Wholeness and Information Processing in Biological Networks: An Algebraic Study of Network Motifs

Taichi Haruna⋆ and Yukio-Pegio Gunji

Department of Earth and Planetary Sciences, Faculty of Science, Kobe University
1-1, Rokkodaicho, Nada, Kobe, 657-8501, Japan
cheetha@kcc.zaq.ne.jp

Abstract. In this paper we address network motifs found in information processing biological networks. Network motifs are local structures in a whole network on one hand, they are materializations of a kind of wholeness to have biological functions on the other hand. We formalize the wholeness by the notion of sheaf. We also formalize a feature of information processing by considering an internal structure of nodes in terms of their information processing ability. We show that two network motifs called bi-fan (BF) and feed-forward loop (FFL) can be obtained by purely algebraic considerations.

Key words: network motifs, wholeness, information processing, sheaves

1 Introduction

Network motifs are local structures found in various biological networks more frequently than random graphs with the same number of nodes and degrees [5, 6]. They are considered to be units of biological functions [2]. Their significance in biological networks such as gene transcription regulations, protein-protein interactions and neural networks are widely discussed (e.g. [2] and references therein). In general, what kinds of network motifs are found depends on the nature of each biological network. However, some common motifs are found in different kinds of biological networks. In particular, motifs called feed-forward loop (FFL) and bi-fan (BF) are common in both gene transcription regulation networks and neural networks [5]. It is pointed out that both networks are information processing networks [5]. There is already an explanation by natural selection about what kinds of motifs arise [9], however, the relationship between motifs and information processing is not yet clear.

In this paper, we investigate the relationship between motifs and information processing by abstract algebra such as theories of sheaves, categories and topoi [7, 8]. It is crucial to represent motifs and information processing by suitable ways. Our formalism is based on two simple ideas. The first idea is that although motifs

⋆ Corresponding Author.

are local structures in a whole network, motifs themselves are coherent wholes to have biological functions. This fact is formalized as a condition related to sheaves, in which coherent parts are glued uniquely as a whole. The second idea is that in information processing networks each node has two roles, receiver and sender of information. Information is processed between reception and sending. Therefore nodes in information processing networks can be considered to have an internal structure in terms of information processing ability. We assume a simple internal structure and formalize it by so-called Grothendieck construction.

This paper is organized as follows. In section 2, the idea that motifs as coherent wholes are formalized by sheaves. However, we will see that no interesting consequence on the emergence of network motifs can be derived by only this idea. In section 3, we assume that each node of an information processing network has information processing ability and their hypothetical simple internal structure is presented. Integrating this idea and the idea described in section 2, we will derive network motifs FFL and BF as conditional statements. Finally in section 4, we give conclusions.

2 Motifs as coherent wholes

The basic structure of networks is just a correspondence between a set of nodes and a set of arrows. Finding motifs in a given network implies introduction of a kind of wholeness. Nodes and arrows in a motif make a coherent whole. In this section we describe this wholeness mathematically.

All networks in this paper are assumed to be directed graphs. A directed graph G consists of a quadruplet $(A, O, \partial_0, \partial_1)$. A is a set of arrows and O is a set of nodes. ∂_0, ∂_1 are maps from A to O. ∂_0 is a source map that sends each arrow to its source node. ∂_1 is a target map that sends each arrow to its target node. A network motif is given by a directed graph $M = (M_A, M_O, \partial_0^M, \partial_1^M)$. We assume that for any node $x \in M_O$ there exists an incoming arrow to x or an outgoing arrow from x (that is, there is no isolated node in M). The category of directed graph $\mathcal{G}rph$ is defined as follows. Objects are directed graphs and morphisms are homomorphisms of directed graphs.

Let G be a directed graph that represents a real network. Given a motif M, we would like to find all local structures found in G that are the same pattern as M. How they can be described mathematically? First let us consider nodes and arrows as local structures of directed graphs. The set of nodes in G can be identified with the set of homomorphisms of directed graphs from the trivial directed graph consisting of a single node without arrows $\{*\}$ to G

$$\mathrm{Hom}(\{*\}, G).$$

As the same way, the set of arrows in G can be identified with the set of homomorphisms of directed graphs from the directed graph with two distinct nodes and a single arrow between them $\{n_0 \to n_1\}$ to G

$$\mathrm{Hom}(\{n_0 \to n_1\}, G).$$

By the analogy with the above identifications, we define the set of all local structures in G that are the same pattern as M by the set of homomorphisms of directed graphs from M to G

$$\mathrm{Hom}(M, G).$$

The above three Hom's can be treated at the same time by the technique called Grothendieck construction. We describe this in the next subsection.

2.1 Grothendieck Construction

Let M be a motif. We define a finite category $\mathcal{C}_M$ as follows. We have three objects $0, 1, 2$. The set of morphisms is generated by identities, two morphisms m_0, m_1 from 0 to 1 and morphisms u_f from 1 to 2 for each $f \in M_A$ with relations $u_f m_i = u_g m_j (i, j \in \{0, 1\})$ when $\partial_i^M f = \partial_j^M g$.

$$0 \underset{m_1}{\overset{m_0}{\rightrightarrows}} 1 \overset{u_f}{\to} 2$$

We define a functor E from $\mathcal{C}_M$ to $\mathcal{G}rph$. The correspondence of objects is defined by

$$E(0) = \{*\},\ E(1) = \{n_0 \to n_1\},\ E(2) = M.$$

The correspondence of morphisms is determined by

$$E(m_0)_O(*) = n_0,\ E(m_1)_O(*) = n_1,\ E(u_f)_A(\to) = f \text{ for } f \in M_A.$$

Here we denote a homomorphism of directed graphs D by a pair of maps $D = (D_A, D_O)$, where D_A is a map between the set of morphisms and D_O is a map between the set of nodes.

The functor E defines a functor R_E from $\mathcal{G}rph$ to the category $\mathcal{S}ets^{\mathcal{C}_M^{op}}$ of presheaves on $\mathcal{C}_M$, where $\mathcal{S}ets$ is the category of sets. Given a directed graph G we define

$$R_E(G) = \mathrm{Hom}(E(-), G).$$

Grothendieck construction [8] says that a tensor product functor is defined as a left adjoint functor to R_E. Here we do not go into general theory but just give a concrete representation of the left adjoint L_E. Let F be a presheaf on $\mathcal{C}_M$. Omitting the calculation, we obtain L_E by

$$L_E(F) = F \otimes_{\mathcal{C}_M} E \cong F(1) \underset{F(m_1)}{\overset{F(m_0)}{\rightrightarrows}} F(0).$$

From this one can see that the composition $L_E R_E$ is isomorphic to the identity functor on $\mathcal{G}rph$. In general, the reverse composition $R_E L_E$ is not isomorphic to the identity functor on $\mathcal{S}ets^{\mathcal{C}_M^{op}}$. However, if we define a suitable Grothendieck topology J_M on $\mathcal{C}_M$ and consider the category of all J_M-sheaves $\mathcal{S}h(\mathcal{C}_M, J_M)$ then the composition $R_E L_E$ can become isomorphic to the identity on $\mathcal{S}h(\mathcal{C}_M, J_M)$. Thus we can obtain an equivalence of categories $\mathcal{S}h(\mathcal{C}_M, J_M) \simeq \mathcal{G}rph$. We describe the topology J_M in the next subsection.

2.2 Grothendieck Topologies

By defining a Grothendieck topology J on a small category $\mathcal{C}$, we can obtain a system of covering in $\mathcal{C}$ and consequently address relationships between parts and whole [8]. J sends each object C in $\mathcal{C}$ to a collection $J(C)$ of sieves on C. A set of morphisms S is called *sieve* on C if any $f \in S$ satisfies $\mathrm{cod}(f) = C$ and the condition $f \in S \Rightarrow fg \in S$ holds. Let S be a sieve on C and $h : D \to C$ be any morphism to C. Then $h^*(S) = \{g|\mathrm{cod}(g) = D,\ hg \in S\}$ is a sieve on D. If $R = \{f_i\}_{i\in I}$ is a family of morphisms with $\mathrm{cod}(f_i) = C$ for any $i \in I$ then $(R) = \{fg|\mathrm{dom}(f) = \mathrm{cod}(g),\ f \in R\}$ is a sieve on C.

Definition 1 *A Grothendieck topology on a small category $\mathcal{C}$ is a function that sends each object C to a collection $J(C)$ of sieves on C such that the following three conditions are satisfied.*

(i)maximality *$t_C \in J(C)$ for any maximal sieve $t_C = \{f|\mathrm{cod}(f) = C\}$.*
(ii)stability *If $S \in J(C)$ then $h^*(S) \in J(D)$ for any morphism $h : D \to C$.*
(iii)transitivity *For any $S \in J(C)$, if R is any sieve on C and $h^*(R) \in J(D)$ for all $h : D \to C \in S$ then $R \in J(C)$.*

We call a sieve S that is an element of $J(C)$ a *cover* of C.

Let M be a motif and $\mathcal{C}_M$ be the category defined by the previous subsection. We define a Grothendieck topology J_M on $\mathcal{C}_M$ by

$$J_M(0) = \{t_0\},\ J_M(1) = \{t_1\},\ J_M(2) = \{t_2, S_M = (\{u_f\}_{f\in M_A})\}.$$

Indeed, J_M satisfies the above three axioms. First maximality is obvious. Second, stability is satisfied since $v^*(t_i) = t_j$ for any arrow $v : j \to i$ and $v^*(S_M) = t_j$ for any $v : j \to 2$. Finally, for transitivity, suppose that for any sieve R on i and $v : j \to i \in t_i$, $v^*(R) \in J_M(i)$ holds for each $t_i \in J_M(i)$. By putting $v = \mathrm{id}_i$ we obtain $R \in J_M(i)$. For $S_M \in J_M(2)$, suppose that $v^*(R) \in J_M(j)$ holds for any sieve R on 2 and any $v : j \to 2 \in S_M$. By putting $v = u_f$, we obtain

$$u_f^*(R) = \{v|u_f v \in R\} \in J_M(1).$$

Hence $\{v|u_f v \in R\} = t_1$. This implies that $u_f = u_f \mathrm{id}_1 \in R$. Since this holds for any $f \in M_A$, we have $S_M = (\{u_f\}_{f\in M_A}) \subseteq R$, which means $R = S_M$ or $R = t_2$. In both cases $R \in J_M(2)$.

2.3 Sheaves

Roughly speaking, sheaves are mechanism that glue coherent parts into a unique whole [8].

Definition 2 *Let $\mathcal{C}$ be a small category and J be a Grothendieck topology on $\mathcal{C}$. Let F be a presheaf on $\mathcal{C}$ and $S \in J(C)$ be a cover of an object C. A matching* *family of F with respect to S is a function that sends each element $f : D \to C$ of S to an element $x_f \in F(D)$ such that*

$$F(g)x_f = x_{fg}$$

holds for all $g : D' \to D$. An amalgamation for such a matching family is an element $x \in F(C)$ such that

$$F(f)x = x_f$$

for all $f \in S$. A presheaf F on $\mathcal{C}$ is called sheaf with respect to J (in short, J-sheaf) if any matching family with respect to any cover $S \in J(C)$ for any object C has a unique amalgamation.

A sieve S on an object C can be identified with a subfunctor of Yoneda embedding $\mathrm{Hom}(-, C)$. Hence a matching family of a presheaf F with respect to S is a natural transformation $S \to F$. We denote the collection of matching family of F with respect to S by $\mathrm{Match}(S, F)$.

The condition of sheaf can be restated as follows. Given a Grothendieck topology J on a small category $\mathcal{C}$, a presheaf F on $\mathcal{C}$ is J-sheaf if and only if the map

$$\kappa_S : F(C) \to \mathrm{Match}(S, F) : x \mapsto F(-)x$$

is bijective for any object C and any cover $S \in J(C)$.

2.4 The Category of Directed Graphs as a Grothendieck Topos

Now we derive a condition in which a presheaf on $\mathcal{C}_M$ becomes J_M-sheaf. Yoneda's lemma says that $F(i) \cong \mathrm{Match}(t_i, F)$ holds by κ_{t_i} for any presheaf F on $\mathcal{C}_M$. Hence we just consider whether

$$F(2) \cong \mathrm{Match}(S_M, F)$$

holds by κ_{S_M} for $S_M \in J_M(2)$. We have the following proposition.

Proposition 3 $\mathrm{Match}(S_M, F) \cong \mathrm{Hom}(M, L_E(F))$.

Proof. Let a natural transformation $\mu : S_M \to F$ be given. Components of μ are

$$\begin{aligned} \mu_2 = \emptyset : S_M(2) = \emptyset \to F(2), \\ \mu_1 : S_M(1) = \{u_f | f \in M_A\} \to F(1), \\ \mu_0 : S_M(0) = \{u_f m_i | f \in M_A,\ i \in \{0, 1\}\} \to F(0). \end{aligned}$$

We define a homomorphism of directed graphs $d : M \to L_E(F)$ by

$$\begin{aligned} d_A : M_A \to F(1) : f \mapsto \mu_1(u_f), \\ d_O : M_O \to F(0) : n \mapsto \mu_0(u_f m_i) \text{ for } n = \partial_i^M f. \end{aligned}$$

d_O is a well-defined map by the definition of $\mathcal{C}_M$.

Conversely, suppose a homomorphism of directed graphs $d : M \to L_E(F)$ is given. A matching family $\mu : S_M \to F$ is defined by

$$\mu_1 : S_M(1) \to F(1) : u_f \mapsto d_A(f),$$
$$\mu_0 : S_M(0) \to F(0) : u_f m_i \mapsto d_O(\partial_i^M f).$$

It is clear that these constructions are the inverse of each other. □

By the proposition, a necessary and sufficient condition that a presheaf F on $\mathcal{C}_M$ is a J_M-sheaf is that the map

$$\tau : F(2) \to \mathrm{Hom}(M, L_E(F)) : \alpha \mapsto d^\alpha$$

is a bijection. d^α is a homomorphism of directed graphs defined by

$$d_A^\alpha : M_A \to F(1) : f \mapsto F(u_f)\alpha,$$
$$d_O^\alpha : M_O \to F(0) : n \mapsto F(u_f m_i)\alpha \text{ for } n = \partial_i^M f.$$

In other words, a presheaf F on $\mathcal{C}_M$ is J_M-sheaf if and only if

$$R_E L_E(F) \cong F$$

holds. Since $L_E R_E$ is isomorphic to the identity functor on $\mathcal{G}rph$, $R_E(G)$ is always J_M-sheaf for any directed graph G. If we denote the category of J_M-sheaves on $\mathcal{C}_M$ by $\mathcal{S}h(\mathcal{C}_M, J_M)$ then we obtain an equivalence of categories

$$\mathcal{S}h(\mathcal{C}_M, J_M) \simeq \mathcal{G}rph.$$

2.5 Sheafification

Given a presheaf F on $\mathcal{C}_M$, what is the best sheaf which "approximates" the presheaf F? The technique which answers this question is called *sheafification* [8]. In this subsection we calculate the sheafification of presheaves on $\mathcal{C}_M$ by a procedure so-called Grothendieck's '+'-construction.

Let F be a presheaf on a small category $\mathcal{C}$ and J a Grothendieck topology on $\mathcal{C}$. A new presheaf F^+ is defined by

$$F^+(C) = \mathrm{colim}_{S \in J(C)} \mathrm{Match}(S, F).$$

The colimit is taken by the reverse inclusion order defined on $J(C)$. This colimit can be described as follows. Elements of the set $F^+(C)$ are equivalence classes of matching families $\mu \in \mathrm{Match}(S, F)$. Two matching families $\mu \in \mathrm{Match}(S, F)$ and $\nu \in \mathrm{Match}(T, F)$ are equivalent if and only if there exists a covering sieve $R \in J(C)$ such that $R \subseteq S \cap T$ such that $\mu|_R = \nu|_R$.

In general, F^+ is not a J-sheaf but it is known that $(F^+)^+$ is a J-sheaf. However, we shall prove that F^+ is already a J_M-sheaf for a presheaf F on $\mathcal{C}_M$ in what follows.

By Yoneda's lemma, we have

$$F^+(i) = \text{colim}_{S \in J_M(i)} \text{Match}(S, F) \cong \text{Match}(t_i, F) \cong F(i)$$

for $i = 0, 1$. For $F^+(2)$, since $\mu|_{S_M} \in \text{Match}(S_M, F)$ for any $\mu \in \text{Match}(S_M, F)$, μ is equivalent to $\mu|_{S_M}$. Besides, because two different elements in $\text{Match}(S_M, F)$ belong to different equivalence classes,

$$F^+(2) = \text{colim}_{S \in J_M(2)} \text{Match}(S, F) \cong \text{Match}(S_M, F) \cong \text{Hom}(M, L_E(F)).$$

This implies that $F^+ \cong R_E(L_E(F))$ which means F^+ is a J_M-sheaf. Since sheafification of a presheaf is unique up to isomorphisms, we can calculate a sheafification of presheaves on $\mathcal{C}_M$ with respect to the topology J_M by applying $R_E L_E$ to them.

3 Information Processing Networks

Let us recall the points in the previous section. Network motifs are coherent wholes. By defining a suitable category and a topology on it, we can address the relationships between parts and whole by sheaves.

In section 2, an object in $\mathcal{G}rph$ is considered to represent a real network. On the other hand, an object in $\mathcal{S}ets^{\mathcal{C}_M^{op}}$ is constructed artificially in relation to finding a motif from the outside of the network. The construction would describe the wholeness of motifs in a mathematically favorable way as an equivalence of categories, however, it does not provide any suggestion what kinds of motifs arise in real networks.

In this section we focus on information processing biological networks such as gene transcription regulation networks or neural networks. We extract a common property of information processing networks in terms of information processing ability and integrate the property into the setting in section 2.

3.1 Information Processing Pattern

In information processing networks, each node in a network can be both a receiver and a sender of information. It processes information between reception and sending. Hence it should be considered to have an internal structure in terms of its information processing ability. One of the simplest candidates for the internal structure is a directed graph consisting of two different nodes and a single arrow between them. The arrow corresponds to information processing, the source of the arrow corresponds to reception of information and the target of the arrow corresponds to sending of information. Suppose two nodes in an information processing network are connected by an arrow. How can we describe this situation with the proposed internal structure of nodes? If we identify the sending of information at the source node with the reception of information at the target node then we could describe the situation by simply identifying the

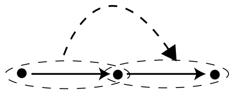

Fig. 1. Broken ellipses denote two nodes at the network level. They have an internal structure that represents information processing ability. A broken curved arrow denotes an arrow connecting them in the network.

target of the arrow corresponding to the source node with the source of the arrow corresponding to the target node. The situation is depicted in Fig. 1.

Now we integrate the above idea into Grothendieck construction in section 2. We make use of the fact that the category of directed graphs is isomorphic to a presheaf category defined by the following diagram.

$$\bullet \underset{t}{\overset{s}{\rightrightarrows}} \bullet$$

We define a pattern M by a directed graph

$$\bullet \overset{e_0}{\to} \bullet \overset{e_1}{\to} \bullet.$$

This pattern is not a motif in the sense in section 2 but is defined in terms of the internal information processing ability of nodes. It represents a specific information processing pattern associated with an arrow in a network. We call the pattern M *information processing pattern*. The motifs in section 2 are defined by an external observer who describes the local structure of networks. On the other hand, the information processing pattern M is defined in terms of an internal perspective on information processing and is relevant to how the specific local structures of information processing networks (BF and FFL) appear as we explain bellow.

Let $\mathcal{C}_M^*$ be a finite category with two objects $1, 2$. We have just two morphisms corresponding to e_0, e_1 from 1 to 2 other than identities. The two morphisms are also denoted by e_0, e_1 since there would be no confusion. $\mathcal{C}_M^*$ is a subcategory of $\mathcal{C}_M$. We denote the restriction of the functor $E : \mathcal{C}_M \to \mathcal{G}rph$ to $\mathcal{C}_M^*$ by the same symbol E. Note that a presheaf on $\mathcal{C}_M^*$ can be seen as a directed graph $F = (F(2), F(1), F(e_0), F(e_1))$. A functor R_E from $\mathcal{G}rph$ to $\mathcal{S}ets^{\mathcal{C}_M^{*\,op}} \cong \mathcal{G}rph$ can be defined by the same way as in section 2. By Grothendieck construction, R_E has a left adjoint L_E. We again just give a concrete description of the left adjoint omitting the calculation.

Let F be a presheaf on $\mathcal{C}_M^*$. We have

$$L_E(F) = F \otimes_{\mathcal{C}_M^*} E \cong F(1) \underset{\partial_1^F}{\overset{\partial_0^F}{\rightrightarrows}} F(1) \times \{0,1\}/\sim,$$

where $\sim$ is an equivalence relation on $F(1) \times \{0,1\}$ generated by the following relation R on $F(1) \times \{0,1\}$. For $(a,1),(b,0) \in F(1) \times \{0,1\}$

$$(a,1)R(b,0) \Leftrightarrow \exists \alpha \in F(2) \text{ s.t. } a = F(e_0)\alpha, b = F(e_1)\alpha,$$

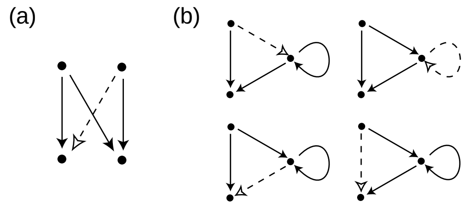

Fig. 2. If real arrows exist then dotted arrows must exist. (a)bi-fan (BF). (b)feed-forward loop (FFL) with a loop.

that is, $(a,1)R(b,0)$ if and only if there is an arrow from a to b. We define $\partial_i^F(a) = [(a,i)]$ $(i = 0,1)$ for $a \in F(1)$, where $[(a,i)]$ is an equivalence class that includes (a,i). The adjunction obtained here is the same one derived heuristically in [3].

3.2 A Derivation of Network Motifs

The wholeness of network motifs is represented by sheaves in section 2. However, it is not useful to consider sheaves in the setting in this section since the category $\mathcal{C}_M^*$ loses information how arrows are connected in M. Instead, we adopt the condition $R_E L_E(F) \cong F$ for a representation of the wholeness. This is equivalent to the condition of sheaf in the setting of section 2. Recall that a presheaf F on $\mathcal{C}_M^*$ can be seen as a directed graph $F = (F(2), F(1), F(e_0), F(e_1))$. We now consider that presheaves on $\mathcal{C}_M^*$ correspond to real networks. Objects in $\mathcal{G}rph$ are supposed to have only auxiliary roles. Roles of the presheaf category and $\mathcal{G}rph$ are reversed from those in section 2.

A necessary and sufficient condition that a binary directed graph F satisfies $R_E L_E(F) \cong F$ is already obtained in [3]. If we write $a \to b$ when there exists an arrow from a to b in F then the condition can be stated as follows.

$$\text{If } a \to b \leftarrow c \to d \text{ then } a \to d.$$

The necessary part is explained in the next paragraph. This implies that if three arrows in F make a sub-pattern of bi-fan (BF) then they are indeed included in a BF (Fig. 2 (a)). If one of four arrows in a BF is a loop then the BF becomes a feed-forward loop (FFL) with a loop (Fig. 2 (b)). Such type of FFL with a loop at the relay point is often observed in real biological networks [1]. Thus we can derive both BF and FFL as conditional statements from algebraic descriptions of wholeness and information processing.

We can interpret the emergence of bi-fan as the stabilization of information processing pattern M. Let F be a directed graph. For nodes $x, y \in F$, $(x,1)R(y,0)$ means that there exists an arrow from x to y, $x \to y$. Suppose $a \to b \leftarrow c \to d$ in F. This implies that

$$(a,1)R(b,0), \ (c,1)R(b,0) \text{ and } (c,1)R(d,0).$$

By the construction of the equivalence relation from R,

$$(a,1)R(b,0) = (b,0)R^{-1}(c,1) = (c,1)R(d,0)$$

implies $(a,1)R(d,0)$, which means $a \to d$. We here use the reflexive law twice, the symmetric law once and the transitive law twice. The reflexive law guarantees the identity of symbol (x,i). (x,i) represents a role (e_0 or e_1) in information processing pattern M. The symmetric law here could be seen as a kind of feedback if we interpret an arrow in a network as a transduction of information, since the symmetric law reverses the relation $(c,1)R(b,0)$ which means $c \to b$ in the network. Finally, the transitive law provides the compositions of relations R and R^{-1}, which are interpreted as propagation of information transduction and feedback. Thus by the construction of the equivalence relation from R, roles $(a,1),(b,0),(c,1)$ and $(d,0)$ in M are integrated as a whole and stabilized. Hence we would like to say that information processing pattern M is stable in F if $R_E L_E(F) \cong F$ holds.

3.3 Another Derivation of the Fixed Point Condition

The condition $R_E L_E(F) \cong F$ says that F is a fixed point of $R_E L_E$ up to an isomorphism of directed graphs. We have just obtained the fixed point condition in relation to the sheaf condition, however, we can derive the fixed condition independent of the sheaf condition. In this subsection we outline the derivation briefly. The details will be presented elsewhere.

Recall that the information processing pattern M represents how two nodes are connected by an arrow at the network level. Hence each connection between two nodes by an arrow at the network level can be seen as an image of M by R_E. This condition can be generalized to any directed graph F:

$$F \cong R_E(G) \text{ for some directed graph } G.$$

We can prove that this condition is equivalent to the fixed point condition $R_E L_E(F) \cong F$. Note that '$\cong$' in '$F \cong R_E(G)$' refers to a directed graph isomorphism in general, however, '$\cong$' in '$R_E L_E(F) \cong F$' stands for that a specific directed graph homomorphism which is a component of the unit of the adjunction $\eta_F : F \to R_E L_E(F)$ is an isomorphism.

The proof proceeds roughly as follows. Suppose $R_E L_E(F) \cong F$ holds. Then we obtain $F \cong R_E(G)$ by putting $G = L_E(F)$. Conversely, suppose $F \cong R_E(G)$ for some directed graph G. If $R_E \cong R_E L_E R_E$ holds then we have $F \cong R_E(G) \cong R_E L_E R_E(G) = R_E L_E(R_E(G)) \cong R_E L_E(F)$. Hence it is sufficient to prove $R_E \cong R_E L_E R_E$. However, one can show that $\eta_{R_E(G)} R_E(\epsilon_G) = \mathrm{id}_{R_E L_E R_E(G)}$ for any directed graph G where η and ϵ are the unit and the counit of the adjunction, respectively. We also have $R_E(\epsilon_G)\eta_{R_E(G)} = \mathrm{id}_{R_E(G)}$, which is just one of the two triangular identities for the adjunction. To be precise, we also need $L_E \cong L_E R_E L_E$ by the natural transformations appeared in the other triangular identity, which can be also checked in our adjunction, for the complete proof.

4 Conclusions

In this paper we derive network motifs found in information processing biological networks from purely algebraic considerations on wholeness and information processing. We first consider the wholeness of network motifs as sheaves on a finite Grothendieck site. This is an external description of the wholeness of network motifs that is not useful to consider the emergence network motifs. Hence we need a kind of internal perspective. We introduce an information processing pattern defined in relation to internal information processing ability of nodes. We show that the wholeness of the information processing pattern is materialized as network motifs such as BF and FFL.

We can generalize the idea of information processing pattern described in this paper. The generalization is presented in [4]. Another example of information processing pattern which seems to be relevant to real networks will be found in [4]. Note that information processing patterns defined in this paper are called *intrinsic motifs* in [4].

The notion of natural computation would be closely related to the idea of information processing pattern introduced in this paper. In this respect, an information processing pattern might be seen as a formal representation of a computation performed by nature *per se*. We believe that algebraic methods including category and topos theory are useful to grasp the formal aspects of natural computation.

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