M) \otimes N \xrightarrow{\alpha_G} G \otimes (M \otimes N)$$

for any $c \in \mathcal{C}$ and morphism $\theta : \text{Int}_M(c) \rightarrow G \otimes M$.

Proof. First, let us check that j is a natural transformation. We have to show that the following diagram commutes for any morphism $f : c \rightarrow c'$ in \mathcal{C} :

$$\begin{array}{ccc} \hat{\mathcal{C}}(\text{Int}_M(c), G \otimes M) & \xrightarrow{j_c} & \mathcal{D}(\text{Int}_{M \otimes N}(c), G \otimes (M \otimes N)) \\ (-) \circ \text{Int}_M(f) \downarrow & & \downarrow (-) \circ \text{Int}_{M \otimes N}(f) \\ \hat{\mathcal{C}}(\text{Int}_M(c'), G \otimes M) & \xrightarrow{j_{c'}} & \mathcal{D}(\text{Int}_{M \otimes N}(c'), G \otimes (M \otimes N)). \end{array}$$

However, for any morphism $\theta : \text{Int}_M(c) \rightarrow G \otimes M$, we have

$$\begin{aligned} j_{c'}(\theta \circ \text{Int}_M(f)) &= \alpha_G \circ ((\theta \circ \text{Int}_M(f)) \otimes N) \circ \beta_{c'}^{-1} \\ &= \alpha_G \circ (\theta \otimes N) \circ (\text{Int}_M(f) \otimes N) \circ \beta_{c'}^{-1} \\ &= \alpha_G \circ (\theta \otimes N) \circ \beta_c^{-1} \circ \text{Int}_{M \otimes N}(f) \\ &= j_c(\theta) \circ \text{Int}_{M \otimes N}(f), \end{aligned}$$

where the second equality follows from the functoriality of $(-) \otimes N$ and the third equality follows from the naturality of β .

Next, we show that the following diagram commutes for any $c \in \mathcal{C}$:

$$\begin{array}{ccc} \hat{\mathcal{C}}(\text{Int}_M(c), G \otimes M) & \xrightarrow{j_c} & \mathcal{D}(\text{Int}_{M \otimes N}(c), G \otimes (M \otimes N)) \\ & \swarrow \varphi_c^{M, G} \quad \searrow \varphi_c^{M \otimes N, G} & \\ & G(c). & \end{array}$$

However, for any $c \in \mathcal{C}$ and $x \in G(c)$, we have

$$\begin{aligned} j_c(\varphi_c^{M, G}(x)) &= j_c(\mu_{(c, x)}^{M, G} \circ \nu_{(c, \text{id}_c)}^{M, c}) \\ &= \alpha_G \circ \left((\mu_{(c, x)}^{M, G} \circ \nu_{(c, \text{id}_c)}^{M, c}) \otimes N \right) \circ \beta_c^{-1} \\ &= \alpha_G \circ \left(\mu_{(c, x)}^{M, G} \otimes N \right) \circ \left(\nu_{(c, \text{id}_c)}^{M, c} \otimes N \right) \circ \beta_c^{-1} \\ &= \mu_{(c, x)}^{M \otimes N, G} \circ \nu_{(c, \text{id}_c)}^{M \otimes N, c} \\ &= \varphi_c^{M \otimes N, G}(x), \end{aligned}$$

where the fourth equality follows from the defining properties of α_G and β_c .

□

Example 2.17 Let \mathcal{C} be a small category. For any representation $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ and $\mathbf{y} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$, let us consider the tensor product $M \otimes \mathbf{y} \cong M$. By $\text{Int}_M(-) \otimes \mathbf{y} \cong \text{Int}_M \cong \text{Int}_{M \otimes \mathbf{y}}$, the morphism $j : \varphi^{M,G} \rightarrow \varphi^{M \otimes \mathbf{y}, G}$ induced in \mathcal{IT}_G is an isomorphism for any $G \in \hat{\mathcal{C}}$.

Example 2.18 Let \mathcal{D} be a bicomplete category. Let us consider the tensor product $\mathbf{y} \otimes N \cong N$ where $N : \uparrow \uparrow \rightarrow \mathcal{D}$ is a representation.

We have $\text{Int}_{\mathbf{y} \otimes N}(1) \cong \text{Int}_N(1)$. Since $\text{Int}_{\mathbf{y}}(1) = \emptyset$, $\text{Int}_{\mathbf{y}}(1) \otimes N = \emptyset \otimes N$ is an initial object of \mathcal{D} . Consequently, $\text{Int}_{\mathbf{y} \otimes N}(1) \not\cong \text{Int}_{\mathbf{y}}(1) \otimes N$ if $\text{Int}_N(1)$ is not an initial object in \mathcal{D} .

Example 2.19 We define the representation $M_{1'} : \uparrow \uparrow \rightarrow \hat{\uparrow \uparrow}$ by

$$\begin{aligned} M_{1'}(0) &= p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} p_2, \\ M_{1'}(1) &= q_0 \xrightarrow{b_0} q_1 \xrightarrow[b_{11}]{b_{10}} q_2 \xrightarrow{b_2} q_3 \end{aligned}$$

and $m_0 \cdot a_0 = b_0$, $m_0 \cdot a_1 = b_{10}$, $m_1 \cdot a_0 = b_{11}$ and $m_1 \cdot a_1 = b_2$.

We have $\text{Int}_{M_{1'}}(1) \cong \{\bullet \bullet\}$ and $\text{Int}_{M_{1'}}(1) \otimes M_0 \cong \{\bullet \longrightarrow \bullet \bullet \longrightarrow \bullet\}$. On the other hand, $\text{Int}_{M_{1'} \otimes M_0}(1) \cong \{\bullet \longrightarrow \bullet \longrightarrow \bullet\}$ because $M_{1'} \otimes M_0 \cong M_0^{\otimes 3}$.

Example 2.20 One can show $\beta : \text{Int}_{M_0^{\otimes n}}(-) \otimes M_0 \xrightarrow{\cong} \text{Int}_{M_0^{\otimes n+1}}$ by direct calculation. Therefore, we have a sequence of morphisms

$$\varphi^{M_0, G} \longrightarrow \varphi^{M_0^{\otimes 2}, G} \longrightarrow \dots \longrightarrow \varphi^{M_0^{\otimes n}, G} \longrightarrow \dots$$

in \mathcal{IT}_G for any directed network $G \in \hat{\uparrow \uparrow}$. In particular, by Example 2.13, we can take every morphism in the following sequence

$$\varphi^{M_0, G} \longrightarrow \varphi^{M_0^{\otimes 3}, G} \longrightarrow \varphi^{M_0^{\otimes 5}, G} \longrightarrow \dots$$

as a non-isomorphism.

2.8 Stability with respect to representations and gluing condition for representations

In this subsection, we study significance of M_0 from a different point of view.

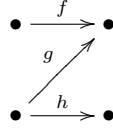
Definition 2.21 Let \mathcal{C} be a small category, \mathcal{D} a bicomplete category and $M : \mathcal{C} \rightarrow \mathcal{D}$ a representation. We say that a presheaf $G \in \hat{\mathcal{C}}$ is **stable** with respect to the representation M if the unit $\eta^{M,G} : G \rightarrow \mathcal{D}(M(-), G \otimes M)$ is an isomorphism⁴.

Stability of $G \in \hat{\mathcal{C}}$ with respect to a representation M concerns when we can recover G from $G \otimes M$, a representation of its dynamic mode.

Example 2.22 Let \mathcal{C} be a small category. Any presheaf $G \in \hat{\mathcal{C}}$ is stable with respect to the representation $\mathbf{y} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$.

Example 2.23 A directed network $G \in \hat{\uparrow}$ is stable with respect to the representation M_0 if and only if it satisfies the following two conditions (Pultr, 1979; Haruna and Gunji, 2007): (i) if $\bullet \xrightarrow[f]{g} \bullet$ in G then $f = g$.

(ii) If



in G , then there exists an arc $\partial_0(f) \xrightarrow{k} \partial_1(h)$ in G .

Example 2.24 $G \in \hat{\uparrow}$ is stable with respect to the representation M_u if and only if G is a graph of an equivalence relation.

A trivial observation is that the representation $\mathbf{y} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ is a minimal representation with respect to stability in the sense that for any representation M , if $G \in \hat{\mathcal{C}}$ is stable with respect to M , then G is stable with respect to \mathbf{y} by Example 2.22. However, if we restrict representations into those with a ‘nice’ property, then the representation M_0 in turn becomes a minimal one for $\mathcal{C} = \hat{\uparrow}$.

Definition 2.25 Let \mathcal{D} be a bicomplete category. We say that a representation $M : \hat{\uparrow} \rightarrow \mathcal{D}$ satisfies the **gluing condition** if the pullback diagram defining $\text{Int}_M(1)$

$$\begin{array}{ccc} \text{Int}_M(1) & \xrightarrow{\nu_{(0,m_1)}^{M,1}} & M(0) \\ \nu_{(0,m_0)}^{M,1} \downarrow & & \downarrow M(m_1) \\ M(0) & \xrightarrow{M(m_0)} & M(1) \end{array}$$

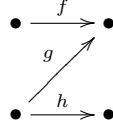
is also a pushout diagram.

⁴A stable directed network G with respect to M is an instance of free algebra of the monad induced by the adjunction $(-) \otimes M \dashv \mathcal{D}(M(-), -)$.

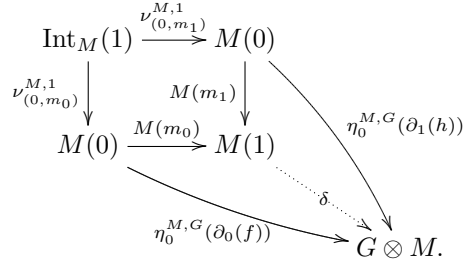
Intuitively, in a representation M satisfying the gluing condition, $M(1)$ precisely consists of two images of $M(0)$ glued on the interface.

Theorem 2.26 *Let \mathcal{D} be a bicomplete category and a representation $M : \uparrow \rightarrow \mathcal{D}$ satisfy the gluing condition. If a directed network $G \in \hat{\uparrow}$ is stable with respect to M , then G is also stable with respect to M_0 .*

Proof. We show that the two conditions (i) and (ii) in Example 2.23 hold for a directed network G when $\eta^{M,G} : G \rightarrow \mathcal{D}(M(-), G \otimes M)$ is an isomorphism. First, we consider (ii). Let us assume



in G . Let us consider the diagram



Note that $\text{Int}_M(m_i) = \nu_{(0,m_i)}^{M,1}$ for $i = 0, 1$.

If the outside square is commutative, then we obtain a unique morphism $\delta : M(1) \rightarrow G \otimes M$ making the diagram commute because the inner square is a pushout. Let us put $k = \left(\eta_1^{M,G}\right)^{-1}(\delta) \in G(1)$. Since $\delta \circ M(m_0) = \eta_0^{M,G}(\partial_0(f))$ and $\delta \circ M(m_1) = \eta_0^{M,G}(\partial_1(h))$, we have $\partial_0(k) = \partial_0\left(\left(\eta_1^{M,G}\right)^{-1}(\delta)\right) = \left(\eta_0^{M,G}\right)^{-1}(\delta \circ M(m_0)) = \partial_0(f)$ and $\partial_1(k) = \partial_1\left(\left(\eta_1^{M,G}\right)^{-1}(\delta)\right) = \left(\eta_0^{M,G}\right)^{-1}(\delta \circ M(m_1)) = \partial_1(h)$ by the naturality of $\left(\eta^{M,G}\right)^{-1}$. This means that we get an arc $\partial_0(f) \xrightarrow{k} \partial_1(h)$ in G .

Thus, it is enough to show that $\eta_0^{M,G}(\partial_0(f)) \circ \text{Int}_M(m_0) = \eta_0^{M,G}(\partial_1(h)) \circ \text{Int}_M(m_1)$. However, since $\varphi_0^{M,G} = \eta_0^{M,G}$ and $\varphi_1^{M,G} = \eta_0^{M,G}(\partial_i(-)) \circ \text{Int}_M(m_i)$ for $i = 0, 1$ by the naturality of $\varphi^{M,G}$, we have $\eta_0^{M,G}(\partial_0(f)) \circ \text{Int}_M(m_0) = \varphi_1^{M,G}(f) = \eta_0^{M,G}(\partial_1(f)) \circ \text{Int}_M(m_1) = \eta_0^{M,G}(\partial_1(g)) \circ \text{Int}_M(m_1) = \varphi_1^{M,G}(g) = \eta_0^{M,G}(\partial_0(g)) \circ \text{Int}_M(m_0) = \eta_0^{M,G}(\partial_0(h)) \circ \text{Int}_M(m_0) = \varphi_1^{M,G}(h) = \eta_0^{M,G}(\partial_1(h)) \circ \text{Int}_M(m_1)$.

Next we consider (i). Assume that $\bullet \xrightarrow[g]{f} \bullet$ in G . Namely, $\partial_0(f) = \partial_0(g)$ and $\partial_1(f) = \partial_1(g)$ for $f, g \in G(1)$. We have $\eta_0^{M,G}(\partial_i(f)) = \eta_0^{M,G}(\partial_i(g))$ for $i = 0, 1$. On the other hand, since $\eta_0^{M,G}(\partial_0(f)) \circ \text{Int}_M(m_0) = \varphi_1^{M,G}(f) = \eta_0^{M,G}(\partial_1(f)) \circ \text{Int}_M(m_1)$, we obtain a unique morphism $\delta : M(1) \rightarrow G \otimes M$ such that $\delta \circ M(m_i) = \eta_0^{M,G}(\partial_i(f))$ for $i = 0, 1$ by the same way as in the proof of (ii) above. Now, $\eta_1^{M,G}(f)$

satisfies the conditions for δ , it follows that $\delta = \eta_1^{M,G}(f)$ by the uniqueness. Since $\partial_i(f) = \partial_i(g)$ for $i = 0, 1$, $\eta_1^{M,G}(g)$ also satisfies the condition. Thus, we also have $\delta = \eta_1^{M,G}(g)$. Consequently, we have $\eta_1^{M,G}(f) = \eta_1^{M,G}(g)$. Since $\eta^{M,G}$ is an isomorphism, the equality $f = g$ holds.

□

The next proposition provides a way to produce a new representation satisfying the gluing condition from an old one.

Proposition 2.27 *Let \mathcal{D} be a bicomplete category. If a representation $M : \uparrow\uparrow \rightarrow \hat{\uparrow}\uparrow$ satisfies the gluing condition, then for any representation $N : \uparrow\uparrow \rightarrow \mathcal{D}$, the tensor product representation $M \otimes N$ also satisfies the gluing condition.*

Proof. Since M satisfies the gluing condition,

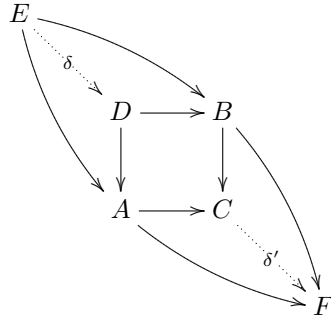
$$\begin{array}{ccc} \text{Int}_M(1) & \xrightarrow{\nu_{(0,m_1)}^{M,1}} & M(0) \\ \nu_{(0,m_0)}^{M,1} \downarrow & & \downarrow M(m_1) \\ M(0) & \xrightarrow{M(m_0)} & M(1) \end{array}$$

is a pushout diagram. Since $(-) \otimes N$ preserves colimits, the outside square of the following commutative diagram is also a pushout:

$$\begin{array}{ccccc} \text{Int}_M(1) \otimes N & & \xrightarrow{\nu_{(0,m_1)}^{M,1} \otimes N} & & M(0) \otimes N \\ & \searrow \nu_{(0,m_0)}^{M,1} \otimes N & & \searrow \nu_{(0,m_1)}^{M \otimes N, 1} & \\ & & \text{Int}_{M \otimes N}(1) & \xrightarrow{\nu_{(0,m_1)}^{M \otimes N, 1}} & M(0) \otimes N \\ & & \downarrow \nu_{(0,m_0)}^{M \otimes N, 1} & & \downarrow M(m_1) \otimes N \\ & & M(0) \otimes N & \xrightarrow{M(m_0) \otimes N} & M(1) \otimes N. \end{array}$$

The inner square is a pullback by the definition of interface. Then, we can show that the inner square is automatically a pushout, which proves that $M \otimes N$ satisfies the gluing condition.

Indeed, let us consider a diagram



in an arbitrary category \mathcal{A} . Let us assume that

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a pullback and

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a pushout. By the universality of the pullback, there exists a unique morphism $\delta : E \rightarrow D$ making the appropriate parts of the diagram commute. If

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & F \end{array}$$

commutes, then

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & F \end{array}$$

also commutes. By the universality of the pushout, there exists a unique morphism $\delta' : C \rightarrow F$ making the appropriate parts of the diagram commute. This means that

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a pushout diagram.

□

One can check that M_0 satisfies the gluing condition easily. Consequently, by Proposition 2.27, we obtain a representation $M_0 \otimes N$ satisfying the gluing condition from M_0 and any representation $N : \mathbb{I} \rightarrow \mathcal{D}$ where \mathcal{D} is a bicomplete category.

3 Static and Dynamic Modes in Real Biological Networks

In Section 2, we have shown that the pair of two network transformations (adjoint functors) R and L considered in Introduction is a canonical one associated with the idea “interaction is interface between functions”. R is associated with the static mode and L is with the dynamic mode. Natural path notions for the two modes have turned out to be the directed path and the lateral path, respectively. The correspondence among the two modes, functors, path notions and their functionality is summarized in the table below.

mode	functor	path	functionality
dynamic	L	lateral	coherence
static	R	directed	transport

In this section, we study how the static and dynamic modes are embedded in real biological networks by measuring importance of arcs with respect to each path notion. The measure we use is **betweenness centrality** (Anthonisse, 1971; Freeman, 1977). In general, betweenness centrality measures importance of a node or an arc in a given network by counting the number of geodesic paths that pass through the node or the arc. In this section and next section, we only consider simple directed networks, namely, directed networks without self-loops and multi-arcs.

Let $G = (A, N, \partial_0, \partial_1)$ be a simple directed network and $f \in A$. The **lateral betweenness centrality (LBC)** of f is defined by

$$\text{LBC}_f = \frac{1}{\sum_{g,h \in A, l_{gh} > 0} (s_{gh} + 1)} \sum_{g,h \in A, l_{gh} > 0} \frac{l_{gh}^f}{l_{gh}},$$

where l_{gh} is the number of geodesic lateral paths from g to h , l_{gh}^f is the number of geodesic lateral paths from g to h that pass through f and s_{gh} is the geodesic lateral distance from g to h . Similarly, the **directed betweenness centrality (DBC)** of f is defined by

$$\text{DBC}_f = \frac{1}{\sum_{g,h \in A, d_{gh} > 0} (t_{gh} + 1)} \sum_{g,h \in A, d_{gh} > 0} \frac{d_{gh}^f}{d_{gh}}.$$

where d_{gh} is the number of geodesic directed paths from g to h , d_{gh}^f is the number of geodesic directed paths from g to h that pass through f and t_{gh} is the geodesic directed distance from g to h . Here, we normalize both LBC_f and DBC_f so that $\sum_{f \in A} \text{LBC}_f = 1$ and $\sum_{f \in A} \text{DBC}_f = 1$. LBC can be seen as a measure of importance with respect to coherence of the network because the lateral path is associated with gluing of functions. On the other hand, DBC is a measure of importance with respect to transport on the network.

In Figure 1 (a)-(c), we show the relationship between DBC and LBC for three directed biological networks: a neuronal network of *C. elegans* (White et al., 1986; Varshney et al., 2011) (nodes: neurons, arcs: chemical synapses, $|N| = 279$, $|A| = 2194$), a gene transcription regulation network of *E. coli* (Shen-Orr et al., 2002) (nodes: genes (operons), arcs: regulation relations, $|N| = 328$, $|A| = 456$) and an ecological flow network of Florida Bay (Ulanowicz et al., 1998) (nodes: taxa, arcs: carbon flows, $|N| = 121$, $|A| = 1767$)⁵. Each point in each figure corresponds to an arc in the network. In any of the three networks, we can see a trade-off relationship between the values of LBC and DBC. Namely, given an arc, if its LBC is large, then its DBC is small and vice versa. Thus, any arc in these networks cannot have large values of LBC and DBC at the same time.

Figure 1 (d) shows the relationship between types of neurons (sensory, inter and motor) and average dominance of LBC or DBC in the neuronal network of *C. elegans*. Each arc is classified into one of the nine groups depending on types of its source and target neurons. For each group, the sum of DBC value minus LBC value over all arcs in the group is shown as a color. We can see that LBC is dominant over DBC on average in the forward direction such as connections from sensory to inter or motor and from inter to motor neurons. On the other hand, DBC is dominant over LBC on average in the backward

⁵Three nodes classified as detritus, one node corresponding to roots and arcs whose source or target are one of these four nodes are removed from the original data to focus on prey-predator interactions.

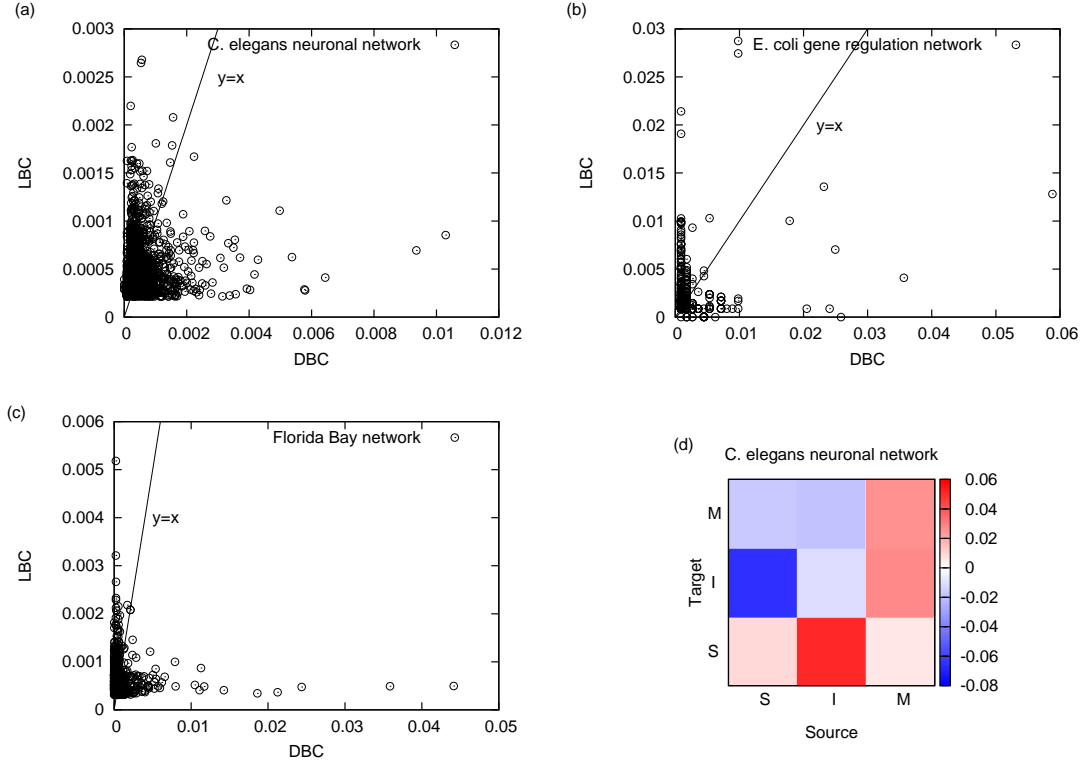


Figure 1: (a)-(c) The relationship between DBC value and LBC value for each arc in the three biological networks is shown. (d) The relationship between types of neurons and dominance of the two betweenness centralities. For each of nine blocks, a group of arcs is assigned based on types of their source neurons (horizontal axis, S: sensory neuron, I: inter neuron, M: motor neuron) and their target neurons (vertical axis). The color of each block indicates the value of the sum of DBC value minus LBC value over all arcs in the corresponding group.

directions such as connections from motor to sensory or inter and from inter to sensory neurons. Thus, in this case, there is a clear relationship between types of synaptic connections and dominance of the two betweenness centralities.

To see whether this kind of trade-off relationship is significant, we compare cumulative frequency distribution functions of LBC and DBC values for real networks with those for randomized networks (Figure 2). In preparing a randomized network for each real network, we repeatedly swap the targets of two randomly chosen arcs. Thus, the degree of each node is preserved in the randomized network. Clearly, the cumulative frequency distribution functions for real networks have longer tails than the averages of those for randomized networks except for that of DBC for the gene transcription regulation network. This suggests that the trade-off relationship between LBC and DBC seen in Figure 1 is not trivial.

The trade-off relationship between LBC and DBC values indicates that a “division of labor” between coherence of the network and transport on the network have emerged through the evolution of the biological networks. In the next section, we discuss how the trade-off relationship emerges through an optimization model of complex networks.

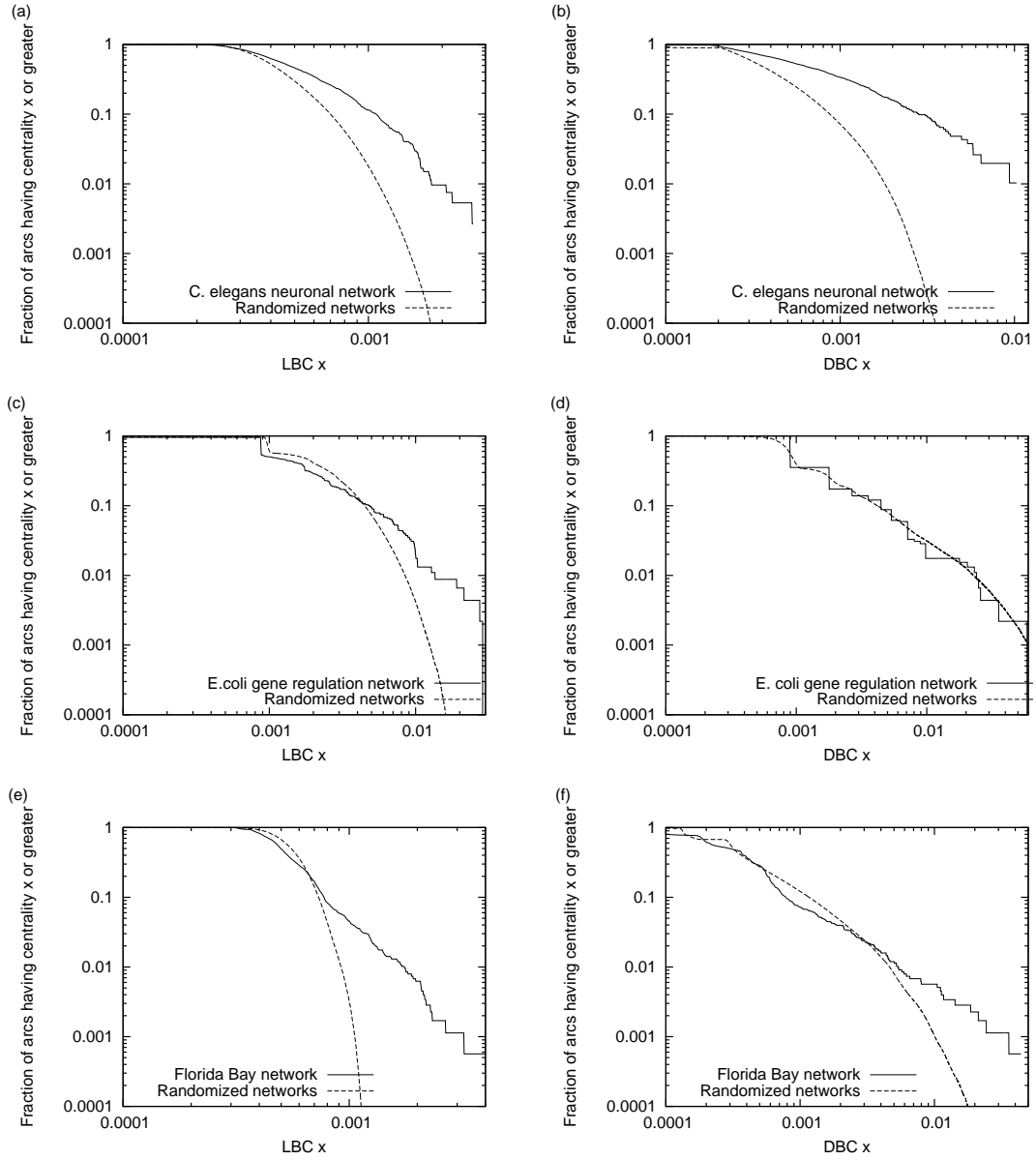


Figure 2: For each of the three biological networks, the cumulative frequency distribution functions of LBC and DBC values and the averages of those for randomized networks preserving the degree of each node are shown. The average is taken over 1000 randomized networks. (a) LBC for the *C. elegans* neuronal network. (b) DBC for the *C. elegans* neuronal network. (c) LBC for the *E. coli* gene regulation network. (d) DBC for the *E. coli* gene regulation network. (e) LBC for the Florida Bay ecological flow network. (f) DBC for the Florida Bay ecological flow network.

4 An Optimization Model of Directed Networks

In the previous section, we found the trade-off relationship between LBC and DBC values in three biological networks such that if an arc has a large value in one of the two betweenness centralities, then the arc has a small value in the other. The comparison with the null-model suggests that this trade-off relationship is not an accidental one. In this section, we propose a simple optimization model of complex networks to study how the trade-off relationship emerges through evolution.

In general, an optimization model of complex networks sets a quality function to be optimized. In the course of evolution, the topology of a network is changed gradually to the direction in which the quality function increases or decreases. Typically, quality functions involve two quantities that trade-off each other. For example, (Ferrer i Cancho and Solé, 2003) took a quality function as a linear combination of average geodesic distance and a normalized number of edges to express the trade-off between transport efficiency and cost to make edges in a network. (Brede and de Vries, 2009) considered a linear combination of average geodesic distance and the largest eigenvalue of adjacency matrix. In this case, the trade-off between transport efficiency and robustness of a network is the target of the optimization. In our model, we take a quality function incorporating two efficiencies (Latora and Marchiori, 2001) with respect to the lateral path and the directed path. This choice is intended to see what kind of networks evolve under a specified balance between intensity of the dynamic mode and that of the static mode in a network. Namely, our target of optimization is the hypothesized trade-off between coherence and transport efficiency of networks.

For a simple directed network $G = (A, N, \partial_0, \partial_1)$, we define its **lateral efficiency** by

$$\text{Eff}_L = \frac{1}{|A|(|A| - 1)} \sum_{f, g \in A, f \neq g} \frac{1}{s_{fg}},$$

where $|A|$ is the number of arcs and s_{fg} is the geodesic lateral distance between two arcs f and g as in the previous section. Eff_L is a measure of the intensity of the dynamic mode or coherence of the network. Similarly, **directed efficiency** of G is defined by

$$\text{Eff}_D = \frac{1}{|A|(|A| - 1)} \sum_{f, g \in A, f \neq g} \frac{1}{t_{fg}},$$

where t_{fg} is the geodesic directed distance between two arcs f and g . Eff_D is a measure of the intensity of the static mode or transport efficiency on the network. The following quality function is used to evolve networks:

$$Q(\lambda) = \lambda \text{Eff}_D + (1 - \lambda) \text{Eff}_L, \quad 0 \leq \lambda \leq 1.$$

λ is the parameter which controls the balance of contributions to the quality function from the two efficiencies. The evolutionary algorithm is given as follows:

- (1) Initial networks are given by random networks with n nodes and the probability of arcs between nodes p . In the following evolutionary process, the number of arcs a is preserved.
- (2) Remove a randomly chosen arc and add an arc randomly.
- (3) If the quality function Q increases in (2), then accept the result, otherwise repeat (2).
- (4) Stop when Q does not increase a (the number of arcs) times successively in (3).

Although we can incorporate other factors such as growing process into the evolutionary algorithm (Brede, 2011), here we consider one of the simplest models to study the basic performance of the evolutionary algorithm using the quality function Q . The results in what follows are obtained from numerical simulations with $n = 100$ and $p = 0.05$. λ is incremented from 0 to 1 by 0.05. For each value of λ , we generated 100 optimized networks evolved from different initial networks.

In Figure 3, the relationship between LBC and DBC for an optimized network for each value of λ is shown. We can see that the trade-off relationship between LBC and DBC holds for the values of λ up to around 0.2. In Figure 4, we plot the correlation coefficient between LBC and DBC averaged over the 100 optimized networks for each λ . It is negative for $0 \leq \lambda \leq 0.10$ and positive for $\lambda \geq 0.15$. A negative correlation coefficient between LBC and DBC indicates the trade-off relationship between them. From visual inspection into Figure 3, one might suppose that the trade-off relationship between LBC and DBC would hold for $\lambda = 0.15, 0.20$. However, there are many arcs that have small values for both LBC and DBC. Correlation coefficients for these values of λ become non-negative due to the existence of these arcs. Thus, the correlation coefficient may not necessarily a good indicator for the trade-off relationship. However, at least we can conclude that dominance of Eff_L in the quality function Q results in the trade-off relationship between LBC and DBC.

We apply the Kolmogorov-Smirnov test for the values of LBC and DBC for each optimized network to see whether their distributions are statistically significant. Here, we take reference cumulative distribution functions for LBC and DBC as the averaged cumulative frequency distribution functions for them over 100 randomly rewired networks preserving the degree of each node, respectively. The averaged value of D static and p -values are shown in Figure 5. In the worst case, the averaged p -value for DBC takes values up to 0.005 around $\lambda = 0.40$ and here the distributions of p -value have large standard deviations. However, on average, we can reject the null-hypothesis at level 0.005 for any λ and for both LBC and DBC. These results indicate that the distributions of LBC and DBC in optimized networks are significantly different from those in the randomized networks.

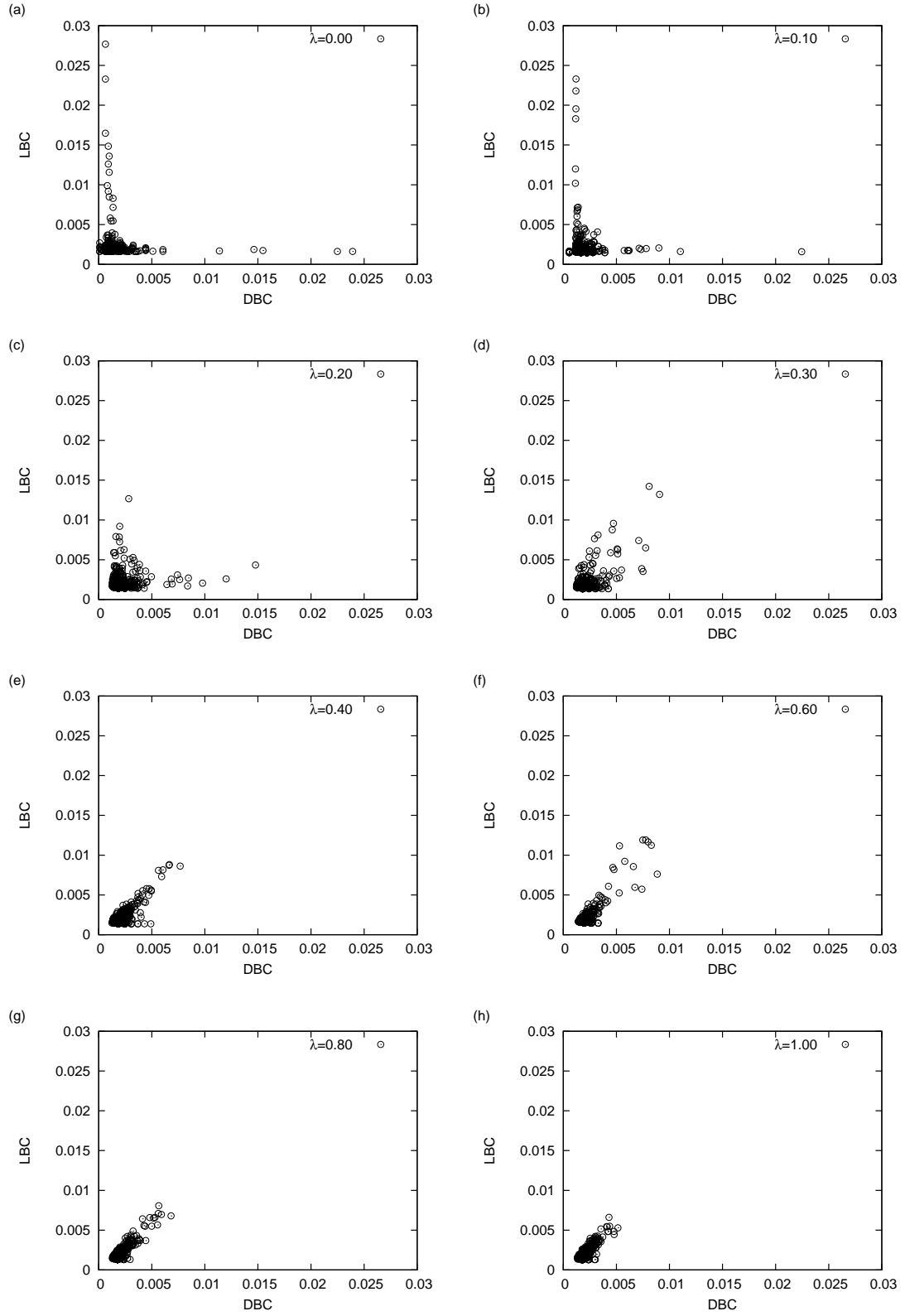


Figure 3: The relationship between LBC and DBC for some values of λ in an optimized network. (a) $\lambda = 0.00$. (b) $\lambda = 0.10$. (c) $\lambda = 0.20$. (d) $\lambda = 0.30$. (e) $\lambda = 0.40$. (f) $\lambda = 0.60$. (g) $\lambda = 0.80$. (h) $\lambda = 1.00$.

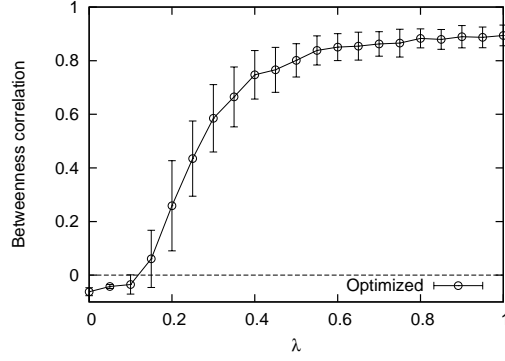


Figure 4: The averaged value of the correlation coefficient between LBC and DBC for each λ . The averages are taken over the 100 optimized networks. Bars are standard deviations.

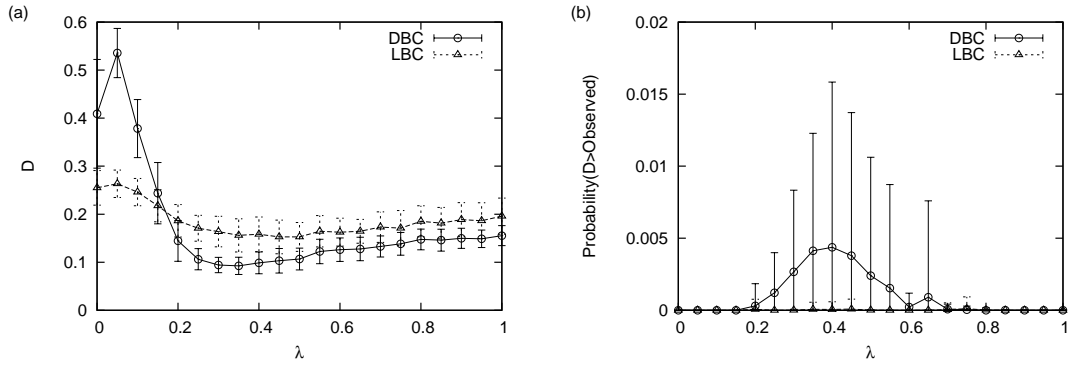


Figure 5: The result of the Kolmogorov-Smirnov test for distribution functions of LBC and DBC in optimized networks. (a) D statics and (b) p -values. See the main text for details.

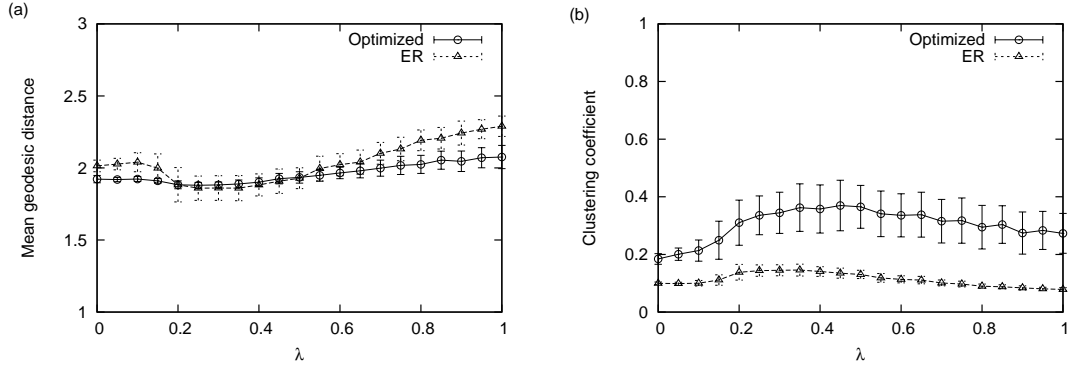


Figure 6: (a) The mean geodesic distance between nodes and (b) the clustering coefficient for each λ averaged over the 100 optimized networks. Bars are standard deviations. Legends with subscript ER are values for corresponding Erdős-Rényi random networks. See the main text for details.

We apply standard complex network analysis (Newman, 2010) to the undirected networks obtained by ignoring the direction of arcs in the optimized networks. By abuse of language, we also refer to these undirected networks as the optimized networks.

Figure 6 shows the mean geodesic distance between nodes l and the clustering coefficient C (which is degree of transitivity in a network defined by the number of triangles times 6 divided by the number of paths of length 2) averaged over the 100 optimized networks for each λ . As references, we also plot the clustering coefficient $C_{ER} = \langle k \rangle / n$ and the mean geodesic distance $l_{ER} = \ln(n) / \ln(\langle k \rangle)$ averaged over 100 Erdős-Rényi random networks with the same number of nodes and edges as the optimized networks, where n is the number of nodes and $\langle k \rangle$ is mean degree. From Figure 6, the mean geodesic distances for the optimized networks are as small as l_{ER} and the clustering coefficients for them are at least about two times larger than C_{ER} for any value of λ . These results suggest that the optimized networks are small-world networks (Watts and Strogatz, 1998). In Figure 7, we show the average value of the Small-world-ness (Humphries and Gurney, 2008) which is a quantitative measure of the degree of small-world for each λ . The Small-world-ness for a network with clustering coefficient C and mean geodesic distance l is defined by

$$S = \frac{l_{ER}}{l} \frac{C}{C_{ER}},$$

where l_{ER} and C_{ER} are as above. It is proposed that if $S > 1$, then we say that the network is small-world. Since the value of S for any λ exceeds about 2 in Figure 7, it is supported that the optimized networks have the small-world property also from this point of view.

Next, we focus on degree distributions of the optimized networks. In Figure 8 (a), we show the average value of exponent for the power law function which best approximates the average degree distribution

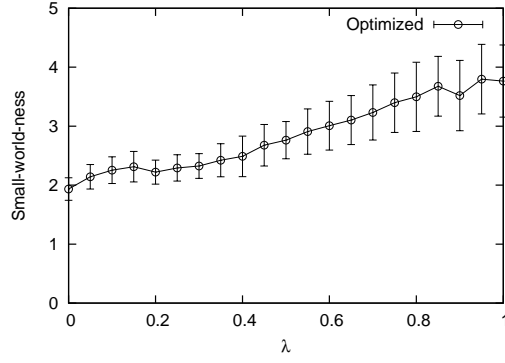


Figure 7: The Small-world-ness for each λ averaged over the 100 optimized networks. Bars are standard deviations.

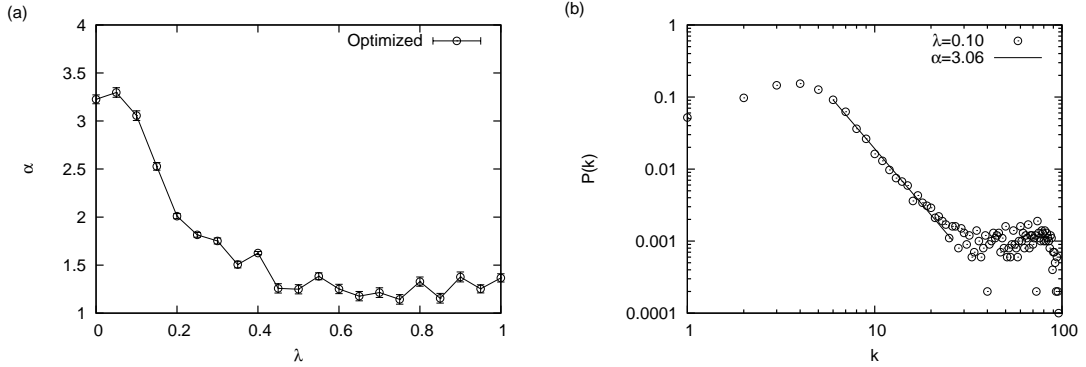


Figure 8: (a) The value of exponent for the power law approximation of the average degree distribution of the optimized networks for each λ . (b) The average degree distribution of the optimized networks for $\lambda = 0.10$.

$P(k)$ of the 100 optimized networks for each λ . We estimated the exponents by applying the maximum likelihood method (Clauset et al., 2009) to the power law function

$$P(k) = k^{-\alpha} / \zeta(\alpha, k_{min}, k_{max}), \quad \zeta(\alpha, k_{min}, k_{max}) = \sum_{k=k_{min}}^{k_{max}} k^{-\alpha},$$

where $[k_{min}, k_{max}]$ is the interval on which we assumed the power law. The interval $[k_{min}, k_{max}]$ was also estimated by the method in (Clauset et al., 2009).

From Figure 8 (a), we can see that α has a decreasing trend as λ increases. It is remarkable that α takes the values about 2 to about 3 which are frequently observed in real world networks whose degree distributions obey power laws when $\lambda \leq 0.2$.

The result of modularity scaling analysis based on degree dependency of local clustering coefficients is shown in Figure 9. The local clustering coefficient C_i for a node i is defined as the number of edges between neighbors of i divided by the number of unordered pairs of neighbors of i . $C(k)$ is the average local clustering coefficients over all nodes with degree k . Figure 9 (a) shows that the exponent β of

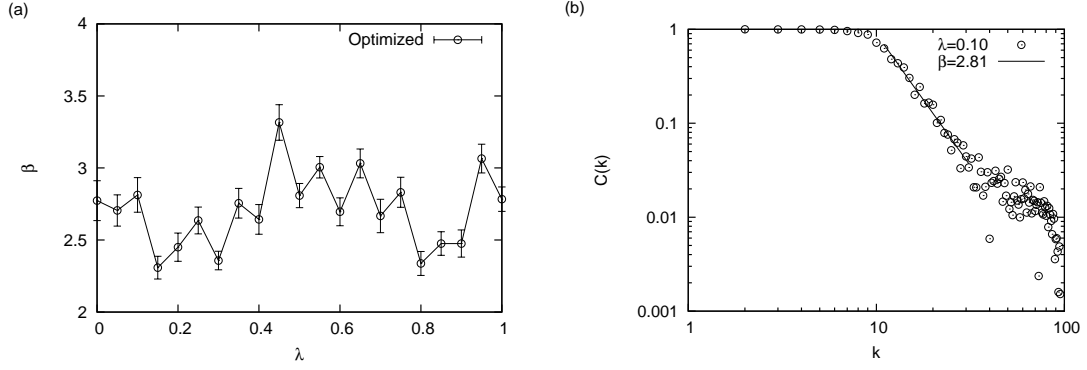


Figure 9: Degree dependency of local clustering coefficients $C(k)$ averaged over the 100 optimized networks for each λ . (a) The exponents β of the approximation $C(k) \propto k^{-\beta}$ on the estimated ranges are shown. (b) $C(k)$ for $\lambda = 0.10$. See the main text for details.

the power law approximation $k^{-\beta}$ for $C(k)$ averaged over the 100 optimized networks for each λ . The exponents were estimated by the least square method. The intervals on which the power laws hold were determined by the method in (Clauset et al., 2009) using χ^2 statistic. The estimated values of β are within the range between 2 and 3.5. However, the values themselves may not be important. Here, what is notable is that $C(k)$ are decreasing functions of k on certain ranges. This property of $C(k)$ can be associated with modularity of networks (Ravasz and Barabási, 2003; Newman, 2010), that is a common feature of biological networks.

Finally, we analyze degree correlation (Newman, 2002) in the optimized networks. The degree correlation coefficient r for a network is defined by

$$r = \frac{\sum_{i,j} (A_{ij} - k_i k_j / 2a) k_i k_j}{\sum_{i,j} (k_i \delta_{ij} - k_i k_j / 2a) k_i k_j},$$

where A is the adjacency matrix of the network, k_i is the degree of node i , a is the number of edges and δ_{ij} is the Kronecker's delta. It is known that the value of r has a certain trend depending on types of networks such as social, technological, biological and so on. In particular, biological networks typically have negative degree correlations (Newman, 2002). Figure 10 shows the degree correlation coefficient r averaged over the 100 optimized networks for each λ . We can see that r takes negative values for each λ .

By combining the result for the relationship between LBC and DBC and that for standard complex networks measures, we conclude that when the lateral efficiency term is dominant ($\lambda < 0.2$) in the quality function Q , networks that have similar qualitative features found typically in biological networks emerge through the optimization process. This suggests that the dynamic mode plays a significant role in the evolution of biological networks.

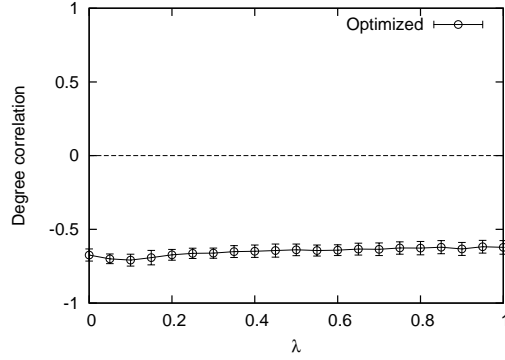


Figure 10: The degree correlation coefficient averaged over the 100 optimized networks for each λ . Bars are standard deviations.

5 Conclusions and Future Work

Let us summarize the results achieved in this paper.

In Introduction, we introduced the two modes of directed networks inspired by biological networks. In the static mode, a network is a passage on which something flows. Thus, the network itself is regarded as a static structure. On the other hand, in the dynamic mode, a network is seen as a pattern constructed by gluing functions of entities constituting the network. In the dynamic mode, interaction between entities is seen as interface between functions.

In Section 2, first we formalized the dynamic mode in the framework of category theory. Then, a canonical representation of the dynamic mode was deduced in terms of the category theoretic universality with respect to the idea “interaction as interface between functions”. A path notion called lateral path is naturally associated with it. The directed path, the path notion corresponding to the static mode, is obtained from the dual (right adjoint) functor to the functor induced by the canonical representation. The results in Section 2 are generalized in Appendix.

In Section 3, we examined how the dynamic and static modes are embedded in real biological networks by using centralities of arcs called lateral betweenness centrality (LBC) and directed betweenness centrality (DBC). DBC is a measure of importance of arcs with respect to the static mode or transport on a network. On the other hand, LBC is a measure of importance of arcs with respect to the dynamic mode or coherence of a network. We found that in several types of real biological networks, there is a trade-off relationship between LBC and DBC, namely, if one has large value, then the value of the other is small, that indicates the existence of “division of labor” with respect to the two modes.

In Section 4, we proposed an evolutionary model of complex networks to see how the trade-off relationship between LBC and DBC can emerge. Evolution of networks is based on an optimization process toward a quality function. We introduced the quality function as the linear combination of the lateral efficiency and the directed efficiency. These quantities are indices of intensities of the dynamic and

static mode in a network, respectively. We found that the trade-off relationship between LBC and DBC emerged only when the lateral efficiency was dominant in the quality function. We also found that the optimized networks have features qualitatively similar to typical real biological networks. It is suggested that the dynamic mode is an important factor in the evolution of biological networks.

Finally, we indicate two applications of the lateral path in complex directed networks to be investigated in future work. One is on robustness of networks and the other is on community detection. For the configuration model of directed networks (a model of random graph with a specified degree sequence), we can compute the size of giant component with respect to the lateral connectedness by applying generating function formalism (Newman et al., 2001). Let us consider a percolation problem when arcs are removed randomly with probability p . The size of the giant component S_{GC} will decrease as p approaches 1. The area A under the graph of S_{GC} as a function of p can be a measure of robustness of networks (Schneider et al., 2011). By comparing A for a real world network and one for the configuration model with the same degree sequence, we can quantify how network structures other than the degree sequence contribute to the robustness of the network and discuss the difference from the cases of strong connectedness or weak connectedness.

For community detection (Fortunato, 2010), we can consider the lateral path version of the Girvan-Newman algorithm (Girvan and Newman, 2002). In this algorithm, arcs with large LBC will be regarded as sitting between communities. An arc that has the largest value of LBC is removed successively as LBC of each arc is recalculated. In this process, if a laterally connected component divides into several components, then we interpret that it is divided into small communities. To obtain a meaningful division into communities, we need a quality function that measures how good the division is to stop the process. A candidate is the density of arcs in the lateral direction which is the lateral path version of the link density (Ahn et al., 2010). We can define it by means of the stability condition with respect to the standard representation M_0 obtained in Section 2. For another community detection algorithm, one can define a new modularity by considering random walk on the set of arcs (Evans and Lambiotte, 2009) with respect to the lateral path. These applications will shed a new light on the meaning of the lateral path which is somewhat abstract in this work.

A General theory of interface

In this appendix, we generalize the results of Section 2.

A.1 Universality of interface transformation induced by a standard representation

Definition A.1 Let \mathcal{C} be a small category and $|\mathcal{C}|$ the set of all objects in \mathcal{C} . The category \mathcal{C} is called **well-founded** if for any nonempty subset $X \subseteq |\mathcal{C}|$ there exists a minimal element in X with respect to the binary relation $<$ on $|\mathcal{C}|$ defined by $c < c' :\Leftrightarrow$ there exists a morphism $f : c \rightarrow c'$ such that f is not an identity, where $x \in X$ is called minimal if $y \not< x$ for all $y \in X$.

Definition A.2 We say that a small category \mathcal{C} has a **dual structure** if there exists an isomorphism $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$.

The isomorphism σ in Definition A.2 gives rise to an isomorphism that goes backward: $\mathcal{C}^{\text{op}} \xrightarrow{\cong} \mathcal{C}$. However, note that σ is not necessarily the inverse of itself in general.

Definition A.3 Let \mathcal{C} be a small category, \mathcal{D} a bicomplete category and $M : \mathcal{C} \rightarrow \mathcal{D}$ a representation. A functor $\text{coInt}_M : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ defined below is called **cointerface** of the representation M : for any object $c \in \mathcal{C}$, we define

$$\text{coInt}_M(c) = \text{colim} \left(\text{Elts}(\mathbf{y}^o(c)) \xrightarrow{\pi_c} \mathcal{C} \xrightarrow{M} \mathcal{D} \right),$$

where $\mathbf{y}^o : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathcal{C}}$ is the Yoneda embedding functor defined by $\mathbf{y}^o(c) = \mathcal{C}(c, -)$ and the category of elements $\text{Elts}(F)$ for any functor $F \in \mathbf{Set}^{\mathcal{C}}$ is defined as the category whose objects are pairs (c, x) such that $c \in \mathcal{C}$ and $x \in F(c)$ and morphisms $f : (c, x) \rightarrow (c', x')$ are morphisms $f : c \rightarrow c'$ in \mathcal{C} such that $F(f)(x) = x'$.

For a morphism $f : c \rightarrow c' \in \mathcal{C}$, let $\{\mu_{(c'',g)}^{M,c'} : M \circ \pi_{c'}(c'', g) = M(c'') \rightarrow \text{coInt}_M(c')\}_{(c'',g) \in \text{Elts}(\mathbf{y}^o(c'))}$ be the colimit on $M \circ \pi_{c'}$ defining $\text{coInt}_M(c')$. Since $\{\mu_{(c'',g \circ f)}^{M,c} : M \circ \pi_c(c'', g \circ f) = M(c'') \rightarrow \text{coInt}_M(c)\}_{(c'',g) \in \text{Elts}(\mathbf{y}^o(c'))}$ is a cocone on $M \circ \pi_{c'}$, there exists a unique morphism $\text{coInt}_M(f) : \text{coInt}_M(c') \rightarrow \text{coInt}_M(c)$ such that $\mu_{(c'',g \circ f)}^{M,c} = \text{coInt}_M(f) \circ \mu_{(c'',g)}^{M,c'}$ for all $(c'', g) \in \text{Elts}(\mathbf{y}^o(c'))$.

Definition A.4 Let \mathcal{C} be a small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. The representation $M_0 : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ defined by the composition $M_0 := \text{coInt}_{\mathbf{y}} \circ \sigma$ is called the **standard representation** of \mathcal{C} with respect to the dual structure σ .

For a small category \mathcal{C} with a dual structure σ , there is a natural transformation $\gamma : \mathbf{y} \circ \sigma \rightarrow \text{Int}_{M_0}$ defined as follows ⁶: For an object $c \in \mathcal{C}$, let $\{\nu_{(c',f)}^{M_0,c} : \text{Int}_{M_0}(c) \rightarrow M_0 \circ \pi_c(c', f) = M_0(c')\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))}$

⁶We regard σ in the domain of γ as the isomorphism $\mathcal{C}^{\text{op}} \xrightarrow{\cong} \mathcal{C}$.

be the limit on $M_0 \circ \pi_c$ defining $\text{Int}_{M_0}(c)$. The family of morphisms

$$\{ \mathbf{y}(\sigma(c)) \xrightarrow{\mathbf{y}(\sigma(f))} \mathbf{y}(\sigma(c')) \xrightarrow{\mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')}} M_0(c') \}_{(c', f) \in \text{Elts}(\mathbf{y}(c))}$$

is a cone on $M_0 \circ \pi_c$, where $\mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')} : \mathbf{y}(\sigma(c')) = \mathbf{y} \circ \pi_{\sigma(c')}(\sigma(c'), \text{id}_{\sigma(c')}) \rightarrow \text{coInt}_{\mathbf{y}}(\sigma(c')) = M_0(c')$ is a component of the colimit defining $M_0(c')$. Indeed, the following diagram commutes for any morphism $h : (c', f) \rightarrow (c'', g)$ in $\text{Elts}(\mathbf{y}(c))$:

$$\begin{array}{ccc} & \mathbf{y}(\sigma(c)) & \\ \mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')} \circ \mathbf{y}(\sigma(f)) \swarrow & & \searrow \mu_{(\sigma(c''), \text{id}_{\sigma(c'')})}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(g)) \\ M_0(c') & \xrightarrow{M_0(h)} & M_0(c''). \end{array}$$

First, note that h satisfies $g \circ h = f$ which is equivalent to $\sigma(h) \circ \sigma(g) = \sigma(f)$, which is also equivalent to $\mathbf{y}(\sigma(h)) \circ \mathbf{y}(\sigma(g)) = \mathbf{y}(\sigma(f))$. Second, by the defining property of $M_0(h)$, we have $M_0(h) \circ \mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')} = \mu_{(\sigma(c'), h)}^{\mathbf{y}, \sigma(c'')}$. Third, since $\sigma(h) : (\sigma(c''), \text{id}_{\sigma(c'')}) \rightarrow (\sigma(c'), \sigma(h))$ is a morphism in $\text{Elts}(\mathbf{y}^o(\sigma(c'')))$, we have $\mu_{(\sigma(c'), \sigma(h))}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(h)) = \mu_{(\sigma(c''), \text{id}_{\sigma(c'')})}^{\mathbf{y}, \sigma(c'')}$. By combining these three equalities, we obtain

$$\begin{aligned} M_0(h) \circ \mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')} \circ \mathbf{y}(\sigma(f)) &= \mu_{(\sigma(c'), \sigma(h))}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(h)) \circ \mathbf{y}(\sigma(g)) \\ &= \mu_{(\sigma(c''), \text{id}_{\sigma(c'')})}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(g)) \end{aligned}$$

as required.

Consequently, by the universality of the limit $\text{Int}_{M_0}(c)$, there exists a unique morphism $\gamma_c : \mathbf{y}(\sigma(c)) \rightarrow \text{Int}_{M_0}(c)$ such that $\nu_{(c', f)}^{M_0, c} \circ \gamma_c = \mu_{(\sigma(c'), \text{id}_{\sigma(c')})}^{\mathbf{y}, \sigma(c')} \circ \mathbf{y}(\sigma(f))$ for all $(c', f) \in \text{Elts}(\mathbf{y}(c))$. γ_c turns out to be natural in c . Indeed, we have the following commutative diagram for morphisms $c'' \xrightarrow{f'} c' \xrightarrow{f} c$ in \mathcal{C} :

$$\begin{array}{ccc} \mathbf{y}(\sigma(c)) & \xrightarrow{\gamma_c} & \text{Int}_{M_0}(c) \\ \downarrow \mathbf{y}(\sigma(f)) & \searrow \delta & \swarrow \nu_{(c'', f \circ f')}^{M_0, c} \\ & M_0(c'') & \\ \uparrow \delta' & \swarrow \nu_{(c'', f')}^{M_0, c'} & \downarrow \text{Int}_{M_0}(f) \\ \mathbf{y}(\sigma(c')) & \xrightarrow{\gamma_{c'}} & \text{Int}_{M_0}(c'), \end{array}$$

where we put $\delta = \mu_{(\sigma(c''), \text{id}_{\sigma(c'')})}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(f') \circ \sigma(f))$ and $\delta' = \mu_{(\sigma(c''), \text{id}_{\sigma(c'')})}^{\mathbf{y}, \sigma(c'')} \circ \mathbf{y}(\sigma(f'))$. By applying Lemma 2.15, we obtain the desired result.

Now, let us obtain a concrete expression of the natural transformation $\gamma : \mathbf{y} \circ \sigma \rightarrow \text{Int}_{M_0}$. For any objects $c, c' \in \mathcal{C}$, we take $\text{Int}_{M_0}(c)(c')$ as the set of tuples $(a_{(d,f)}) \in \prod_{(d,f) \in \text{Els}(\mathbf{y}(c))} M_0(d)(c')$ such that $h \cdot a_{(d,f)} = a_{(e,g)}$ for any morphism $h : d \rightarrow e$ in \mathcal{C} such that $g \circ h = f$, where $h \cdot a_{(d,f)} = M_0(h)(c')(a_{(d,f)})$. Maps $\nu_{(d,f)}^{M_0, c}(c') : \text{Int}_{M_0}(c)(c') \rightarrow (M_0 \circ \pi_c(d, f))(c') = M_0(d)(c')$ are projections.

Putting $\bar{c}' = \sigma^{-1}(c')$, we take

$$M_0(d)(c') = \left(\sum_{c'' \in \mathcal{C}} \mathbf{y}^o(\sigma(d))(\sigma(c'')) \times \mathbf{y}(\sigma(c''))(\sigma(\bar{c}')) \right) / \sim,$$

where \sim is the equivalence relation generated by the relation r defined by $(\sigma(i), \sigma(j))r(\sigma(k) \circ \sigma(i), \sigma(k) \circ \sigma(j))$ for morphisms

$$\begin{array}{ccc} \sigma(d) & & \\ & \searrow \sigma(i) & \\ & \sigma(c'') & \xrightarrow{\sigma(k)} \sigma(c''') \\ & \nearrow \sigma(j) & \\ \sigma(\bar{c}') & & \end{array}$$

in \mathcal{C} . For any morphism $\sigma(f') : \sigma(d) \rightarrow \sigma(c')$ in \mathcal{C} , the map

$$\mu_{(\sigma(c''), \sigma(f'))}^{\mathbf{y}, \sigma(d)}(\sigma(\bar{c}')) : \mathbf{y}(\sigma(c''))(\sigma(\bar{c}')) \rightarrow M_0(d)(\sigma(\bar{c}'))$$

is given by $\sigma(g') \mapsto [(\sigma(f'), \sigma(g'))]$.

By this expression of M_0 , we have $h \cdot a_{(d,f)} = [(\sigma(i) \circ \sigma(h), \sigma(j))]$ for $a_{(d,f)} = [(\sigma(i), \sigma(j))]$.

The map $\gamma_c(c') : \mathbf{y}(\sigma(c))(c') \rightarrow \text{Int}_{M_0}(c)(c')$ is uniquely determined by the following commutative diagram:

$$\begin{array}{ccc} \mathbf{y}(\sigma(c))(c') & \xrightarrow{\gamma_c(c')} & \text{Int}_{M_0}(c)(c') \\ \downarrow \mathbf{y}(\sigma(f))(c') & & \downarrow \nu_{(d,f)}^{M_0, c}(c') \\ \mathbf{y}(\sigma(d))(c') & \xrightarrow{\mu_{(\sigma(d), \text{id}_{\sigma(d)})}^{\mathbf{y}, \sigma(d)}(c')} & M_0(d)(c'). \end{array}$$

Hence, for a morphism $\sigma(s) : \sigma(\bar{c}') \rightarrow \sigma(c)$ in \mathcal{C} , if we put $\gamma_c(c')(\sigma(s)) = (a_{(d,f)})$, then $a_{(d,f)} = [(\text{id}_{\sigma(d)}, \sigma(f) \circ \sigma(s))]$.

Proposition A.5 *Let \mathcal{C} be a small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. For any presheaf $G \in \hat{\mathcal{C}}$ and $c \in \mathcal{C}$, if $\gamma_c : \mathbf{y}(\sigma(c)) \rightarrow \text{Int}_{M_0}(c)$ is an isomorphism, then we have the following expression for the component of the natural transformation $\varphi_c^{M_0, G} : G(c) \rightarrow G \otimes M_0(\sigma(c))$: for any*

$x \in G(c)$, $\varphi_c^{M_0, G}(x) = [(x, [(\text{id}_{\sigma(c)}, \text{id}_{\sigma(c)})])]$, where we identify $\hat{\mathcal{C}}(\text{Int}_{M_0}(c), G \otimes M_0)$ with $G \otimes M_0(\sigma(c))$ by

$$\hat{\mathcal{C}}(\text{Int}_{M_0}(c), G \otimes M_0) \xrightarrow[(-) \circ \gamma_c]{\cong} \hat{\mathcal{C}}(\mathbf{y}(\sigma(c)), G \otimes M_0) \xrightarrow[\theta_c]{\cong} G \otimes M_0(\sigma(c))$$

and θ_c is a natural isomorphism given by the Yoneda Lemma.

Proof. By definition, $\varphi_c^{M_0, G} : G(c) \rightarrow G \otimes M_0(\sigma(c))$ is the map given by

$$\begin{aligned} \varphi_c^{M_0, G}(x) &= \theta_c \left(\mu_{(c, x)}^{M_0, G} \circ \nu_{(c, \text{id}_c)}^{M_0, c} \circ \gamma_c \right) \\ &= \left(\mu_{(c, x)}^{M_0, G} \circ \nu_{(c, \text{id}_c)}^{M_0, c} \circ \gamma_c \right) (\sigma(c))(\text{id}_{\sigma(c)}) \end{aligned}$$

for each $x \in G(c)$. However, the composition

$$\mathbf{y}(\sigma(c))(\sigma(c)) \xrightarrow{\gamma_c(\sigma(c))} \text{Int}_{M_0}(c)(\sigma(c)) \xrightarrow{\nu_{(c, \text{id}_c)}^{M_0, c}} M_0(c)(\sigma(c)) \xrightarrow{\mu_{(c, x)}^{M_0, G}} G \otimes M_0(\sigma(c))$$

gives

$$\text{id}_{\sigma(c)} \mapsto \left([(\text{id}_{\sigma(d)}, \sigma(f))] \right)_{(d, f) \in \text{Els}(\mathbf{y}(c))} \mapsto [(\text{id}_{\sigma(c)}, \sigma(\text{id}_c))] \mapsto [(x, [(\text{id}_{\sigma(c)}, \text{id}_{\sigma(c)})])].$$

□

Theorem A.6 *Let \mathcal{C} be a well-founded small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$ and $M_0 : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ the standard representation with respect to σ . If the natural transformation $\gamma : \mathbf{y} \circ \sigma \rightarrow \text{Int}_{M_0}$ is an natural isomorphism, then $(\hat{\mathcal{C}}(\text{Int}_{M_0}(-), G \otimes M_0), \varphi^{M_0, G})$ is an initial object of \mathcal{IT}_G for any presheaf $G \in \hat{\mathcal{C}}$.*

Proof. Let $\psi : G \rightarrow H$ be an object in \mathcal{IT}_G . We shall show that there exists a unique morphism $\iota : (\hat{\mathcal{C}}(\text{Int}_{M_0}(-), G \otimes M_0), \varphi^{M_0, G}) \rightarrow (H, \psi)$ such that $\psi = \iota \circ \varphi^{M_0, G}$.

We appeal to a well-founded induction. Recall that a well-founded induction on a set X with a well-founded binary relation $<$ (a binary relation $<$ on X is called well-founded if any nonempty subset of X has a minimal element with respect to $<$) asserts that for a condition $P(x)$ on X

$$\forall x \in X ((\forall y \in X (y < x \Rightarrow P(y))) \Rightarrow P(x))$$

implies $P(x)$ holds for any $x \in X$.

We apply a well-founded induction on the set $|\mathcal{C}|$ with the binary relation $<$ on it defined in Definition A.1 to the following condition $P(c)$: $P(c')$ is true for any $c' < c$ and there exists a unique map $\iota_c : G \otimes M_0(\sigma(c)) \rightarrow H(c)$ such that $\iota_c \circ G \otimes M_0(\sigma(f)) = H(f) \circ \iota_{c'}$ and $\psi_c = \iota_c \circ \varphi_c^{M_0, G}$ for any morphism $f : c' \rightarrow c$, namely, the following two diagrams commute for any morphism $f : c' \rightarrow c$:

$$\begin{array}{ccc} G \otimes M_0(\sigma(c)) & \xrightarrow{\iota_c} & H(c) \\ \uparrow G \otimes M_0(\sigma(f)) & & \uparrow H(f) \\ G \otimes M_0(\sigma(c')) & \xrightarrow{\iota_{c'}} & H(c'). \end{array}$$

and

$$\begin{array}{ccc} G \otimes M_0(\sigma(c)) & \xrightarrow{\iota_c} & H(c) \\ & \swarrow \varphi_c^{M_0, G} \quad \searrow \psi_c & \\ & G(c). & \end{array}$$

If $c \in |\mathcal{C}|$ is a minimal element, then $\sigma(c)$ is maximal and thus any morphism $\sigma(f) : \sigma(c) \rightarrow \sigma(c')$ must be $\text{id}_{\sigma(c)}$. Consequently, we have

$$\begin{aligned} G \otimes M_0(\sigma(c)) &= \left(\sum_{c' \in \mathcal{C}} G(c') \times M_0(c')(\sigma(c)) \right) / \sim \\ &\cong \left(\sum_{c' \in \mathcal{C}} G(c') \times \mathbf{y}(\sigma(c'))(\sigma(c)) \right) / \sim \\ &= G(c) \times \{\text{id}_{\sigma(c)}\} \cong G(c). \end{aligned}$$

By the concrete expression of $\varphi_c^{M_0, G}$ given in Proposition A.5, one can see that $\varphi_c^{M_0, G}$ is an isomorphism. Hence, we obtain a unique map that satisfies $P(c)$ by $\iota_c := \psi_c \circ (\varphi_c^{M_0, G})^{-1}$.

Now, let $c \in |\mathcal{C}|$ is not minimal and assume that $P(c')$ is true for any $c' < c$. We would like to prove that $P(c)$ is also true.

For any $[(x, \omega)] \in G \otimes M_0(\sigma(c))$, if $\omega = [(\sigma(g), \sigma(f))]$ with

$$\begin{array}{ccc} \sigma(c'') & & \\ \searrow \sigma(g) & & \\ & \sigma(c') & \\ \nearrow \sigma(f) & & \\ \sigma(c) & & \end{array}$$

in \mathcal{C} , then for $\omega' := [(\sigma(g), \text{id}_{\sigma(c')})] \in M_0(c'')(\sigma(c'))$ we have

$$G \otimes M_0(\sigma(f))([(x, \omega')]) = [(x, \omega)],$$

where $G \otimes M_0(\sigma(f))$ sends $[(x, [(\sigma(i), \sigma(j))])] \in G \otimes M_0(\sigma(c'))$ to $[(x, [(\sigma(i), \sigma(j) \circ \sigma(f))])] \in G \otimes M_0(\sigma(c))$ in general.

$$\begin{array}{ccc}
 & \sigma(c'') & \\
 & \searrow \sigma(i) & \\
 & \sigma(c''') & \\
 \nearrow \sigma(j) & & \\
 \sigma(c) \xrightarrow[\sigma(f)]{} \sigma(c') & &
 \end{array}$$

Note that we can always take f as a non-identity because c is not minimal by the assumption of the well-founded induction and hence we can take $\text{dom}(f) = c' < c$.

Consequently, we define the map $\iota_c : G \otimes M_0(\sigma(c)) \rightarrow H(c)$ as follows: for each $[(x, \omega)] \in G \otimes M_0(\sigma(c))$, we take c' , f and ω' as above and define

$$\iota_c([(x, \omega)]) = H(f)(\iota_{c'}([(x, \omega')])).$$

Now, let us check ι_c is a well-defined map. To do so, we take two steps. First, we show that $\iota_c([(x, \omega)])$ is independent of the choice of representative of $[(x, \omega)]$ and second it does not depend on the choice of representative of ω .

Let $[(x, \omega)] = [(y, \omega')] \in G \otimes M_0(\sigma(c))$. It is sufficient to consider the case when $y = x \cdot f$ and $\omega = f \cdot \omega'$ for some morphism $f : c_2 \rightarrow c_1$ in \mathcal{C} , where $x \in G(c_1)$, $y \in G(c_2)$, $\omega \in M_0(c_1)(\sigma(c))$ and $\omega' \in M_0(c_2)(\sigma(c))$.

Let $(\sigma(i), \sigma(j)) \in \mathbf{y}^o(\sigma(c_2))(\sigma(c')) \times \mathbf{y}(\sigma(c'))(\sigma(c))$ be a representative of ω' such that j is a non-identity.

$$\begin{array}{ccc}
 \sigma(c_1) \xrightarrow{\sigma(f)} \sigma(c_2) & & \\
 & \searrow \sigma(i) & \\
 & \sigma(c') & \\
 \nearrow \sigma(j) & & \\
 \sigma(c) & &
 \end{array}$$

Since $\omega = f \cdot \omega' = f \cdot [(\sigma(i), \sigma(j))] = [(\sigma(i) \circ \sigma(f), \sigma(j))]$, $(\sigma(i) \circ \sigma(f), \sigma(j))$ is a representative of ω . Putting $\chi := [(\sigma(i), \text{id}_{\sigma(c')})]$, we have $[(y, \chi)] = [(x \cdot f, \chi)] = [(x, f \cdot \chi)]$. This completes the first step.

For the second step, let $[(x, \omega)] \in G \otimes M_0(\sigma(c))$ and $(x, \omega) \in G(c') \times M_0(c')(\sigma(c))$. It is enough to show that if $(\sigma(i), \sigma(j)) \in \omega$ and j is a non-identity, then

$$H(j \circ k) \circ \iota_{c'''}([(x, [(\sigma(k) \circ \sigma(i), \text{id}_{\sigma(c''')})])]) = H(j) \circ \iota_{c''}([(x, [(\sigma(i), \text{id}_{\sigma(c'')})])])$$

for any morphism k in \mathcal{C} such that

$$\begin{array}{ccc} \sigma(c') & \xrightarrow{\sigma(i)} & \sigma(c'') \\ & \nearrow \sigma(j) & \xrightarrow{\sigma(k)} \sigma(c'''). \end{array}$$

However, by the assumption of the well-founded induction, we have

$$\begin{aligned} & H(j \circ k) \circ \iota_{c'''}([x, [(\sigma(k) \circ \sigma(i), \text{id}_{\sigma(c''')})]]) \\ &= H(j) \circ H(k) \circ \iota_{c'''}([x, [(\sigma(k) \circ \sigma(i), \text{id}_{\sigma(c''')})]]) \\ &= H(j) \circ \iota_{c''} \circ G \otimes M_0(\sigma(k))([x, [(\sigma(k) \circ \sigma(i), \text{id}_{\sigma(c''')})]]) \\ &= H(j) \circ \iota_{c''}([x, [(\sigma(k) \circ \sigma(i), \sigma(k))]]) \\ &= H(j) \circ \iota_{c''}([x, [(\sigma(i), \text{id}_{\sigma(c'')})]]) \end{aligned}$$

By the construction of ι_c , it is clear that we have $\iota_c \circ G \otimes M_0(\sigma(f)) = H(f) \circ \iota_{c'}$ for any morphism $f : c' \rightarrow c$ and ι_c is a unique map satisfying this condition.

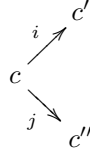
Finally, let us show that $\psi_c = \iota_c \circ \varphi_c^{M_0, G}$. For any $x \in G(c)$, we have $\varphi_c^{M_0, G}(x) = [(x, [(\text{id}_{\sigma(c)}, \text{id}_{\sigma(c)})])]$ by Proposition A.5. Since c is not minimal, we can take a morphism $f : c' \rightarrow c$ in \mathcal{C} such that $c' \neq c$. Consequently, we have

$$\begin{aligned} \iota_c \circ \varphi_c^{M_0, G}(x) &= \iota_c([x, [(\text{id}_{\sigma(c)}, \text{id}_{\sigma(c)})]]) \\ &= \iota_c([x, [(\sigma(f), \sigma(f))]]) \\ &= \iota_c \circ G \otimes M_0(\sigma(f))([x, [(\sigma(f), \text{id}_{\sigma(c')})]]) \\ &= H(f) \circ \iota_{c'}([x \cdot f, [(\text{id}_{\sigma(c')}, \text{id}_{\sigma(c')})]]) \\ &= H(f) \circ \iota_{c'} \circ \varphi_{c'}^{M_0, G}(x \cdot f) \\ &= H(f) \circ \psi_{c'}(x \cdot f) \\ &= \psi_c(x), \end{aligned}$$

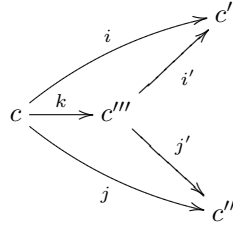
where we use the naturality ι in the fourth equality, Proposition A.5 in the fifth equality, the assumption of the well-founded induction in the sixth equality and the naturality of ψ in the last equality. □

One can check that the category $\uparrow\uparrow$ satisfies the assumption of Theorem A.6. Hence, Theorem A.6 is a generalization of Theorem 2.6. However, the condition that the natural transformation $\gamma : \mathbf{y} \circ \sigma \rightarrow \text{Int}_{M_0}$ is a natural isomorphism seems to be too abstract. In the following, we shall translate this condition into a condition on the structure of the small category \mathcal{C} . However, translation in general situation is not so informative. Here, we focus on the case \mathcal{C} is a category freely generated by a finite acyclic directed graph Γ with a directed graph isomorphism $\sigma : \Gamma \xrightarrow{\cong} \Gamma^{\text{op}}$, where Γ^{op} is a directed graph obtained by reversing the direction of all the arcs in Γ . σ induces an isomorphism of categories $\mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$ which is also denoted by σ . Thus, \mathcal{C} is a well-founded category with a dual structure σ .

Definition A.7 Let \mathcal{C} be a category. A **span** in \mathcal{C} is a pair of morphisms (i, j) in \mathcal{C} with the common domain.



A span (i, j) is called **maximal** if there is no non-identity morphism k and span (i', j') such that $i = i' \circ k$ and $j = j' \circ k$, namely, if we cannot find morphisms k, i', j' with k non-identity that make the following diagram commute:



A **maximal span for a span** (i, j) is a maximal span (\tilde{i}, \tilde{j}) such that there exists a morphism k such that $i = \tilde{i} \circ k$ and $j = \tilde{j} \circ k$.

Let us consider the following condition (A) for a category \mathcal{C} :

(A) for any span (i, j) , if there is a maximal span for it, then it is unique.

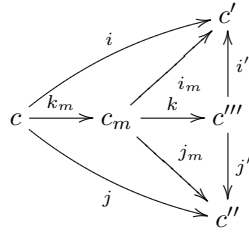
Lemma A.8 Let \mathcal{C} be a well-founded small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. For any span (i, j) there is a maximal span for it.

Proof. Let (i, j) be a span and put

$$X = \{\text{cod}(k) \mid \exists (i', j') : \text{span such that } i = i' \circ k \text{ and } j = j' \circ k\}.$$

Since $c := \text{dom}(i) = \text{dom}(j) \in X$ with $i' = i$, $j' = j$ and $k = \text{id}_c$, we have $X \neq \emptyset$. X has a maximal element c_m with respect to the binary relation $<$ defined in Definition A.1 because \mathcal{C} is well-founded and $\mathcal{C} \cong \mathcal{C}^{\text{op}}$. Consequently, there exist a morphism $k_m : c \rightarrow c_m$ and span (i_m, j_m) with $\text{dom}(i_m) = \text{dom}(j_m) = c_m$ in \mathcal{C} such that $i = i_m \circ k_m$ and $j = j_m \circ k_m$.

Suppose that (i_m, j_m) is not a maximal span. Then, there exist a non-identity morphism k and a span (i', j') such that $i_m = i' \circ k$ and $j_m = j' \circ k$. We have $i = i_m \circ k_m = i' \circ (k \circ k_m)$ and $j = j_m \circ k_m = j' \circ (k \circ k_m)$. Hence, $\text{cod}(k \circ k_m) \in X$. On the other hand, we have $c_m < \text{cod}(k) = \text{cod}(k \circ k_m)$ because k is not an identity. However, this contradicts to the maximality of c_m in X . Thus, (i_m, j_m) is a maximal span for (i, j) .



□

Lemma A.9 Let \mathcal{C} be a well-founded small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. Assume that the condition (A) holds for \mathcal{C} . If maximal spans (i, j) and (i', j') satisfies $[(\sigma(i), \sigma(j))] = [(\sigma(i'), \sigma(j'))] \in M_0(c)(\sigma(c'))$ where $c := \text{cod}(i) = \text{cod}(i')$ and $c' := \text{cod}(j) = \text{cod}(j')$, then we have $(i, j) = (i', j')$.

Proof. If $[(\sigma(i), \sigma(j))] = [(\sigma(i'), \sigma(j'))]$, then there exist a sequence of spans

$$(i, j) = (i_0, j_0), (i_1, j_1), \dots, (i_n, j_n) = (i', j')$$

and a sequence of morphisms k_1, k_2, \dots, k_n such that for each $p = 1, 2, \dots, n$ either of the following two conditions holds: (1) $i_p = i_{p-1} \circ k_p$ and $j_p = j_{p-1} \circ k_p$ or (2) $i_{p-1} = i_p \circ k_p$ and $j_{p-1} = j_p \circ k_p$. Since (i_0, j_0) and (i_n, j_n) are maximal spans, we have (1) for $p = 1$ and (2) for $p = n$. We can assume that (1) and (2) occur alternately without loss of generality. Thus, n is an even number.

We can also assume that (i_p, j_p) is maximal if p is an even number. Indeed, for example, if $(\tilde{i}_2, \tilde{j}_2)$ is a maximal span for (i_2, j_2) with a morphism \tilde{k}_2 such that $i_2 = \tilde{i}_2 \circ \tilde{k}_2$ and $j_2 = \tilde{j}_2 \circ \tilde{k}_2$, then we have $i_1 = \tilde{i}_2 \circ (\tilde{k}_2 \circ k_2)$ and $j_1 = \tilde{j}_2 \circ (\tilde{k}_2 \circ k_2)$ for $p = 2$ and $i_3 = \tilde{i}_2 \circ (\tilde{k}_2 \circ k_3)$ and $j_3 = \tilde{j}_2 \circ (\tilde{k}_2 \circ k_3)$ for $p = 3$.

Now, both (i_0, j_0) and (i_2, j_2) are maximal spans for (i_1, j_1) . It follows that $(i_0, j_0) = (i_2, j_2)$ by (A). By the same manner, we have $(i_0, j_0) = (i_2, j_2) = (i_4, j_4) = \dots = (i_n, j_n)$.

□

Proposition A.10 *Let \mathcal{C} be a well-founded small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. If the condition (A) holds for \mathcal{C} , then the map $\gamma_c(\sigma(c')) : \mathbf{y}(\sigma(c))(\sigma(c')) \rightarrow \text{Int}_{M_0}(c)(\sigma(c'))$ is injective for any $c, c' \in \mathcal{C}$.*

Proof. Assume that $\gamma_c(\sigma(c'))(\sigma(s)) = \gamma_c(\sigma(c'))(\sigma(s'))$ for morphisms $s, s' : c \rightarrow c'$ in \mathcal{C} . Then, we have

$$[(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s))] = [(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s'))]$$

for all $(c'', f) \in \text{Elts}(\mathbf{y}(c))$. Both $(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s))$ and $(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s'))$ are maximal spans because of the well-foundedness of \mathcal{C} . By Lemma A.9, we have $\sigma(f) \circ \sigma(s) = \sigma(f) \circ \sigma(s')$, namely, $s \circ f = s' \circ f$ for each morphism $f : c'' \rightarrow c$. If we take $f = \text{id}_c$, then we have $s = s'$.

□

Proposition A.11 *Let \mathcal{C} be a well-founded small category equipped with a dual structure $\sigma : \mathcal{C} \xrightarrow{\cong} \mathcal{C}^{\text{op}}$. Assume that the condition (A) holds for \mathcal{C} . For any $c, c' \in \mathcal{C}$, the map $\gamma_c(\sigma(c')) : \mathbf{y}(\sigma(c))(\sigma(c')) \rightarrow \text{Int}_{M_0}(c)(\sigma(c'))$ is surjective if and only if the following condition (B) holds:*

(B) *For any family of maximal spans $\{(i_f, j_f)\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ such that $\text{cod}(i_f) = c''$ and $\text{cod}(j_f) = c'$ for each $(c'', f) \in \text{Elts}(\mathbf{y}(c))$, assume that if $f = g \circ h$, then (i_g, j_g) is a unique maximal span for the span $(h \circ i_f, j_f)$. Then, $i_f = \text{id}_{c''}$ for any $(c'', f) \in \text{Elts}(\mathbf{y}(c))$ and there exists a morphism $s : c \rightarrow c'$ such that $j_f = s \circ f$ for any $(c'', f) \in \text{Elts}(\mathbf{y}(c))$.*

$$\begin{array}{ccccc} c''' & \xrightarrow{i_f} & c'' & & \\ j_f \downarrow & & \downarrow f & \searrow h & \\ c' & \xleftarrow{s} & c & \xleftarrow{g} & d \end{array}$$

Proof. Suppose that the condition (B) holds for $c, c' \in \mathcal{C}$. For an element $[(\sigma(i_f), \sigma(j_f))]_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$, we can assume that each representative (i_f, j_f) of $[(\sigma(i_f), \sigma(j_f))]$ is a maximal span by taking its unique maximal span if necessary because a span and its maximal span are contained in the same equivalence class. For a morphism $f : c'' \rightarrow c$, if $f = g \circ h$, then $[(\sigma(i_f) \circ \sigma(h), \sigma(j_f))] = h \cdot [(\sigma(i_f), \sigma(j_f))] = [(\sigma(i_g), \sigma(j_g))]$. If $(\tilde{i}_f, \tilde{j}_f)$ is a unique maximal span for the span $(h \circ i_f, j_f)$, then we have $\tilde{i}_f = i_g$ and $\tilde{j}_f = j_g$ by Lemma A.9. By the condition (B), $i_f = \text{id}_{c''}$ for any morphism $f : c'' \rightarrow c$ and there exists

$s : c \rightarrow c'$ such that $j_f = s \circ f$ for any morphism $f : c'' \rightarrow c$. Consequently, we have $[(\sigma(i_f), \sigma(j_f))] = [(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s))]$ for any $(c'', f) \in \text{Elts}(\mathbf{y}(c))$ which means that $\gamma_c(\sigma(c'))$ is surjective.

For the reverse direction, suppose that $\gamma_c(\sigma(c'))$ is surjective. Take any family of maximal spans $\{(i_f, j_f)\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ such that $\text{cod}(i_f) = c''$ and $\text{cod}(j_f) = c'$ for each $(c'', f) \in \text{Elts}(\mathbf{y}(c))$ and assume that if $f = g \circ h$, then (i_g, j_g) is a unique maximal span for the span $(h \circ i_f, j_f)$. Clearly, $p := [(\sigma(i_f), \sigma(j_f))]\big|_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ is an element of $\text{Int}_{M_0}(c)(\sigma(c'))$. Since $\gamma_c(\sigma(c'))$ is a surjection, there exists a morphism $s : c \rightarrow c'$ such that $\gamma_c(\sigma(c'))(\sigma(s)) = p$. Then, we have $[(\sigma(i_f), \sigma(j_f))] = [(\text{id}_{\sigma(c'')}, \sigma(f) \circ \sigma(s))]$ for any $(c'', f) \in \text{Elts}(\mathbf{y}(c))$. By Lemma A.9, we obtain $i_f = \text{id}_{c''}$ and $j_f = s \circ f$ for any $(c'', f) \in \text{Elts}(\mathbf{y}(c))$.

□

Lemma A.12 *If a category \mathcal{C} is freely generated by a finite acyclic directed graph, then (A) holds for \mathcal{C} .*

Proof. For a span (i, j) , let $(\tilde{i}_1, \tilde{j}_1)$ and $(\tilde{i}_2, \tilde{j}_2)$ be its two maximal spans. Then, there exist morphisms k_1 and k_2 such that $i = \tilde{i}_1 \circ k_1 = \tilde{i}_2 \circ k_2$ and $j = \tilde{j}_1 \circ k_1 = \tilde{j}_2 \circ k_2$. Assume that i can be written as $i = f_n \circ f_{n-1} \circ \cdots \circ f_1$, where f_l , $l = 1, 2, \dots, n$ are generating morphisms for \mathcal{C} , namely, morphisms corresponding to arcs in the base graph. If (i, j) is maximal, then we have nothing to do. Hence, we assume that (i, j) is not a maximal span. Consequently, morphisms i , k_1 and k_2 are not identities.

We can write $k_1 = f_{n_1} \circ \cdots \circ f_1$ and $k_2 = f_{n_2} \circ \cdots \circ f_1$. Suppose that $n_1 \neq n_2$. Without loss of generality, we can assume that $n_1 > n_2$. Putting $g = f_{n_1} \circ \cdots \circ f_{n_2+1}$, we have $k_1 = g \circ k_2$. It follows that $\tilde{i}_2 = \tilde{i}_1 \circ g$ and $\tilde{j}_2 = \tilde{j}_1 \circ g$. Since g is not an identity, this contradicts to the maximality of the span $(\tilde{i}_2, \tilde{j}_2)$. Consequently, we must have $k_1 = k_2$ which implies that $(\tilde{i}_1, \tilde{j}_1) = (\tilde{i}_2, \tilde{j}_2)$.

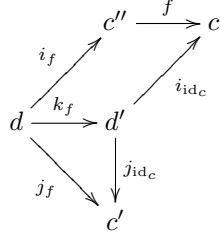
□

Lemma A.13 *Let \mathcal{C} be a category freely generated by a finite acyclic directed graph Γ with a directed graph isomorphism $\sigma : \Gamma \xrightarrow{\cong} \Gamma^{\text{op}}$. For any $c, c' \in \mathcal{C}$, the condition (B) is equivalent to the following condition (B'):*

(B') *For any family of maximal spans $\{(i_f, j_f)\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ such that $\text{cod}(i_f) = c''$ and $\text{cod}(j_f) = c'$ for each $(c'', f) \in \text{Elts}(\mathbf{y}(c))$, assume that if $f = g \circ h$, then (i_g, j_g) is a unique maximal span for the span $(h \circ i_f, j_f)$. Then, $i_{\text{id}_c} = \text{id}_c$.*

Proof. Assume that (B') holds for $c, c' \in \mathcal{C}$. Take any family of maximal spans $\{(i_f, j_f)\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ such that $\text{cod}(i_f) = c''$ and $\text{cod}(j_f) = c'$ for each $(c'', f) \in \text{Elts}(\mathbf{y}(c))$ and assume that if $f = g \circ h$,

then (i_g, j_g) is a unique maximal span for the span $(h \circ i_f, j_f)$. Then, for any morphism $f : c'' \rightarrow c$, $(i_{\text{id}_c}, j_{\text{id}_c})$ is a maximal span for $(f \circ i_f, j_f)$ and there exists a morphism k_f such that $f \circ i_f = i_{\text{id}_c} \circ k_f$ and $j_f = j_{\text{id}_c} \circ k_f$ because $f = \text{id}_c \circ f$.



Now, suppose that $i_f \neq \text{id}_{c''}$. Let us write $i_f = a_n \circ \dots \circ a_1$ and $j_f = b_m \circ \dots \circ b_1$, where a_p and b_q are generating morphisms for $p = 1, \dots, n$ and $q = 1, \dots, m$. Then, we have $n \geq 1$. If $m = 0$, then $j_f = \text{id}_{c'}$, which in turn implies that k_f is an identity. If $m \geq 1$, then we must have $a_1 \neq b_1$ because (i_f, j_f) is a maximal span. If k_f is not an identity, then we can write $k_f = c_l \circ \dots \circ c_1$ with $l \geq 1$, where c_1, \dots, c_l are generating morphisms. However, by the equalities $f \circ i_f = i_{\text{id}_c} \circ k_f$ and $j_f = j_{\text{id}_c} \circ k_f$, it follows that $a_1 = c_1 = b_1$. This is a contradiction. Therefore, k_f is an identity also in this case. On the other hand, we have $i_{\text{id}_c} = \text{id}_c$ by the condition (B'). Thus, we obtain $f \circ i_f = \text{id}_c$. However, this contradicts to the well-foundedness of \mathcal{C} . It follows that $i_f = \text{id}_{c''}$, $f = k_f$ and $j_f = j_{\text{id}_c} \circ f$. The condition (B) holds with $s = j_{\text{id}_c}$.

The converse is obvious.

□

Proposition A.14 *Let \mathcal{C} be a category freely generated by a finite acyclic directed graph Γ with a directed graph isomorphism $\sigma : \Gamma \xrightarrow{\cong} \Gamma^{\text{op}}$. For an object $c \in \mathcal{C}$, assume that there exist two different generating morphisms f and g such that $\text{cod}(f) = c = \text{cod}(g)$ if $c \in |\mathcal{C}|$ is not minimal with respect to $<$. Then, (B') holds for c and any $c' \in \mathcal{C}$.*

Proof. Take any family of maximal spans $\{(i_f, j_f)\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c))}$ such that $\text{cod}(i_f) = c''$ and $\text{cod}(j_f) = c'$ for each $(c'', f) \in \text{Elts}(\mathbf{y}(c))$ and assume that if $f = g \circ h$, then (i_g, j_g) is a unique maximal span for the span $(h \circ i_f, j_f)$. If c is a minimal element of $|\mathcal{C}|$, then we have no choice but to put $i_{\text{id}_c} = \text{id}_c$. Otherwise, we have two different generating morphisms f and g such that $\text{cod}(f) = c = \text{cod}(g)$ by the assumption of the claim. As in the proof of Lemma A.13, we have $i_{\text{id}_c} \circ k_f = f \circ i_f$. If $i_{\text{id}_c} \neq \text{id}_c$, then we can write $i_{\text{id}_c} = a_n \circ \dots \circ a_1$ for generating morphisms a_1, \dots, a_n and $n \geq 1$. Since f is a generating morphism, it follows that $f = a_n$. By the same manner, we obtain $g = a_n$. Thus, we obtain $f = g$ but this is impossible because of the assumption of the claim. Consequently, we have $i_{\text{id}_c} = \text{id}_c$.

□

Theorem A.15 *Let \mathcal{C} be a category freely generated by a finite acyclic directed graph Γ with a directed graph isomorphism $\sigma : \Gamma \xrightarrow{\cong} \Gamma^{\text{op}}$. Let $M_0 : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ be the standard representation with respect to σ . Then, $(\hat{\mathcal{C}}(\text{Int}_{M_0}(-), G \otimes M_0), \varphi^{M_0, G})$ is an initial object of \mathcal{IT}_G for any presheaf $G \in \hat{\mathcal{C}}$.*

Proof. For any $c, c' \in \mathcal{C}$, the map $\gamma_c(\sigma(c')) : \mathbf{y}(\sigma(c))(\sigma(c')) \rightarrow \text{Int}_{M_0}(c)(\sigma(c'))$ is injective by Lemma A.12 and Proposition A.10. If c is minimal or there exist two different generating morphisms whose codomains are c , then $\gamma_c(\sigma(c'))$ is surjective by Proposition A.11, Lemma A.12, Lemma A.13 and Proposition A.14. Thus, if there exist two different generating morphisms whose codomains are c for any non-minimal element $c \in |\mathcal{C}|$, then the natural transformation $\gamma : \mathbf{y} \circ \sigma \rightarrow \text{Int}_{M_0}$ is a natural isomorphism. Consequently, the claim follows by Theorem A.6.

If there is an object c such that the number of generating morphism f such that $\text{cod}(f) = c$ is 1, then γ is not necessarily a natural isomorphism. Indeed, this happens in the simplest case $\Gamma = \{\bullet \rightarrow \bullet\}$. However, if $f : c' \rightarrow c$ is a unique generating morphism such that $\text{cod}(f) = c$, then we have $M_0(c') \cong M_0(c)$ and $\text{Int}_{M_0}(c') \cong \text{Int}_{M_0}(c)$. Consequently, ι_c is uniquely determined by $\iota_{c'}$, where ι_c and $\iota_{c'}$ are as in the proof of Theorem A.6. Thus, the claim follows by slightly modifying the well-founded induction used in the proof of Theorem A.6.

□

The representation M_0 defined in Section 2.2 is isomorphic to the standard representation for the isomorphism $\sigma : \uparrow\uparrow \xrightarrow{\cong} \uparrow\uparrow^{\text{op}}$ defined by $\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(m_0) = m_1$, $\sigma(m_1) = m_0$. In general, different dual structure on the same category gives rise to non-isomorphic standard representations. However, if \mathcal{C} satisfies the assumption of Theorem A.6 or Theorem A.15, then all standard representations induce the isomorphic objects in \mathcal{IT}_G and they all are initial objects. For example, the dual structure σ' on $\uparrow\uparrow$ given by $\sigma(m_0) = m_0$, $\sigma(m_1) = m_1$ induces a non-isomorphic standard representation to that defined in Section 2.2.

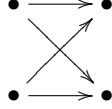
Let us see some examples to which Theorem A.15 are applied. We denote the category freely generated by the graph

$$\Gamma = 0 \xrightarrow{m} 1$$

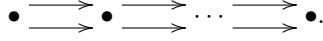
by \uparrow . The presheaf category $\hat{\uparrow}$ consists of all functions between two sets and pairs of functions between them satisfying an obvious commutative diagram. Let σ be the unique dual structure on \uparrow . It is obvious

that the standard representation M_0 with respect to σ is given by maps from a singleton to a singleton for both $M_0(0)$ and $M_0(1)$. Hence, for any $G \in \hat{\uparrow}$, $\varphi_0^{M_0, G} : G(0) \rightarrow \hat{\uparrow}(\text{Int}_{M_0}(0), G \otimes M_0) \cong G(0)$ is a bijection and $\varphi_1^{M_0, G} : G(1) \rightarrow \hat{\uparrow}(\text{Int}_{M_0}(1), G \otimes M_0) \cong G(0)$ is isomorphic to the map $G(m) : G(1) \rightarrow G(0)$. Theorem A.15 implies that the partition of the set $G(1)$ by the fibers of the map $G(m)$ is the finest one among those induced by all representations into bicomplete categories.

Potentially important categories in applications to which Theorem A.15 are applied include the categories freely generated by the following graphs:



and



In the former case, its presheaf category is the category of bipartite directed networks. Bipartite networks are frequently used to represent metabolic networks. In the latter case, we obtain the category consisting of “hierarchical directed networks”. This could be used to represent hierarchical systems. However, here, we do not pursue these examples further.

A.2 Stability with respect to representations and gluing condition for representations

Now, we proceed to generalizations of Theorem 2.26 and Proposition 2.27. Throughout this subsection, \mathcal{C} is a well-founded small category equipped with a dual structure σ .

For an object $c \in \mathcal{C}$, let $\text{Elts}(\mathbf{y}(c))^*$ be a category whose objects are the same as those in $\text{Elts}(\mathbf{y}(c))$ and morphisms are defined as follows: any morphism $h : (c', f) \rightarrow (c'', g)$ in $\text{Elts}(\mathbf{y}(c))$ such that $(c'', g) \neq (c, \text{id}_c)$ is a morphism in $\text{Elts}(\mathbf{y}(c))^*$. We denote it by h^* to indicate it is a morphism in $\text{Elts}(\mathbf{y}(c))^*$. Note that if $(c'', g) \neq (c, \text{id}_c)$, then (c', f) cannot be (c, id_c) because of the well-foundedness of \mathcal{C} for a morphism $h : (c', f) \rightarrow (c'', g)$ in $\text{Elts}(\mathbf{y}(c))$. For any morphism $f : (c', f) \rightarrow (c, \text{id}_c)$ in $\text{Elts}(\mathbf{y}(c))$, we add a morphism f^* which goes in the reverse way to $\text{Elts}(\mathbf{y}(c))^*$: $f^* : (c, \text{id}_c) \rightarrow (c', f)$. The composition of morphisms $f^* : (c, \text{id}_c) \rightarrow (c', f)$ and $h^* : (c', f) \rightarrow (c'', g)$ in $\text{Elts}(\mathbf{y}(c))^*$ is defined as $h^* \circ f^* := g^*$.

Let \mathcal{D} be a complete category and $M : \mathcal{C} \rightarrow \mathcal{D}$ a representation. For each $c \in \mathcal{C}$, we define a functor $I_{M,c} : \text{Elts}(\mathbf{y}(c))^* \rightarrow \mathcal{D}$ as follows: for objects in $\text{Elts}(\mathbf{y}(c))^*$, $I_{M,c}(c, \text{id}_c) = \text{Int}_M(c)$ and $I_{M,c}(c', f) = M(c')$ if $(c', f) \neq (c, \text{id}_c)$. For morphisms, $\text{id}_c : (c, \text{id}_c) \rightarrow (c, \text{id}_c)$ is sent to $\text{id}_{\text{Int}_M(c)}$, $f^* : (c, \text{id}_c) \rightarrow (c', f)$

is sent to $\nu_{(c',f)}^{M,c} : \text{Int}_M(c) \rightarrow M(c')$ if $(c', f) \neq (c, \text{id}_c)$ and $h^* : (c', f) \rightarrow (c'', g)$ is sent to $M(h) : M(c') \rightarrow M(c'')$ if $(c', f) \neq (c, \text{id}_c)$ and $(c'', g) \neq (c, \text{id}_c)$.

One can check that $I_{M,c}$ is indeed a functor by using the facts that M is a functor and $\nu^{M,c}$ is a cone on $M \circ \pi_c$.

The following definition generalizes the gluing condition given in Definition 2.25.

Definition A.16 Let \mathcal{C} be a well-founded small category with a dual structure σ and \mathcal{D} a bicomplete category. We say that a representation $M : \mathcal{C} \rightarrow \mathcal{D}$ satisfies the **gluing condition** if the cocone

$$\begin{aligned} & \{\nu_{(c, \text{id}_c)}^{M,c} : I_{M,c}(c, \text{id}_c) = \text{Int}_M(c) \rightarrow M(c)\} \\ & \cup \{M(f) : I_{M,c}(c', f) = M(c') \rightarrow M(c)\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))^*, (c',f) \neq (c, \text{id}_c)} \end{aligned}$$

is a colimit on $I_{M,c}$.

Proposition A.17 Let \mathcal{C} be a well-founded small category with a dual structure σ . Then, the standard representation M_0 with respect to σ satisfies the gluing condition.

Proof. Let

$$\{\theta_{(c, \text{id}_c)} : \text{Int}_{M_0}(c) \rightarrow D\} \cup \{\theta_{(c', f)} : M_0(c') \rightarrow D\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))^*, (c',f) \neq (c, \text{id}_c)}$$

be a cocone on $I_{M_0,c}$. Since colimits in $\hat{\mathcal{C}}$ can be computed pointwise, it is sufficient to show the universality of $M_0(c)(d)$ for each $d \in \mathcal{C}$. In what follows, we put $d = \sigma(\bar{d})$.

Let us define a map $\rho : M_0(c)(d) \rightarrow D(d)$ in the following way: when c is non-minimal element of $|\mathcal{C}|$ with respect to $<$, then for any element of $M_0(c)(d)$ we can take a representative $(\sigma(i), \sigma(j))$ of it such that $i \neq \text{id}_c$. Then, we define $\rho([\sigma(i), \sigma(j)]) = \theta_{(e,i)}(d)([\text{id}_{\sigma(e)}, \sigma(j)])$, where $e = \text{dom}(i) = \text{dom}(j)$.

$$\begin{array}{ccc} & M_0(e)(d) & \\ \theta_{(e,i)}(d) \swarrow & \downarrow M_0(i)(d) & \\ D(d) & \xleftarrow{\rho} & M_0(c)(d) \end{array}$$

If c is minimal, then any element of $M_0(c)(d)$ takes a form $[(\text{id}_{\sigma(c)}, \sigma(j))] = \{(\text{id}_{\sigma(c)}, \sigma(j))\}$ for some $j : c \rightarrow \bar{d}$ and $\text{Int}_{M_0}(c) = M_0(c)$. Thus, we define $\rho([\text{id}_{\sigma(c)}, \sigma(j)]) = \theta_{(c, \text{id}_c)}(d)([(\text{id}_{\sigma(c)}, \sigma(j))])$. Note that in either case, we have the same form for ρ . It is straightforward to check that ρ is a well-defined map.

Now, we show that $\rho \circ M_0(f)(d) = \theta_{(c',f)}(d)$ for any $(c', f) \in \text{Elts}(\mathbf{y}(c))^*$ such that $(c', f) \neq (c, \text{id}_c)$

and $\rho \circ \nu_{(c, \text{id}_c)}^{M_0, c}(d) = \theta_{(c, \text{id}_c)}(d)$. For any morphism $f : c' \rightarrow c$ such that $c' \neq c$, we have

$$\begin{aligned}
\rho \circ M_0(f)(d)([(\sigma(i), \sigma(j))]) &= \rho([(\sigma(i) \circ \sigma(f), \sigma(j))]) \\
&= \theta_{(e, f \circ i)}(d)([(\text{id}_{\sigma(e)}, \sigma(j))]) \\
&= \theta_{(c', f)}(d) \circ M_0(i)(d)([(\text{id}_{\sigma(e)}, \sigma(j))]) \\
&= \theta_{(c', f)}(d)([(\sigma(i), \sigma(j))])
\end{aligned}$$

for any $[(\sigma(i), \sigma(j))] \in M_0(c')(d)$, where the third equality follows from the assumption that θ is a cocone on $I_{M_0, c}$.

$$\begin{array}{ccc}
\sigma(c) & \xrightarrow{\sigma(f)} & \sigma(c') & \xrightarrow{\sigma(i)} & \sigma(e) \\
& & \nearrow \sigma(j) & & \\
& & \sigma(\bar{d}) & &
\end{array}$$

For $\text{id}_c : c \rightarrow c$, if c is minimal, then $\nu_{(c, \text{id}_c)}^{M_0, c}(d)$ is an identity. Hence, we have nothing to do. If c is not minimal, then there exists a morphism $f : c' \rightarrow c$ such that $c' \neq c$. The desired equality follows because the following diagram commutes:

$$\begin{array}{ccccc}
& & \text{Int}_{M_0}(c)(d) & & \\
& \swarrow \theta_{(c, \text{id}_c)}(d) & \downarrow \nu_{(c', f)}^{M_0, c}(d) & \searrow \nu_{(c, \text{id}_c)}^{M_0, c}(d) & \\
& & M_0(c')(d) & & \\
& \swarrow \theta_{(c', f)}(d) & \searrow M_0(f)(d) & & \\
D(d) & \xleftarrow{\rho} & M_0(c)(d) & &
\end{array}$$

Finally, we show the uniqueness of ρ . Suppose that there exists a map $\rho' : M_0(c)(d) \rightarrow D(d)$ such that $\rho' \circ M_0(f)(d) = \theta_{(c', f)}(d)$ for any $(c', f) \in \text{Elts}(\mathbf{y}(c))^*$ such that $(c', f) \neq (c, \text{id}_c)$ and $\rho' \circ \nu_{(c, \text{id}_c)}^{M_0, c}(d) = \theta_{(c, \text{id}_c)}(d)$. Then, for any $[(\sigma(i), \sigma(j))] \in M_0(c)(d)$, we have

$$\begin{aligned}
\rho'([(\sigma(i), \sigma(j))]) &= \rho' \circ M_0(i)(d)([(\text{id}_{\sigma(e)}, \sigma(j))]) \\
&= \theta_{(e, i)}(d)([(\text{id}_{\sigma(e)}, \sigma(j))]) \\
&= \rho([(\sigma(i), \sigma(j))]),
\end{aligned}$$

where we take i as non-identity when c is not minimal.

□

The next theorem is a generalization of Theorem 2.26.

Theorem A.18 *Let \mathcal{C} be a well-founded small category with a dual structure σ . For a presheaf $G \in \hat{\mathcal{C}}$, assume that the standard representation M_0 with respect to σ induces an initial object in the category \mathcal{IT}_G . Let $M : \mathcal{C} \rightarrow \mathcal{D}$ be a representation into a bicomplete category \mathcal{D} satisfying the gluing condition. If G is stable with respect to M , then G is also stable with respect to M_0 .*

Proof. For a presheaf $G \in \hat{\mathcal{C}}$, assume that $\eta_c^{M,G}$ is an isomorphism for any $c \in \mathcal{C}$. We show that $\eta_c^{M_0,G}$ is also an isomorphism for any $c \in \mathcal{C}$ by appealing to a well-founded induction with respect to $<$ on $|\mathcal{C}|$.

If c is a minimal element of $|\mathcal{C}|$, then we have $M_0(c) = \text{Int}_{M_0}(c) \cong \mathbf{y}(\sigma(c))$. Hence, we can identify $\eta_c^{M_0,G}$ with $\varphi_c^{M_0,G}$ which has been shown to be an isomorphism in the proof of Theorem A.6.

Let c be a non-minimal element of $|\mathcal{C}|$. Assume that $\eta_{c'}^{M_0,G}$ is an isomorphism for any $c' \in \mathcal{C}$ such that $c' < c$. We first show that $\eta_c^{M_0,G}$ is surjective. For any morphism $\delta : M_0(c) \rightarrow G \otimes M_0$, we shall construct a morphism $\bar{\delta} : M(c) \rightarrow G \otimes M$ and then show that $\eta_c^{M_0,G} \left((\eta_c^{M,G})^{-1} (\bar{\delta}) \right) = \delta$.

Let us construct a cocone

$$\{\theta_{(c, \text{id}_c)} : \text{Int}_M(c) \rightarrow G \otimes M\} \cup \{\theta_{(c', f)} : M(c') \rightarrow G \otimes M\}_{(c', f) \in \text{Elts}(\mathbf{y}(c))^*, (c', f) \neq (c, \text{id}_c)}$$

on $I_{M,c}$ by defining $\theta_{(c', f)} = \eta_{c'}^{M,G} \left((\eta_{c'}^{M_0,G})^{-1} (\delta \circ M_0(f)) \right)$ for $(c', f) \neq (c, \text{id}_c)$ and $\theta_{(c, \text{id}_c)} = \iota_c \left(\delta \circ \nu_{(c, \text{id}_c)}^{M_0, c} \right)$, where ι is a unique morphism from $\varphi^{M_0,G}$ to $\varphi^{M,G}$ in \mathcal{IT}_G . To show that θ is indeed a cocone on $I_{M,c}$, we have to show that $\theta_{(c', f)} = \theta_{(c'', g)} \circ M(h)$ for any morphism $h^* : (c', f) \rightarrow (c'', g)$ in $\text{Elts}(\mathbf{y}(c))^*$ such that $(c'', g) \neq (c, \text{id}_c)$ and $\theta_{(c, \text{id}_c)} = \theta_{(c', f)} \circ \nu_{(c', f)}^{M, c}$ for any morphism $f^* : (c, \text{id}_c) \rightarrow (c', f)$ in $\text{Elts}(\mathbf{y}(c))^*$ such that $(c, \text{id}_c) \neq (c', f)$.

For the former equality, we have

$$\begin{aligned} \theta_{(c'', g)} \circ M(h) &= \eta_{c''}^{M,G} \left((\eta_{c''}^{M_0,G})^{-1} (\delta \circ M_0(g)) \right) \circ M(h) \\ &= \eta_{c'}^{M,G} \left(G(h) \left((\eta_{c''}^{M_0,G})^{-1} (\delta \circ M_0(g)) \right) \right) \\ &= \eta_{c'}^{M,G} \left((\eta_{c''}^{M_0,G})^{-1} (\delta \circ M_0(g) \circ M_0(h)) \right) \\ &= \eta_{c'}^{M,G} \left((\eta_{c''}^{M_0,G})^{-1} (\delta \circ M_0(f)) \right) \\ &= \theta_{(c', f)}, \end{aligned}$$

where we use the naturality of $\eta^{M,G}$ in the second equality and that of $\eta^{M_0,G}$ in the third equality. For

the latter equality,

$$\begin{aligned}
\theta_{(c, \text{id}_c)} &= \iota_c \left(\delta \circ \nu_{(c, \text{id}_c)}^{M_0, c} \right) \\
&= \iota_c \left(\delta \circ M_0(f) \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \circ \text{Int}_{M_0}(f) \right) \\
&= \iota_{c'} \left(\delta \circ M_0(f) \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \right) \circ \text{Int}_M(f) \\
&= \iota_{c'} \left(\varphi_{c'}^{M_0, G}(x) \right) \circ \text{Int}_M(f) \\
&= \varphi_{c'}^{M, G}(x) \circ \text{Int}_M(f) \\
&= \mu_{(c', x)}^{M, G} \circ \nu_{(c', \text{id}_{c'})}^{M, c'} \circ \text{Int}_M(f) \\
&= \mu_{(c', x)}^{M, G} \circ \nu_{(c', f)}^{M, c'} \\
&= \theta_{(c', f)} \circ \nu_{(c', f)}^{M, c'},
\end{aligned}$$

where the second equality follows because the diagram

$$\begin{array}{ccc}
\text{Int}_{M_0}(c) & \xrightarrow{\text{Int}_{M_0}(f)} & \text{Int}_{M_0}(c') \\
\downarrow \nu_{(c, \text{id}_c)}^{M_0, c} & \searrow \nu_{(c', f)}^{M_0, c} & \downarrow \nu_{(c', \text{id}_{c'})}^{M_0, c'} \\
M_0(c) & \xleftarrow{M_0(f)} & M_0(c')
\end{array} \tag{1}$$

commutes and the third equality is due to the naturality of ι . In the fourth equality, we put $x := \left(\eta_{c'}^{M_0, G} \right)^{-1} (\delta \circ M_0(f)) \in G(c')$. Thus, we have $\varphi_{c'}^{M_0, G}(x) = \mu_{(c', x)}^{M, G} \circ \nu_{(c', \text{id}_{c'})}^{M, c'} = \eta_{c'}^{M_0, G}(x) \circ \nu_{(c', \text{id}_{c'})}^{M, c'}$. For the remaining equalities, we use the naturality of ι in the fifth equality, the definition of $\varphi_{c'}^{M, G}(x)$ in the sixth equality, the definition of $\text{Int}_M(f)$ in the seventh equality and $\theta_{(c', f)} = \eta_{c'}^{M, G}(x) = \mu_{(c', x)}^{M, G}$ in the last equality.

Consequently, we obtain a unique morphism $\bar{\delta} : M(c) \rightarrow G \otimes M$ such that $\theta_{(c, \text{id}_c)} = \bar{\delta} \circ \nu_{(c, \text{id}_c)}^{M, c}$ and $\theta_{(c', f)} = \bar{\delta} \circ M(f)$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$.

Now, we show that $\eta_c^{M_0, G} \left(\left(\eta_c^{M, G} \right)^{-1} (\bar{\delta}) \right) = \delta$. We put the left-hand side by δ' in what follows. We appeal to the fact that M_0 satisfies the gluing condition (Proposition A.17). If we can show that $\delta' \circ \nu_{(c, \text{id}_c)}^{M_0, c} = \delta \circ \nu_{(c, \text{id}_c)}^{M_0, c} =: \zeta_{(c, \text{id}_c)}$ and $\delta' \circ M_0(f) = \delta \circ M_0(f) =: \zeta_{(c', f)}$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$, then ζ is a cocone on $I_{M_0, c}$. Since M_0 satisfies the gluing condition by Proposition A.17, there is a unique morphism $\kappa : M_0(c) \rightarrow G \otimes M_0$ such that $\kappa \circ \nu_{(c, \text{id}_c)}^{M_0, c} = \zeta_{(c, \text{id}_c)}$ and $\kappa \circ M_0(f) = \zeta_{(c', f)}$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$. However, both δ and δ' satisfy the condition for κ , we conclude $\delta = \kappa = \delta'$.

Let $f : c' \rightarrow c$ be a morphism such that $c' \neq c$. We want to show that $\delta' \circ M_0(f) = \delta \circ M_0(f)$ holds. Putting $x = \left(\eta_c^{M, G} \right)^{-1} (\bar{\delta}) \in G(c)$, we have $\delta' \circ M_0(f) = \eta_{c'}^{M_0, G}(x) \circ M_0(f) = \eta_{c'}^{M_0, G}(x \cdot f)$, where we use

the naturality of $\eta^{M_0, G}$ in the second equality. Now, we have

$$\begin{aligned}
x \cdot f &= G(f) \left((\eta_c^{M, G})^{-1} (\bar{\delta}) \right) \\
&= (\eta_{c'}^{M, G})^{-1} (\bar{\delta} \circ M(f)) \\
&= (\eta_{c'}^{M, G})^{-1} (\theta_{(c', f)}) \\
&= (\eta_{c'}^{M, G})^{-1} \left(\eta_{c'}^{M, G} \left((\eta_{c'}^{M_0, G})^{-1} (\delta \circ M_0(f)) \right) \right) \\
&= (\eta_{c'}^{M_0, G})^{-1} (\delta \circ M_0(f)),
\end{aligned}$$

where we use the naturality of $\eta^{M, G}$ in the second equality. Thus, we obtain the desired equality.

To show the equality $\delta' \circ \nu_{(c, \text{id}_c)}^{M_0, c} = \delta \circ \nu_{(c, \text{id}_c)}^{M_0, c}$, take a morphism $f : c' \rightarrow c$ such that $c' \neq c$ whose existence is guaranteed by the assumption that c is not minimal. We have

$$\begin{aligned}
\delta' \circ \nu_{(c, \text{id}_c)}^{M_0, c} &= \eta_c^{M_0, G}(x) \circ \nu_{(c, \text{id}_c)}^{M_0, c} \\
&= \mu_{(c, x)}^{M_0, G} \circ \nu_{(c, \text{id}_c)}^{M_0, c} \\
&= \varphi_c^{M_0, G}(x) \\
&= \varphi_{c'}^{M_0, G}(x \cdot f) \circ \text{Int}_{M_0}(f) \\
&= \mu_{(c', x \cdot f)}^{M_0, G} \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \circ \text{Int}_{M_0}(f) \\
&= \eta_{c'}^{M_0, G}(x \cdot f) \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \circ \text{Int}_{M_0}(f) \\
&= \eta_{c'}^{M_0, G} \left((\eta_{c'}^{M, G})^{-1} (\delta \circ M_0(f)) \right) \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \circ \text{Int}_{M_0}(f) \\
&= \delta \circ M_0(f) \circ \nu_{(c', \text{id}_{c'})}^{M_0, c'} \circ \text{Int}_{M_0}(f) \\
&= \delta \circ \nu_{(c, \text{id}_c)}^{M_0, c},
\end{aligned}$$

where the fourth equality follows from the naturality of $\varphi^{M_0, G}$ and the last equality is due to the commutative diagram (1). This complete the proof that $\eta_c^{M_0, G}$ is a surjection.

Next, we show that $\eta_c^{M_0, G}$ is injective. For any $x, y \in G(c)$, suppose that $\eta_c^{M_0, G}(x) = \eta_c^{M_0, G}(y)$ holds. We put the both sides of the equality by δ . If we construct a unique morphism $\bar{\delta} : M(c) \rightarrow G \otimes M$ such that $\bar{\delta} \circ \nu_{(c, \text{id}_c)}^{M, c} = \theta_{(c, \text{id}_c)}$ and $\bar{\delta} \circ M(f) = \theta_{(c', f)}$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$ by the same manner as in the proof of the surjectivity.

We shall show that both $\eta_c^{M, G}(x)$ and $\eta_c^{M, G}(y)$ satisfy the condition for $\bar{\delta}$. Then, we will obtain $\eta_c^{M, G}(x) = \bar{\delta} = \eta_c^{M, G}(y)$ which implies $x = y$ because $\eta_c^{M, G}$ is a bijection.

For any morphism $f : c' \rightarrow c$ such that $c' \neq c$, we have

$$\begin{aligned}\eta_c^{M,G}(x) \circ M(f) &= \eta_{c'}^{M,G}(x \cdot f) \\ &= \eta_{c'}^{M,G} \left(\left(\eta_{c'}^{M_0,G} \right)^{-1} (\delta \circ M_0(f)) \right) \\ &= \theta_{(c',f)},\end{aligned}$$

where we use the naturality of $\eta^{M_0,G}$ in the second equality. By the same manner, we obtain $\eta_c^{M,G}(y) \circ M(f) = \bar{\delta} \circ M(f)$. We also have

$$\begin{aligned}\eta_c^{M,G}(x) \circ \nu_{(c,\text{id}_c)}^{M,c} &= \mu_{(c,x)}^{M,G} \circ \nu_{(c,\text{id}_c)}^{M,c} \\ &= \varphi_c^{M,G}(x) \\ &= \iota_c \circ \varphi_c^{M_0,G}(x) \\ &= \iota_c \left(\eta_c^{M_0,G}(x) \circ \nu_{(c,\text{id}_c)}^{M_0,c} \right) \\ &= \iota_c \left(\delta \circ \nu_{(c,\text{id}_c)}^{M_0,c} \right) \\ &= \theta_{(c,\text{id}_c)}\end{aligned}$$

and $\eta_c^{M,G}(y) \circ \nu_{(c,\text{id}_c)}^{M,c} = \theta_{(c,\text{id}_c)}$ by the same manner, as desired. This completes the proof of the injectivity. \square

The next proposition is a generalization of Proposition 2.27.

Proposition A.19 *Let \mathcal{C} be a well-founded small category with a dual structure σ and $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ a representation satisfying the gluing condition. Then, the tensor product representation $M \otimes N$ also satisfies the gluing condition for any representation $N : \mathcal{C} \rightarrow \mathcal{D}$ into a bicomplete category \mathcal{D} .*

Proof. We show that the cocone

$$\begin{aligned}&\{\nu_{(c,\text{id}_c)}^{M \otimes N, c} : \text{Int}_{M \otimes N}(c) \rightarrow M \otimes N(c)\} \\ &\cup \{M \otimes N(f) : M \otimes N(c') \rightarrow M \otimes N(c)\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))^*, (c',f) \neq (c,\text{id}_c)}\end{aligned}$$

is a colimit on $I_{M \otimes N, c}$ for any $c \in \mathcal{C}$.

Let

$$\{\theta_{(c,\text{id}_c)} : \text{Int}_{M \otimes N}(c) \rightarrow X\} \cup \{\theta_{(c',f)} : M \otimes N(c') \rightarrow X\}_{(c',f) \in \text{Elts}(\mathbf{y}(c))^*, (c',f) \neq (c,\text{id}_c)}$$

be a cocone on $I_{M \otimes N, c}$. Since M satisfies the gluing condition,

$$\{\nu_{(c, \text{id}_c)}^{M, c} : \text{Int}_M(c) \rightarrow M(c)\} \cup \{M(f) : M(c') \rightarrow M(c)\}_{(c', f) \in \text{Elts}(\mathbf{y}(c))^*, (c', f) \neq (c, \text{id}_c)}$$

is a colimit on $I_{M, c}$. Since $(-) \otimes N$ preserves colimits,

$$\begin{aligned} & \{\nu_{(c, \text{id}_c)}^{M, c} \otimes N : \text{Int}_M(c) \otimes N \rightarrow M(c) \otimes N\} \\ & \cup \{M(f) \otimes N : M(c') \otimes N \rightarrow M(c) \otimes N\}_{(c', f) \in \text{Elts}(\mathbf{y}(c))^*, (c', f) \neq (c, \text{id}_c)} \end{aligned}$$

is also a colimit on $(-) \otimes N \circ I_{M, c}$. Note that $M(c) \otimes N = M \otimes N(c)$ and $M(f) \otimes N = M \otimes N(f)$.

Now, for the canonical map $\beta_c : \text{Int}_M(c) \otimes N \rightarrow \text{Int}_{M \otimes N}(c)$,

$$\begin{aligned} & \{\theta_{(c, \text{id}_c)} \circ \beta_c : \text{Int}_M(c) \otimes N \rightarrow X\} \\ & \cup \{\theta_{(c', f)} : M \otimes N(c') \rightarrow X\}_{(c', f) \in \text{Elts}(\mathbf{y}(c))^*, (c', f) \neq (c, \text{id}_c)} \end{aligned}$$

is a cocone on $(-) \otimes N \circ I_{M, c}$ because the following diagram commutes for any morphism $f^* : (c, \text{id}_c) \rightarrow (c', f)$ in $\text{Elts}(\mathbf{y}(c))^*$ such that $(c', f) \neq (c, \text{id}_c)$:

$$\begin{array}{ccc} \text{Int}_{M \otimes N}(c) & \xrightarrow{\theta_{(c, \text{id}_c)}} & X \\ \beta_c \uparrow & \searrow \nu_{(c', f)}^{M \otimes N, c} & \uparrow \theta_{(c', f)} \\ \text{Int}_M(c) \otimes N & \xrightarrow{\nu_{(c', f)}^{M, c} \otimes N} & M \otimes N(c'). \end{array}$$

Consequently, there exists a unique map $\xi_c : M \otimes N \rightarrow X$ such that (i) $\theta_{(c, \text{id}_c)} \circ \beta_c = \xi_c \circ \nu_{(c, \text{id}_c)}^{M, c} \otimes N$ and (ii) $\theta_{(c', f)} = \xi_c \circ M(f) \otimes N$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$.

We shall show that the equality (iii) $\theta_{(c, \text{id}_c)} = \xi_c \circ \nu_{(c, \text{id}_c)}^{M \otimes N, c}$ holds. Indeed, if c is minimal with respect to $<$ on $|\mathcal{C}|$, then β_c is a bijection. Thus, we have

$$\begin{aligned} \xi_c \circ \nu_{(c, \text{id}_c)}^{M \otimes N, c} &= \xi \circ \nu_{(c, \text{id}_c)}^{M, c} \otimes N \circ \beta_c^{-1} \\ &= \theta_{(c, \text{id}_c)} \circ \beta_c \circ \beta_c^{-1} = \theta_{(c, \text{id}_c)}, \end{aligned}$$

where we use (i) in the second equality. If c is not minimal, then there exists a morphism $f : c' \rightarrow c$ such

that $c' \neq c$. In this case, (iii) holds because the following diagram commutes:

$$\begin{array}{ccc}
 M \otimes N(c) & \xrightarrow{\xi_c} & X \\
 \uparrow \nu_{(c, \text{id}_c)}^{M \otimes N, c} & \swarrow M \otimes N(f) \quad \searrow \theta_{(c', f)} & \\
 \text{Int}_{M \otimes N}(c) & \xrightarrow{\nu_{(c', f)}^{M \otimes N, c}} & M \otimes N(c') \\
 & \searrow \theta_{(c, \text{id}_c)} & \nearrow \\
 & & X
 \end{array}$$

Moreover, ξ_c satisfying both (ii) and (iii) is unique. Suppose a map $\xi'_c : M \otimes N(c) \rightarrow X$ satisfies (ii) and (iii). By composing (iii) and β_c , we obtain (i) with ξ_c replaced by ξ'_c . Thus, (i) and (ii) also holds for ξ'_c . Since ξ_c satisfying both (i) and (ii) is unique, we conclude that $\xi_c = \xi'_c$. This completes the proof of the claim. \square

Finally, we study the relationship between the stability with respect to a representation M satisfying the gluing condition and the stability with respect to a tensor product representation $M \otimes N$.

Proposition A.20 *Let \mathcal{C} be a well-founded small category with a dual structure σ , $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ a representation satisfying the gluing condition and $N : \mathcal{C} \rightarrow \mathcal{D}$ a representation into a bicomplete category \mathcal{D} . For a presheaf $G \in \hat{\mathcal{C}}$, assume that the map $(-) \otimes N : \hat{\mathcal{C}}(M(c), G \otimes M) \rightarrow \mathcal{D}(M \otimes N(c), (G \otimes M) \otimes N)$ is injective for any minimal element $c \in |\mathcal{C}|$ with respect to $<$. Then, if G is stable with respect to $M \otimes N$, then G is also stable with respect to M .*

Proof. Let G be stable with respect to $M \otimes N$ and assume that the map $(-) \otimes N : \hat{\mathcal{C}}(M(c), G \otimes M) \rightarrow \mathcal{D}(M \otimes N(c), (G \otimes M) \otimes N)$ is injective for any minimal element $c \in |\mathcal{C}|$ with respect to $<$. Define the natural transformation $j : \hat{\mathcal{C}}(M(-), G \otimes M) \rightarrow \mathcal{D}(M \otimes N(-), G \otimes (M \otimes N))$ by $j_c(\delta) := \alpha_G \circ \delta \otimes N$ for any $c \in \mathcal{C}$ and morphism $\delta : M(c) \rightarrow G \otimes M$, where α_G is the canonical isomorphism $\alpha_G : (G \otimes M) \otimes N \xrightarrow{\cong} G \otimes (M \otimes N)$. Then, we have $\eta_c^{M \otimes N, G} = j_c \circ \eta_c^{M, G}$ because $\eta_c^{M \otimes N, G}(x) = \mu_{(c, x)}^{M \otimes N, G} = \alpha_G \circ \mu_{(c, x)}^{M, G} \otimes N = j_c(\mu_{(c, x)}^{M, G}) = j_c(\eta_c^{M, G}(x))$ for any $x \in G(c)$. From this, we can show that $\eta_c^{M, G}$ is an injection for any $c \in \mathcal{C}$. Indeed, if $\eta_c^{M, G}(x) = \eta_c^{M, G}(y)$ for $x, y \in G(c)$, then $\eta_c^{M \otimes N, G}(x) = j_c(\eta_c^{M, G}(x)) = j_c(\eta_c^{M, G}(y)) = \eta_c^{M \otimes N, G}(y)$. Since $\eta_c^{M \otimes N, G}$ is assumed to be a bijection, we obtain $x = y$.

To show that $\eta_c^{M, G}$ is a surjection for any $c \in \mathcal{C}$, we appeal to a well-founded induction on $|\mathcal{C}|$ with respect to $<$. If c is a minimal element of $|\mathcal{C}|$ with respect to $<$, then $\eta_c^{M, G}$ turns out to be a surjection. Indeed, for any morphism $\delta : M(c) \rightarrow G \otimes M$, if we put $x := (\eta_c^{M \otimes N, G})^{-1}(j_c(\delta)) \in G(c)$, then $j_c(\delta) = \eta_c^{M \otimes N, G}(x) = j_c(\eta_c^{M, G}(x))$. By the assumption of the claim, j_c is an injection. Hence, we obtain $\delta = \eta_c^{M, G}(x)$.

When c is not minimal, assume that $\eta_{c'}^{M, G}$ is surjective for any $c' < c$. Since we have shown that $\eta_d^{M, G}$

is injective for any $d \in \mathcal{C}$, $\eta_{c'}^{M,G}$ is a bijection for any $c' < c$. For any morphism $\delta : M(c) \rightarrow G \otimes M$, put $x := (\eta_c^{M \otimes N, G})^{-1}(j_c(\delta)) \in G(c)$ and $\delta' = \eta_c^{M,G}(x)$. Suppose that we have $\delta \circ \nu_{(c, \text{id}_c)}^{M,c} = \delta' \circ \nu_{(c, \text{id}_c)}^{M,c} =: \zeta_{(c, \text{id}_c)}$ and $\delta \circ M(f) = \delta' \circ M(f) =: \zeta_{(c', f)}$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$. Then, ζ is a cocone on $I_{M,c}$. Since M satisfies the gluing condition, there exists a unique morphism $\kappa : M(c) \rightarrow G \otimes M$ such that $\zeta_{(c, \text{id}_c)} = \kappa \circ \nu_{(c, \text{id}_c)}^{M,c}$ and $\zeta_{(c', f)} = \kappa \circ M(f)$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$. Since both δ and δ' satisfies the condition for κ , we conclude that $\delta = \kappa = \delta'$ which implies that $\eta_c^{M,G}$ is a surjection.

Thus, our task is to show that $\delta \circ \nu_{(c, \text{id}_c)}^{M,c} = \delta' \circ \nu_{(c, \text{id}_c)}^{M,c}$ and $\delta \circ M(f) = \delta' \circ M(f)$ for any morphism $f : c' \rightarrow c$ such that $c' \neq c$. For the latter equality, fix any morphism $f : c' \rightarrow c$ such that $c' \neq c$. Then, we have

$$\begin{aligned}
\delta' \circ M(f) &= \eta_c^{M,G}(x) \circ M(f) \\
&= \eta_{c'}^{M,G}(x \cdot f) \\
&= \eta_{c'}^{M,G} \left(G(f) \left((\eta_c^{M \otimes N, G})^{-1}(j_c(\delta)) \right) \right) \\
&= \eta_{c'}^{M,G} \left(\left(\eta_{c'}^{M \otimes N, G} \right)^{-1}(j_c(\delta) \circ M \otimes N(f)) \right) \\
&= \eta_{c'}^{M,G} \left(\left(\eta_{c'}^{M \otimes N, G} \right)^{-1}(j_{c'}(\delta \circ M(f))) \right) \\
&= \delta \circ M(f),
\end{aligned}$$

where we use the naturality of $\eta^{M,G}$ in the second equality, that of $\eta^{M \otimes N, G}$ in the fourth equality and that of j in the fifth equality. The last equality follows from the equality $\eta_{c'}^{M,G} = j_{c'}^{-1} \circ \eta_{c'}^{M \otimes N, G}$ which in turn obtained from the invertibility of $\eta_{c'}^{M,G}$ and $j_{c'}$.

The former equality also follows from a similar calculation showing $\delta \circ \nu_{(c, \text{id}_c)}^{M_0, c} = \delta' \circ \nu_{(c, \text{id}_c)}^{M_0, c}$ in the proof of Theorem A.18.

□

Corollary A.21 *Let \mathcal{C} be a well-founded small category with a dual structure σ , $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ a representation satisfying the gluing condition and $N : \mathcal{C} \rightarrow \mathcal{D}$ a representation into a bicomplete category \mathcal{D} . For a presheaf $G \in \hat{\mathcal{C}}$, assume that the component of the unit $\eta_c^{M,G} : G(c) \rightarrow \hat{\mathcal{C}}(M(c), G \otimes M)$ is an isomorphism for any minimal element $c \in |\mathcal{C}|$ with respect to $<$. Then, if G is stable with respect to $M \otimes N$, then G is also stable with respect to M .*

To obtain instances of Corollary A.21, let us consider the case when $\mathcal{C} = \uparrow\uparrow$. If a representation $M : \uparrow\uparrow \rightarrow \hat{\uparrow\uparrow}$ satisfies $M(0) = \mathbf{y}(1)$ and the gluing condition, then we have $\eta_0^{M,G} : G(0) \xrightarrow{\cong} \hat{\mathcal{C}}(M(0), G \otimes M)$ for any directed network G . This can be extended to a general case and proved as follows.

Proposition A.22 *Let \mathcal{C} be a well-founded small category with a dual structure σ . We assume that there exists the minimum element c in $|\mathcal{C}|$ with respect to $<$. Then, if a representation $M : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ satisfies the gluing condition and $M(c) \cong \mathbf{y}(\sigma(c))$, then we have $\eta_c^{M,G} : G(c) \xrightarrow{\cong} \hat{\mathcal{C}}(M(c), G \otimes M)$ for any presheaf $G \in \hat{\mathcal{C}}$.*

Proof. For any $G \in \hat{\mathcal{C}}$, we have

$$\begin{aligned} \hat{\mathcal{C}}(M(c), G \otimes M) &\cong \hat{\mathcal{C}}(\mathbf{y}(\sigma(c)), G \otimes M) \\ &\cong G \otimes M(\sigma(c)) \\ &= \left(\sum_{c' \in \mathcal{C}} G(c') \times M(c')(\sigma(c)) \right) / \sim. \end{aligned}$$

Now, suppose that $\{M(f) : M(c'') \rightarrow M(c')\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c')), c'' \neq c'}$ is an epimorphic family for any $c' \neq c$. Then, each component of the morphism $\sum M(f) : \sum M(c'') \rightarrow M(c')$ is a surjection, where \sum is taken over all $(c'', f) \in \text{Elts}(\mathbf{y}(c'))$ such that $c'' \neq c'$. In particular, for any $a \in M(c')(\sigma(c))$ there exists $f : c'' \rightarrow c'$ and $b \in M(c'')(\sigma(c))$ such that $c'' \neq c'$ and $f \cdot b = a$. By a well-founded induction, we can show that for any $c' \neq c$ and $a \in M(c')(\sigma(c))$, there exists a morphism $f : c \rightarrow c'$ such that $f \cdot \text{id}_{\sigma(c)} = a$ because $M(c)(\sigma(c)) \cong \mathbf{y}(\sigma(c))(\sigma(c)) = \{\text{id}_{\sigma(c)}\}$. Consequently, we obtain $G(c) \cong \hat{\mathcal{C}}(M(c), G \otimes M)$. One can see that this isomorphism is given by $\eta_c^{M,G}$.

Let us show that $\{M(f) : M(c'') \rightarrow M(c')\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c')), c'' \neq c'}$ is an epimorphic family for any $c' \neq c$. In general, if a functor $F : \mathcal{I} \rightarrow \mathcal{A}$ has a colimit, then it is an epimorphic family. Thus, the cocone

$$\begin{aligned} &\{\nu_{(c', \text{id}_{c'})}^{M, c'} : I_{M, c'}(c', \text{id}_{c'}) = \text{Int}_M(c') \rightarrow M(c')\} \\ &\cup \{M(f) : I_{M, c'}(c'', f) = M(c'') \rightarrow M(c')\}_{(c'', f) \in \text{Elts}(\mathbf{y}(c'))^*, (c'', f) \neq (c', \text{id}_{c'})} \end{aligned}$$

is an epimorphic family, since M satisfies the gluing condition. This is equivalent to saying that the morphism $\nu_{(c', \text{id}_{c'})}^{M, c'} + \sum M(f) : \text{Int}_M(c') + \sum M(c'') \rightarrow M(c')$ is an epimorphism, where \sum is taken over all $(c'', f) \in \text{Elts}(\mathbf{y}(c'))^*$ such that $(c'', f) \neq (c', \text{id}_{c'})$. Since c' is not minimal, there exists a morphism $f : c'' \rightarrow c'$ such that $c'' \neq c'$ and $\nu_{(c', \text{id}_{c'})}^{M, c'} = M(f) \circ \nu_{(c'', f)}^{M, c'}$. Thus, $\sum M(f) : \sum M(c'') \rightarrow M(c')$ is an epimorphism. □

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References

- Ahn, Y.Y., Bagrow, J.P., Lehmann, S., 2010. Link communities reveal multiscale complexity in networks. *Nature* 466, 761–764.
- Albert, R., Barabási, A.L., 2002. Statistical mechanics of complex networks. *Rev. Mod. Phys.* 74, 47–97.
- Anthonisse, J.M., 1971. The rush in a directed graph, technical report bn 9/71, stichting mathematisch centrum, amsterdam .
- Arenas, A., Dáz-Guimera, A., Kurths, J., Moreno, Y., Zhou, C., 2008. Synchronization in complex networks. *Physics Reports* 469, 93–153.
- Arita, M., 2004. The metabolic world of *Escherichia coli* is not small. *Proc. Natl. Acad. Sci. USA* 101, 1543–1547.
- Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D.U., 2006. Complex networks: Structure and dynamics. *Phys. Rep.* 424, 175–308.
- Borceux, F., 1994. *Handbook of Categorical Algebra 1*. Cambridge University Press.
- Brede, M., 2011. Growth and optimality in network evolution. *Artificial Life* 17, 281–291.
- Brede, M., de Vries, B.J.M., 2009. Networks that optimize a trade-off between efficiency and dynamical resilience. *Physics Letters A* 373, 3910–3914.
- Ferrer i Cancho, R., Solé, R.V., 2003. Optimization in complex networks, in: et al., R.P.S. (Ed.), *Statistical Mechanics of Complex Networks*, No. 625 in *Lecture Notes in Physics*, Springer, Berlin. pp. 114–125.
- Clauset, A., Shalizi, C.R., Newman, M.E.J., 2009. Power-law distributions in empirical data. *SIAM Review* 51, 661–703.
- Crofts, J.J., Estrada, E., Higham, D.H., Taylor, A., 2010. Mapping directed networks. *Elec. Trans. Num. Anal.* 37, 337–350.

- Dorogovtsev, S.N., Goltsev, A.V., Mendes, J.F.F., 2008. Critical phenomena in complex networks. *Review of Modern Physics* 80, 1275–1335.
- Ehresmann, A.C., Vanbreemersch, J.P., 1987. Hierarchical evolutive systems: a mathematical model for complex systems. *Bull. Math. Biol.* 49, 13–50.
- Evans, T.S., Lambiotte, R., 2009. Line graphs, link partitions, and overlapping communities. *Phys. Rev. E* 80, 016105.
- Fortunato, S., 2010. Community detection in graphs. *Phys. Rep.* 486, 75–174.
- Freeman, L.C., 1977. A set of measures of centrality based upon betweenness. *Sociometry* 40, 35–41.
- Girvan, M., Newman, M.E.J., 2002. Community structure in social and biological networks. *Proc. Natl. Acad. Sci. USA* 99, 7821–7826.
- Haruna, T., 2008. Algebraic Theory of Biological Organization. Doctoral Dissertation, Kobe University.
- Haruna, T., 2010. An application of category theory to the study of complex networks. *International Journal of Computing Anticipatory Systems* 23, 146–157.
- Haruna, T., 2011. Global structure of directed networks emerging from a category theoretical formulation of the idea “objects as processes, interactions as interfaces”, in: et al., T.L. (Ed.), *Advances in Artificial Life, ECAL 2011, Proceedings of the Eleventh European Conference on the Synthesis and Simulation of Living Systems*, pp. 310–317.
- Haruna, T., Gunji, Y.P., 2007. Duality between decomposition and gluing: A theoretical biology via adjoint functors. *BioSystems* 90, 716–727.
- Haruna, T., Gunji, Y.P., 2009. Wholeness and information processing in biological networks: an algebraic study of network motifs, in: et al., Y.S. (Ed.), *Natural Computing, 2nd International Workshop on Natural Computing*, pp. 70–80.
- Holme, P., Saramäki, J., 2012. Temporal networks. *Physics Reports* 519, 97–125.
- Humphries, M.D., Gurney, K., 2008. Network ‘small-world-ness’: a quantitative method for determining canonical network equivalence. *PLoS ONE* 3, e0002051.
- Latora, V., Marchiori, M., 2001. Efficient behavior of small-world networks. *Phys. Rev. Lett.* 87, 198701.
- MacLane, S., 1998. *Categories for the Working Mathematician*, 2nd edition. Springer-Verlag, New York.
- Newman, M.E.J., 2002. Assortative mixing in networks. *Phys. Rev. Lett.* 89, 208701.

- Newman, M.E.J., 2003. The structure and function of complex networks. *SIAM Review* 45, 167–256.
- Newman, M.E.J., 2010. *Networks: An Introduction*. Oxford University Press Inc., New York.
- Newman, M.E.J., Strogatz, S.H., Watts, D.J., 2001. Random graphs with arbitrary degree distributions and their applications. *Phys. Rev. E* 64, 026118.
- Pultr, A., 1979. On linear representations of graphs, in: *Fundamentals of computation theory (Proc. Conf. Algebraic, Arith. And Categorical Methods in Comput. Theory, Berlin/Wendisch-Riets, 1979)*, *Math. Res.* 2, pp. 362–369.
- Ravasz, E., Barabási, A.L., 2003. Hierarchical organization in complex networks. *Phys. Rev. E* 67, 026112.
- Rosen, R., 1958a. A relational theory of biological systems. *Bull. Math. Biophys.* 20, 245–260.
- Rosen, R., 1958b. The representation of biological systems from the standpoint of the theory of categories. *Bull. Math. Biophys.* 20, 317–341.
- Schneider, C.M., Moreira, A.A., Andrade, J.J.S., Havlin, S., Herrmann, H.J., 2011. Mitigation of malicious attacks on networks. *Proc. Natl. Acad. Sci. USA* 108, 3838–3841.
- Shen-Orr, S., Milo, R., Mangan, S., Alon, U., 2002. Network motifs in the transcriptional regulation network of escherichia coli. *Nature Genetics* 31, 64–68.
- Szabó, G., Fáth, G., 2007. Evolutionary games on graphs. *Physics Reports* 446, 97–216.
- Ulanowicz, R., Bondavalli, C., Egnatovich, M., 1998. Network analysis of trophic dynamics in south florida ecosystem, fy 97: The florida bay ecosystem. Ref. No. [UMCES]CBL 98-123. Chesapeake Biological Laboratory, Solomons, MD 20688-0038 USA .
- Varshney, L.R., Chen, B.L., Paniagua, E., Hall, D.H., Chklovskii, D.B., 2011. Structural properties of the caenorhabditis elegans neuronal network. *PLoS computational biology* 7, e1001066.
- Watts, D.J., Strogatz, S.H., 1998. Collective dynamics of ‘small-world’ networks. *Nature* 393, 440–442.
- White, J.G., Southgate, E., Thomson, J.N., Brenner, S., 1986. The structure of the nervous system of the nematode caenorhabditis elegans. *Phil. Trans. R. Soc. London B* 314, 1–340.
- Wolkenhauer, O., Hofmeyr, J.H.S., 2007. An abstract cell model that describes the self-organization of cell function in living systems. *J. Theor. Biol.* 246, 461–476.