Duality between Decomposition and Gluing: A Theoretical Biology via Adjoint Functors

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Abstract

Two ideas in theoretical biology, 'decomposition into functions' and 'gluing functions', are formalized as endofunctors on the category of directed graphs. We prove that they constitute an adjunction. The invariant structures of the adjunction are obtained. They imply two biologically significant conditions: the existence of cycles in finite graphs and anticipatory diagrams.

keywords: line-graph, glue, adjunction, cycle, anticipation

1 Introduction

The use of category theory (MacLane, 1998) in theoretical biology dates from Robert Rosen's pioneering works in the late 1950s (Rosen, 1958a, 1958b, 1959). Describing biological systems using category theory, he analyzes their properties in terms of optimality principles (Rosen, 1959), sequential machines (Rosen, 1964a, 1964b, 1966), category theory itself and so on. Rosen (1972) gives a summary. We believe that his use of category theory is very effective. However, those who are familiar with category theory may question the fact that there is no direct use of adjoint functors in his work. Although adjoint functors are central to category theory, they do not appear explicitly in Rosen's works. Baianu et al. (2006) consider an adjunction related to the category of metabolism-repair systems. However, it appears to be an addition to Rosen's work. Louie (1985) refers to Galois theory, which is a special case of adjunctions, in relation to the categorical analysis of dynamical system theory. The Galois connection is used in wider context than that considered by Rosen.

One of the motivations of this paper is to provide an intrinsic link between Rosen's ideas and adjoint functors. However, we do not deal directly with Rosen's works. In particular, we do not consider its computational aspects (Louie, 2005). Instead, we consider the idea behind Rosen's work and formalize it in the category of directed graphs. We will show that Rosen's idea is one half of an adjunction. After constructing the adjunction, we find the invariant structures of the adjunction which turn out to have significant consequences for theoretical biology. They imply the existence of a cycle and conditions for anticipation.

Formally a biological system is considered to have a circular organization (Letelier et al., 2006). If a biological system is represented by a finite directed graph then the existence of a cycle indicates it has a circular organization. On the other hand, the existence of cycles in finite directed graphs is a weaker result than the condition for closure to efficient cause in metabolism-repair systems (Rosen, 1991). However, we can provide a more general framework in which an alternative logical route to these topics can be introduced by focusing on the adjunction that is established in this paper.

Anticipation is another important issue in theoretical biology because it seems that anticipation is intrinsic to biological systems in relation to learning, adaptation, evolution and so on (Rosen, 1985). This subject has a broad scope; however, we restrict ourselves here to treating only its formal aspects.

Natural systems are usually expressed as dynamical systems that contain the temporal dimension explicitly. At first glance category theory seems to be incompatible with the temporal dimension. For example, a composite arrow in a category must exist before the composition. If one attempts to include the temporal dimension in a category, one has to consider the dynamical change of the category (Ehresmann and Vanbremeersch, 1987). Such an approach regards a category as the structural pattern of a concrete system. However, this is not the only way to view a category. One can view a category as an analytical tool for investigating the common properties of certain objects. Here we take this latter point of view. In particular, an adjunction that is independent of the temporal dimension is the primary tool in the following discussion. Analysis in terms of an adjunction can be applicable to any temporally changing object as long as the object belongs to the category on which the adjunction holds.

The organization of this paper is as follows. In section two we review two ideas in theoretical biology: 'decomposition into functions' and 'gluing functions'. In sections three and four we formalize these ideas as functors on the category of directed graphs. In section five adjunctions are derived from these functors and their invariant structures are obtained. In section six we discuss the invariant structures in relation to the existence of a cycle and anticipation. In section seven we give a summary and outlook. In appendix A we provide a slightly different formalization of 'decomposition into functions' and 'gluing functions'.

2 Ideas in Theoretical Biology

In the framework of Rosen's theoretical biology, an object in a system is defined by its functions. The units of a system are the functions of its constituent objects. This idea of 'decomposition into functions' is the idea behind constructing the abstract block diagram of a system (Rosen, 1958b, 1959). Abstract block diagrams contain other structures (e.g. products) but here we only consider the idea of 'decomposition into functions' which we believe is central. A simple example of the idea is shown in Figure 1. On the left hand side of the picture, M represents a black box (an enzyme, a machine, etc.) that transforms input x into two outputs y and z. On the right hand side of the picture M is decomposed into two functions. The one labeled m_1 transforms x into y and the other, labeled m_2, transforms x into z. Note that the second graph is dual to the first: x, y and z are directed edges of a graph before 'decomposition into functions' while they are nodes of a graph after the transformation.

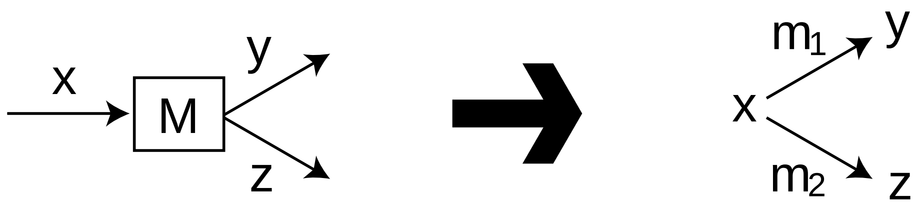

Figure 1: The idea in Rosen's abstract block diagrams. The functions of an object M appear as directed edges after 'decomposition into functions '.

The other idea we consider in this paper is the operation which is the inverse of 'decomposition into functions': that is, the construction of an object by gluing its functions. This idea is found in Paton (2002). Paton represents a system by a pair of undirected graphs called a star graph and a tetrahedron graph (Figure 2). The star graph is the extent part of the pair. In the ecosystem example in Figure 2, the nodes of the star graph are the names of concrete agents in the real world, like plants, animals, bacteria and so on. Their roles label the edges. Note that objects belonging to different levels (e.g. ecosystem and the others) are mixed up in the set of nodes of the star graph. On the other hand, the tetrahedron graph which is the line-graph of the star graph is the intent part. A line-graph of a graph is obtained by making old edges into new nodes and linking two new nodes if they are tied by an old node. The nodes are now named after the verbs on the edges of the star graph, that is, the functional roles of agents in the ecosystem.

We have so far used the term 'function' in a loose way. However, we shall use the term in a formal way hereafter: the function of a node in a graph is connecting a pair of edges. Thus, in our terminology, the functions of nodes in the star graph become edges in the tetrahedron graph. This is the same idea as 'decomposition into functions' in abstract block diagrams. In addition to 'decomposition into functions', Paton looks at the operation in the other direction: the transformation of the tetrahedron graph into the star graph. Under this operation the ecosystem implicit in the tetrahedron graph is made explicit by gluing its distributed functions. We can find the idea of 'gluing functions' as the inverse operation to 'decomposition into functions' in Paton's work. In the following two sections we formalize these two ideas in the category of directed graphs. While Paton's star graph and tetrahedron graph are undirected graphs, we will work

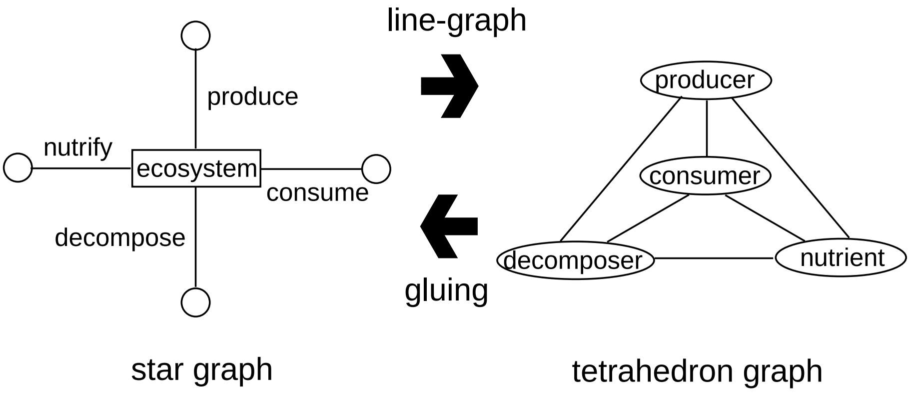

Figure 2: Paton's star graph and tetrahedron graph. He describes a concept like an ecosystem by a pair of undirected graphs. The star graph is converted to the tetrahedron graph by making a line-graph. The star graph is produced from the tetrahedron graph by gluing functions.

in the category of directed graphs for simplicity. The formalization in the category of undirected graphs might become easier after we develop the theory in the category of directed graphs. However, we do not treat that topic in this paper.

3 Decomposition into Functions

We work in the category of directed graphs $\mathcal{G}rph$ in order to formalize the two ideas reviewed in the previous section. The objects in $\mathcal{G}rph$ are directed graphs. (An example of a directed graph is given in Figure 3.) A directed graph G consists of a quadruplet $G = (A, O, \partial_0, \partial_1)$ where A is a set of directed edges (or arrows), O is a set of nodes (or objects) and ∂_i $(i = 0, 1)$ are maps from A to O. ∂_0 is a source map that sends each directed edge to its source. ∂_1 is a target map that sends each directed edge to its target. The arrows in $\mathcal{G}rph$ are the homomorphisms of directed graphs. Given directed graphs $G = (A, O, \partial_0, \partial_1)$ and $G' = (A', O', \partial'_0, \partial'_1)$, a homomorphism of directed graphs $D : G \to G'$ consists of two maps $D_O : O \to O'$ and $D_A : A \to A'$ that satisfy the equations $D_O \partial_i = \partial'_i D_A$ $(i = 0, 1)$. As usual, these equations can be represented by the

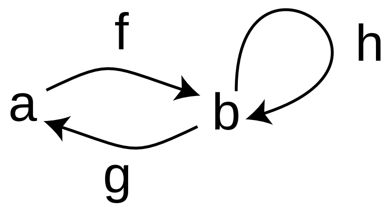

Figure 3: An example of a directed graph $G = (A, O, \partial_0, \partial_1)$. The set of directed edges is $A = \{f, g, h\}$. The set of nodes is $O = \{a, b\}$. The source and target maps are defined by $\partial_0 f = a, \partial_1 f = b, \partial_0 g = b, \partial_1 g = a, \partial_0 h = b$ and $\partial_1 h = b$.

following commutative diagram ($i = 0, 1$).

$$\begin{array}{ccc} A & \xrightarrow{D_A} & A' \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial'_i} \\ O & \xrightarrow{D_O} & O' \end{array}$$

Here we merely consider the functions of a node to be the connection of directed edges. Then the result of the operation of 'decomposition into function' is the so called directed line-graph of a directed graph. This can be seen as a functor from $\mathcal{G}rph$ to itself.

Definition 3.1 *Let $\mathcal{R}$ be an operation that transforms given directed graph $G = (A, O, \partial_0, \partial_1)$ into a new directed graph $\mathcal{R}G = (\mathcal{R}A, \mathcal{R}O, \partial_0^{\mathcal{R}}, \partial_1^{\mathcal{R}})$ by taking its line-graph, where*

$$\begin{aligned} \mathcal{R}A &= \{(f, g) \in A \times A | \partial_1 f = \partial_0 g\} \\ \mathcal{R}O &= A \\ \partial_0^{\mathcal{R}}(f, g) &= f \textit{ and } \quad \partial_1^{\mathcal{R}}(f, g) = g \textit{ for } (f, g) \in \mathcal{R}A. \end{aligned}$$

It is straightforward to verify that $\mathcal{R}$ is an endofunctor (i.e. a functor from $\mathcal{G}rph$ to itself).

The functions of a node in G that connect directed edges become multiple directed edges of $\mathcal{R}G$. For example, node x in Figure 4 connects f to h, f to i, g to h and g to i. These functions become four directed edges in $\mathcal{R}G$.

As described above, 'decomposition into functions' can be formalized as a functor on $\mathcal{G}rph$. In the next section we also formalize 'gluing functions' as a functor on $\mathcal{G}rph$.

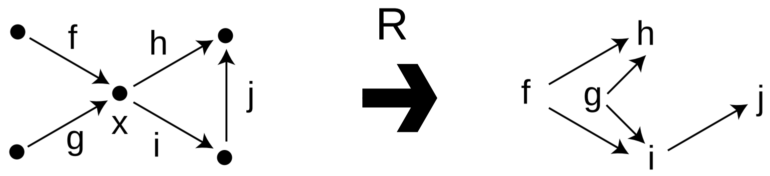

Figure 4: The directed edges f, g, h, i and j become nodes under the operation of $\mathcal{R}$. While the node x is decomposed into four directed edges.

4 Gluing Functions

A functor that represents 'gluing functions' would be a kind of inverse operation to $\mathcal{R}$. Under the operation of $\mathcal{R}$ directed edges become nodes. Therefore, a new node created by the operation of 'gluing functions' would be obtained by gluing the distributed functions on directed edges.

Motivated by the above consideration, we formalize the inverse operation to $\mathcal{R}$ as follows. Given a directed graph $G = (A, O, \partial_0, \partial_1)$, an operation $\mathcal{L}''$ constructs a new directed graph $\mathcal{L}''G = (\mathcal{L}''A, \mathcal{L}''O, \partial_0^{\mathcal{L}''}, \partial_1^{\mathcal{L}''})$ as follows.

$$\begin{aligned}
\mathcal{L}''A &= O \\
\mathcal{L}''O &= T/\sim'' \\
T &= \{(x, y) \in O \times O | \exists f \in A \ \partial_0 f = x, \partial_1 f = y\}
\end{aligned}$$

Here $\sim''$ is an equivalence relation generated by the relation R'' on T defined by

$$(x, y)R''(z, w) \Leftrightarrow x = z \text{ or } y = w.$$

The motivation for the definition of R'' is explained schematically in Figure 5. We might expect the source and target maps for $x \in \mathcal{L}''A = O$ to be defined by

$$\partial_0^{\mathcal{L}''} x = [(\partial_0 f, \partial_1 f)]_{\sim''} \quad \text{and} \quad \partial_1^{\mathcal{L}''} x = [(\partial_0 g, \partial_1 g)]_{\sim''}$$

where $\partial_1 f = x, \partial_0 g = x, \ f, g \in A$ and $[\alpha]_{\sim''}$ is the equivalence class that includes α. However, there does not necessarily exist an $f \in A$ such that $\partial_1 f = x$ (or a $g \in A$ such that $\partial_0 g = x$) for all $x \in O$. The problem is that we cannot define a source map for x with 0 in-degree and cannot define a target map for x with 0 out-degree.

There are at least two possible strategies for coping with the problem:

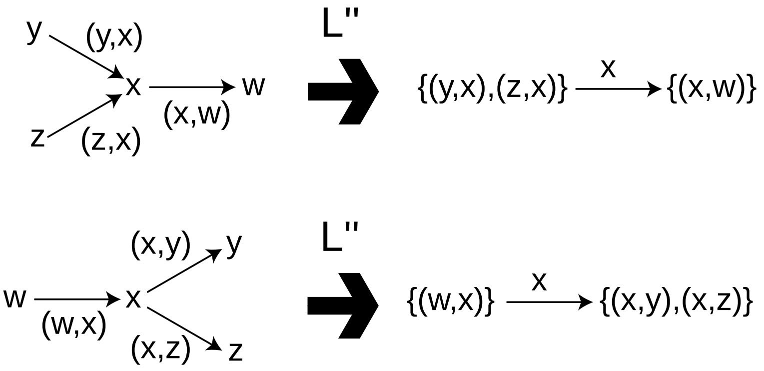

Figure 5: Because (y, x) and (z, x) have the same target x, they must be equivalent when x becomes a directed edge (above). Because (x, y) and (x, z) have the same source x, they must be equivalent when x becomes a directed edge (below).

(I) Modifying $\mathcal{L}''O$ while keeping the category $\mathcal{G}rph$.

(II) Restricting the category in which we work while maintaining the construction of $\mathcal{L}''O$.

In the first strategy we have to modify $\mathcal{L}''O$ so that source and target maps work for x with 0 in-degree or 0 out-degree. As we will see below, the idea of 'gluing functions' becomes implicit in the first strategy. On the other hand the idea of 'gluing functions' remains explicit in the second strategy. In this strategy we find the largest subcategory in which $\mathcal{L}''$ becomes a functor. Because we are interested in the duality between 'decomposition into functions' and 'gluing functions', our emphasis is on the second strategy. Nevertheless, it is also convenient to work following the first strategy. For this reason, we begin by formalizing the first strategy.

Our problem is to modify the definition of $\mathcal{L}''O$ so that source and target maps can be defined on all x including 0 in-degree and 0 out-degree nodes. A solution can be obtained by extending the set T which appears in the definition of $\mathcal{L}''O$. T is a set of functions which connect two directed edges. The definition of T mentioned above ignores situations in which there are no incoming directed edges to a node or no outgoing directed edges from a node. Hence we add new elements to T to represent the source or target of such edges.

Definition 4.1 *We construct a new directed graph* $\mathcal{L}'G = (\mathcal{L}'A, \mathcal{L}'O, \partial_0^{\mathcal{L}'}, \partial_1^{\mathcal{L}'})$ *from a*

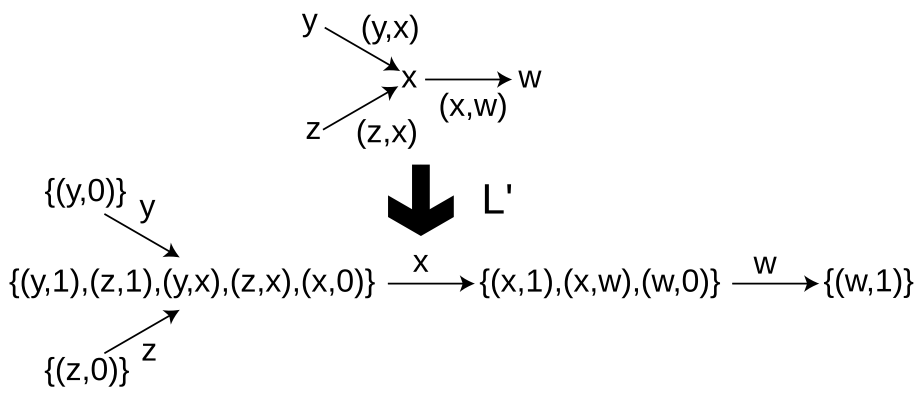

Figure 6: An example of the operation of $\mathcal{L}'$. We can define the source and target maps for any new directed edges (unlike $\mathcal{L}''$).

given directed graph $G = (A, O, \partial_0, \partial_1)$ as follows.

$$\begin{aligned} \mathcal{L}'A &= O \\ \mathcal{L}'O &= S/\sim' \\ S &= T \cup (O \times 2) \\ T &= \{(x,y) \in O \times O | \exists f \in A\ \partial_0 f = x, \partial_1 f = y\} \end{aligned}$$

Here $\sim'$ is an equivalence relation on S generated by the following relation R' on S.

$$(x,y)R'(z,w) \Leftrightarrow x = z \text{ or } y = w, (x,y)R'(z,0) \Leftrightarrow y = z, (x,y)R'(z,1) \Leftrightarrow x = z$$

The source and target maps are

$$\partial_0^{\mathcal{L}'} x = [(x,0)]_{\sim'} \quad \text{and } \partial_1^{\mathcal{L}'} x = [(x,1)]_{\sim'}.$$

It can be verified that $\mathcal{L}'$ is a functor from $\mathcal{G}rph$ to itself.

We can define a functor $\mathcal{L}$ that is naturally isomorphic to $\mathcal{L}'$ without T (see Figures 6 and 7). The objects (y,x) and (z,x) in the source of x and (x,w) in the target of x in Figure 6 are redundant. We can obtain the same graph without them (see Figure 7).

Definition 4.2 *A functor $\mathcal{L}$ from $\mathcal{G}rph$ to itself that sends a directed graph $G = (A, O, \partial_0, \partial_1)$ to a directed graph $\mathcal{L}G = (\mathcal{L}A, \mathcal{L}O, \partial_0^{\mathcal{L}}, \partial_1^{\mathcal{L}})$ is defined as follows.*

$$\begin{aligned} \mathcal{L}A &= O \\ \mathcal{L}O &= (O \times 2)/\sim \end{aligned}$$

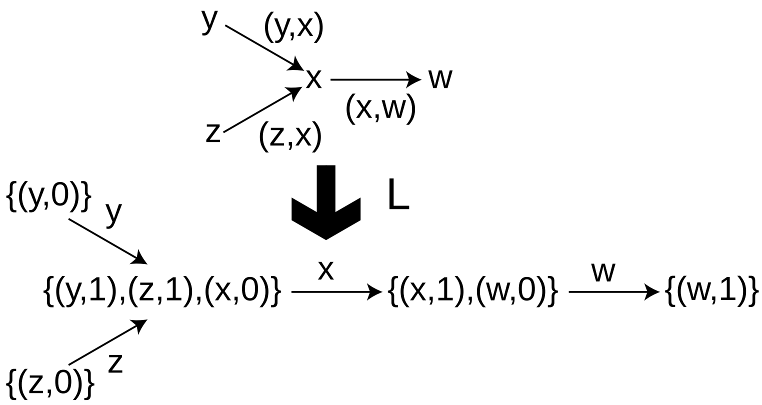

Figure 7: $\mathcal{L}'$ can be simplified.

Here $\sim$ is an equivalence relation on $O \times 2$ generated by the relation R on $O \times 2$.

$$(x, 1)R(y, 0) \Leftrightarrow \exists f \in A \ \partial_0 f = x, \partial_1 f = y.$$

The source and target maps are the same as those of $\mathcal{L}'$.

$$\partial_0^{\mathcal{L}} x = [(x, 0)]_\sim \quad \textit{and } \partial_1^{\mathcal{L}} x = [(x, 1)]_\sim$$

Proposition 4.3 *The two functors $\mathcal{L}$ and $\mathcal{L}'$ are naturally isomorphic.*

Proof. A natural isomorphism $\psi : \mathcal{L} \to \mathcal{L}'$ is defined as follows. For each directed graph $G = (A, O, \partial_0, \partial_1)$, $\psi_G : \mathcal{L}G \to \mathcal{L}'G$ consists of two maps. The arrow part is defined by $(\psi_G)_A := id_O : (\mathcal{L}G)_A = O \to (\mathcal{L}'G)_A = O$. The object part $(\psi_G)_O : (\mathcal{L}G)_O \to (\mathcal{L}'G)_O$ is defined by sending $[(x, i)]_\sim$ to $[(x, i)]_{\sim'}$ for $i = 0, 1$. $(\psi_G)_O$ is a well-defined map because $(x, 1)R(y, 0) \Leftrightarrow (x, 1)R'^{-1}(x, y)R'(y, 0)$. Bijectivity and naturality can easily be checked. $\square$

5 The Adjunctions and its Invariant Structures

In the previous two sections, we defined two functors $\mathcal{R}$ and $\mathcal{L}$. $\mathcal{L}$ is constructed as the inverse operation of $\mathcal{R}$ in some sense. In this section we reveal the precise mathematical meaning of ‘inverse’. In fact they are not inverses of each other in the precise meaning of the word but they form an adjunction on $\mathcal{G}rph$. In particular, $\mathcal{L}$ is a left adjoint functor to $\mathcal{R}$.

Theorem 5.1 *$\mathcal{L}$ is a left adjoint to $\mathcal{R}$. That is, we have a natural isomorphism*

$$\mathcal{G}rph(\mathcal{L}G, G') \simeq \mathcal{G}rph(G, \mathcal{R}G')$$

for any pair of directed graphs G, G'.

Proof. First we construct a map $\varphi_{G,G'} : \mathcal{G}rph(\mathcal{L}G, G') \to \mathcal{G}rph(G, \mathcal{R}G')$ where $G = (A, O, \partial_0, \partial_1)$ and $G' = (A', O', \partial'_0, \partial'_1)$. Given a directed graph homomorphism $D : \mathcal{L}G \to G'$, we have two maps $D_O : \mathcal{L}O \to O'$ and $D_A : \mathcal{L}A = O \to A'$. We define a directed graph homomorphism $\varphi_{G,G'}(D) : G \to \mathcal{R}G'$ from these maps. For the object part we define

$$\varphi_{G,G'}(D)_O := D_A : O \to \mathcal{R}O' = A'.$$

For the arrow part

$$\varphi_{G,G'}(D)_A : A \to \mathcal{R}A' = \{(f,g) \in A' \times A' | \partial'_1 f = \partial'_0 g\}$$

is defined as a map that sends each $f \in A$ to $(D_A\partial_0 f, D_A\partial_1 f)$. In order to verify $(D_A\partial_0 f, D_A\partial_1 f) \in \mathcal{R}A'$, we have to show that $\partial'_1 D_A\partial_0 f = \partial'_0 D_A\partial_1 f$. This result is obtained by the following calculation.

$$\begin{aligned}
\partial'_1 D_A \partial_0 f &= D_O \partial^{\mathcal{L}}_1 \partial_0 f \\
&= D_O[(\partial_0 f, 1)]_\sim = D_O[(\partial_1 f, 0)]_\sim \\
&= D_O \partial^{\mathcal{L}}_0 \partial_1 f = \partial'_0 D_A \partial_1 f.
\end{aligned}$$

Next we define the inverse map of $\varphi_{G,G'}$, that is, $\varphi^{-1}_{G,G'} : \mathcal{G}rph(G, \mathcal{R}G') \to \mathcal{G}rph(\mathcal{L}G, G')$. Let $\hat{D} : G \to \mathcal{R}G'$ be a directed graph homomorphism. We need to construct $\varphi^{-1}_{G,G'}(\hat{D}) : \mathcal{L}G \to G'$ from $\hat{D}_O : O \to \mathcal{R}O = A'$ and $\hat{D}_A : A \to \mathcal{R}A'$. The arrow part is defined by

$$\varphi^{-1}_{G,G'}(\hat{D})_A := \hat{D}_O : \mathcal{L}A = O \to A'.$$

The object part

$$\varphi^{-1}_{G,G'}(\hat{D})_O : \mathcal{L}O \to O'$$

is defined as a map that sends $[(x,0)]_\sim$ to $\partial_1'\hat{D}_O x$ and $[(y,1)]_\sim$ to $\partial_0'\hat{D}_O y$. The well-definedness of this map can be verified as follows. It is sufficient to show that if $(x,1)R(y,0)$ then $\partial_1'\hat{D}_O x = \partial_0'\hat{D}_O y$. If $(x,1)R(y,0)$ holds then there exists $f \in A$ such that $\partial_0 f = x, \partial_1 f = y$. Therefore, we have

$$\begin{aligned}
\partial_1'\hat{D}_O x &= \partial_1'\hat{D}_O\partial_0 f \\
&= \partial_1'\partial_0^{\mathcal{R}}\hat{D}_A f = \partial_0'\partial_1^{\mathcal{R}}\hat{D}_A f \\
&= \partial_0'\hat{D}_O\partial_1 f = \partial_0'\hat{D}_O y.
\end{aligned}$$

lt is easily verified that $\varphi_{G,G'}^{-1}$ is in fact the inverse of $\varphi_{G,G'}$. Naturality is also a routine calculation. □

This adjunction is essentially the same as the one described by Pultr (1979). However, ours is slightly different to that of Pultr (1979) who constructed an adjunction between $\mathcal{G}rph$ and the category of directed graphs without multiple directed edges.

Now we describe the unit and counit of the adjunction. The unit is a natural transformation $\eta : \mathcal{I} \to \mathcal{RL}$ where $\mathcal{I}$ is the identity functor on $\mathcal{G}rph$. Given a directed graph $G = (A, O, \partial_0, \partial_1)$, we have $\mathcal{L}G = (\mathcal{L}A = O, \mathcal{L}O = (O \times 2)/\sim, \partial_0^{\mathcal{L}}, \partial_1^{\mathcal{L}})$. Hence $\mathcal{RL}G$ consists of the following data.

$$\begin{aligned}
\mathcal{RL}A &= \{(x,y) \in \mathcal{L}A \times \mathcal{L}A = O \times O | \partial_1^{\mathcal{L}} x = \partial_0^{\mathcal{L}} y \text{ i.e. } (x,1) \sim (y,0)\} \\
\mathcal{RL}O &= O \\
\partial_0^{\mathcal{RL}}(x,y) &= x \text{ and } \quad \partial_1^{\mathcal{RL}}(x,y) = y
\end{aligned}$$

The components of the natural transformation $\eta_G : G \to \mathcal{RL}G$ are defined by the following two maps.

$$\begin{aligned}
(\eta_G)_O = id_O &: O \to \mathcal{RL}O = O \\
(\eta_G)_A &: A \to \mathcal{RL}A : f \mapsto (\partial_0 f, \partial_1 f)
\end{aligned}$$

The counit is also a natural transformation on $\mathcal{G}rph$, $\epsilon : \mathcal{LR} \to \mathcal{I}$. For a directed graph $G = (A, O, \partial_0, \partial_1)$ we have $\mathcal{R}G = (\mathcal{R}A = \{(f,g) \in A \times A | \partial_1 f = \partial_0 g\}, \mathcal{R}O = A, \partial_0^{\mathcal{R}}, \partial_1^{\mathcal{R}})$.

Hence we find that $\mathcal{LR}G$ consists of the following data.

$$\begin{aligned}
\mathcal{LR}A &= \mathcal{R}O = A \\
\mathcal{LR}O &= (\mathcal{R}O \times 2)/\sim = (A \times 2)/\sim \\
\partial_i^{\mathcal{LR}} f &= [(f,i)]_\sim \ (i = 0,1)
\end{aligned}$$

Here $\sim$ is an equivalence relation on $A \times 2$ generated by the relation defined by

$$\begin{aligned}
(f,1)R(g,0) &\Leftrightarrow \exists\, \alpha \in \mathcal{R}A \ (\partial_0^{\mathcal{R}}\alpha = f, \partial_1^{\mathcal{R}}\alpha = g) \\
&\Leftrightarrow \partial_1 f = \partial_0 g
\end{aligned}$$

Each component $\epsilon_G : \mathcal{LR}G \to G$ is defined by

$$\begin{aligned}
(\epsilon_G)_A = id_A &: \ \mathcal{LR}A = A \to A \\
(\epsilon_G)_O &: \ \mathcal{LR}O = (A \times 2)/\sim \to O : [(f,i)]_\sim \mapsto \partial_i f \ \ (i = 0,1).
\end{aligned}$$

The map $(\epsilon_G)_O$ is well-defined because $(f,1)R(g,0) \Leftrightarrow \partial_1 f = \partial_0 g$ as shown above.

The unit and the counit are not natural isomorphisms in general. (An example for the counit is shown in Figure 8.) However, directed graphs that are biologically interesting might be those G such that $\eta_G : G \simeq \mathcal{RL}G$ or $\epsilon_G : \mathcal{LR}G \simeq G$. That is, if $\eta_G : G \simeq \mathcal{RL}G$ holds for a directed graph G then G is invariant under the sequential operations of first 'gluing functions' and second 'decomposition into functions'. On the other hand, if $\epsilon_G : \mathcal{LR}G \simeq G$ holds then G is invariant under the sequential operations of first 'decomposition into functions' and second 'gluing functions'. Hence, we consider the conditions for η_G or ϵ_G to be natural isomorphisms in what follows.

Let $G = (A, O, \partial_0, \partial_1)$ be a directed graph. First we consider η_G. Because $(\eta_G)_O$ is an identity we only need the conditions for $(\eta_G)_A$. We write $x \to y$ if there exists a directed edge from x to y.

Lemma 5.2 *Consider the following two conditions.*

(i-a) For $f, g \in A$ if $\partial_i f = \partial_i g$ $(i = 0,1)$ then $f = g$. That is, there is at most one directed edge between each pair of nodes.

(i-b) If $x \to y, z \to y, z \to w$ then $x \to w$.

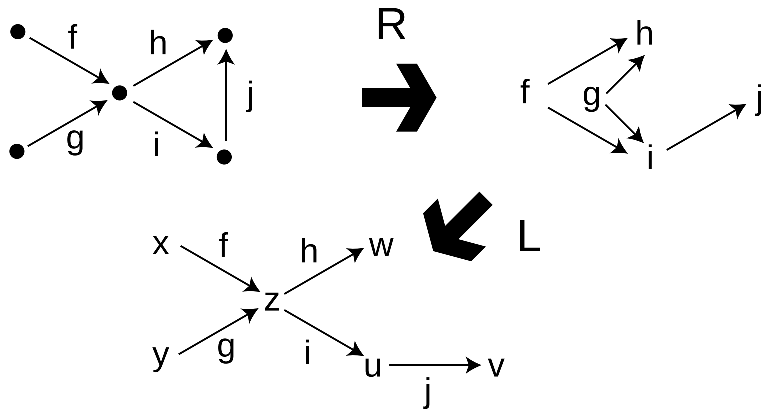

Figure 8: Sequential operations of first $\mathcal{R}$ and second $\mathcal{L}$. Directed edges h and j have the same target at first. However, after the operation of $\mathcal{LR}$, this target is divided into w and v. Here $x = \{(f, 0)\}, y = \{(g, 0)\}, z = \{(f, 1), (g, 1), (h, 0), (i, 0)\}, w = \{(h, 1)\}, u = \{(i, 1), (j, 0)\}$ and $v = \{(j, 1)\}$.

Then $(\eta_G)_A$ is injective if and only if condition (i-a) holds and $(\eta_G)_A$ is surjective if and only if (i-b) holds. Condition (i-b) is depicted in Figure 9.

Proof. For the injective part,

$$\begin{aligned}(\eta_G)_A \text{ is an injection} \quad &\Leftrightarrow \quad \text{if } (\eta_G)_A(f) = (\eta_G)_A(g) \text{ then } f = g \\ &\Leftrightarrow \quad \text{if } (\partial_0 f, \partial_1 f) = (\partial_0 g, \partial_1 g) \text{ then } f = g \\ &\Leftrightarrow \quad \text{condition (i-a) holds.}\end{aligned}$$

For the subjective part, we first prove necessity. Suppose $(\eta_G)_A$ is a surjection. Then there exists $f \in A$ such that $(\eta_G)_A(f) = (x, y)$ for any $(x, y) \in O \times O$ with $(x, 1) \sim (y, 0)$. That is, if $(x, 1) \sim (y, 0)$ holds then $x \to y$. Suppose $x \to y, z \to y$ and $z \to w$ hold. Then we have $(x, 1)R(y, 0)R^{-1}(z, 1)R(w, 0)$. Hence $(x, 1) \sim (w, 0)$ holds. By the condition for surjectivity we have $x \to w$. This is condition (i-b).

Next we show sufficiency. Suppose condition (i-b) holds. In order to prove that $(\eta_G)_A$ is a surjection, we have to show the existence of $f \in A$ such that $(\eta_G)_A(f) = (x, y)$ for any $(x, y) \in \mathcal{RL}A$. This is equivalent to showing that if $(x, 1) \sim (y, 0)$ then $x \to y$. If $(x, 1) \sim (y, 0)$ holds then there exist $s_1, \cdots, s_n \in O \times 2$ such that $(x, 1) = s_1, s_i R \cup R^{-1} s_{i+1}, s_n = (y, 0)$ $(i = 1, 2, \cdots, n-1)$. Because we have $s_1 = (x, 1), s_n = (y, 0)$, the chain must be $s_1 R s_2 R^{-1} s_3 R \cdots R s_n$ with R and R^{-1} appearing alternately in the chain. n takes values $n = 2k+2$ ($k = 0, 1, 2, \cdots$). If $k = 0$ then the claim is trivial. If $k = 1$ then there exist $x', y' \in O$ such that $(x, 1)R(x', 0)R^{-1}(y', 1)R(y, 0) \Leftrightarrow x \to x', y' \to x', y' \to y$. By condition (i-b) we get $x \to y$. For the general case, the claim can be verified by mathematical induction. $\square$

Now we consider the counit. Because $(\epsilon_G)_A$ is an identity, we only have to obtain the conditions for $(\epsilon_G)_O$.

Lemma 5.3 *Consider the following conditions.*

(ii-a) If $\partial_i f = \partial_i g$ and $f \neq g$ holds then there exists $h \in A$ such that $\partial_{i+1 \ mod \ 2} h = \partial_i f (= \partial_i g)$.

(ii-b) For any $x \in O$ there exists $f \in A$ such that $\partial_0 f = x$ or $\partial_1 f = x$.

$(\epsilon_G)_O$ is injective if and only if condition (ii-a) holds. $(\epsilon_G)_O$ is surjective if and only if condition (ii-b) holds. Both conditions are depicted in Figure 9.

Proof. First we prove the injective part. Suppose condition (ii-a) holds. Given $\alpha, \beta \in (A \times 2)/\sim$ we have to show that if $(\epsilon_G)_O(\alpha) = (\epsilon_G)_O(\beta)$ then $\alpha = \beta$. We have three cases depending on the representative elements of α and β. The first case we consider is when $\alpha = [(f, 1)]_\sim$ and $\beta = [(g, 0)]_\sim$. If $(\epsilon_G)_O(\alpha) = (\epsilon_G)_O(\beta)$ holds then we have

$$\partial_1 f = \partial_0 g \Leftrightarrow (f, 1)R(g, 0) \Rightarrow \alpha = \beta.$$

Next consider the case when α and β have the forms $\alpha = [(f, 0)]_\sim$ and $\beta = [(g, 0)]_\sim$. If $(\epsilon_G)_O(\alpha) = (\epsilon_G)_O(\beta)$ holds then $\partial_0 f = \partial_0 g$. If $f \neq g$ then there exists $h \in A$ such that $\partial_0 f = \partial_0 g = \partial_1 h$ by condition (ii-a). It follows that $(f, 0)R^{-1}(h, 1)R(g, 0)$. This implies $(f, 0) \sim (g, 0)$, that is, $\alpha = \beta$. The remaining case $\alpha = [(f, 1)]_\sim$ and $\beta = [(g, 1)]_\sim$ is similar.

For the opposite direction, suppose $(\epsilon_G)_O$ is an injection. Then $(f, 0) \sim (g, 0)$ if $\partial_0 f = \partial_0 g$. Hence if $f \neq g$ then there exist $s_1, \cdots, s_n \in A \times 2, (f, 0) = s_1 R^{-1} s_2 R \cdots R s_n = (g, 0)$. Here n takes values $n = 2k+3$ ($k = 0, 1, 2, \cdots$). Because $n \geq 3$, there exist $f, g, h \in A$

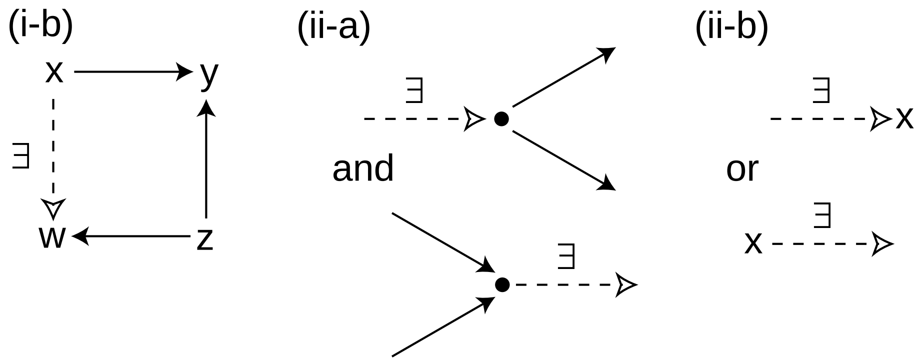

Figure 9: The conditions in Lemma 5.2 and Lemma 5.3 are shown schematically.

such that $(f,0)R^{-1}(h,1) \sim (g,0)$. We get $\partial_0 f = \partial_1 h = \partial_0 g$. Thus we obtain condition (ii-a) for $i = 0$. The condition for $i = 1$ follows similarly.

For surjectivity, if there exists a node that is neither a source nor a target of any directed edge then $(\epsilon_G)_O$ is not a surjection because the value of $(\epsilon_G)_O$ is a source or a target of a directed edge. This proves necessity. For sufficiency, suppose condition (ii-b) holds. Then for any $x \in O$ there exist $i \in \{0,1\}$ and $f \in A$ such that $(\epsilon_G)_O([(f,i)]_\sim) = \partial_i f = x$. Hence $(\epsilon_G)_O$ is a surjection. $\square$

Using Lemma 5.2 and Lemma 5.3, we obtain the following theorem.

Theorem 5.4 *The largest subcategory of $\mathcal{G}rph$ on which the unit $\eta : \mathcal{I} \to \mathcal{RL}$ is a natural isomorphism consists of directed graphs that satisfy conditions (i-a) and (i-b) and have directed graph homomorphisms between them. This is a full subcategory of $\mathcal{G}rph$. Similarly the full subcategory of $\mathcal{G}rph$ whose objects are directed graphs that satisfy conditions (ii-a) and (ii-b) is the largest subcategory of $\mathcal{G}rph$ on which the counit $\epsilon : \mathcal{LR} \to \mathcal{I}$ is a natural isomorphism.*

The results for strategy (I) are summarized in Theorem 5.1 and Theorem 5.4. However, the functor $\mathcal{L}$ does not represent the idea of 'gluing functions' explicitly. In order to give an explicit representation of 'gluing functions' as a functor, we must restrict the category in which we work. This is strategy (II) which we consider in what follows. In contrast to the first strategy, we can obtain an explicit adjunction between 'decomposition into functions' and 'gluing functions'. The operation which represents 'gluing functions' directly is $\mathcal{L}''$

which was defined in the previous section. First we define the category with which we work hereafter.

Definition 5.5 *A subcategory $\mathcal{H}$ of $\mathcal{G}rph$ is defined as follows.*

The objects in $\mathcal{H}$ are directed graphs $G = (A, O, \partial_0, \partial_1)$ that satisfy the following condition.

(H) For all $x \in O$ there exist $f, g \in A$ such that $\partial_1 f = x = \partial_0 g$.

The arrows in $\mathcal{H}$ are homomorphisms of directed graphs. That is, $\mathcal{H}$ is a full subcategory of $\mathcal{G}rph$.

Condition (H) is the weakest condition under which the operation $\mathcal{L}''$ becomes a functor. This is justified by the following proposition.

Proposition 5.6 *A directed graph $G = (A, O, \partial_0, \partial_1)$ satisfies condition (H) if and only if any equivalence class of $\mathcal{L}'O$ includes an element of T, where $T = \{(x, y) \in O \times O | \exists f \in A\ \partial_0 f = x, \partial_1 f = y\}$.*

Proof. Suppose $G = (A, O, \partial_0, \partial_1)$ satisfies condition (H). For any $\alpha \in \mathcal{L}'O$ there exist $x \in O$ and $i \in \{0, 1\}$ such that $\alpha = [(x, i)]_{\sim'}$. Consider the case $\alpha = [(x, 0)]_{\sim'}$. By condition (H), there exists $f \in A$ such that $x = \partial_1 f$. We have $(\partial_0 f, \partial_1 f) \in \alpha$ because $(\partial_0 f, \partial_1 f)R'(x, 0)$, where R' is the relation on $S = T \cup (O \times 2)$ given in definition 4.1. The other case can be proved similarly.

For the opposite direction, suppose any $\alpha \in \mathcal{L}'O$ includes an element of T. Fix any element $z \in O$. By assumption, there exists $(x, y) \in T$ such that $(z, 0) \sim' (x, y)$. There also exists $(x', y') \in T$ such that $(x', y')R'(z, 0)$. By the definition of R', we have $y' = z$. We obtain $z = \partial_1 f$ for $f \in A$ such that $\partial_0 f = x'$ and $\partial_1 f = y'$. The existence of $g \in A$ such that $z = \partial_0 g$ can be shown in the same way by considering $(x'', y'') \in T$ such that $(x'', y'')R'(z, 1)$. □

Now we collect some facts about $\mathcal{H}$ and $\mathcal{L}''$.

Proposition 5.7 *Let $\mathcal{H}$ be the subcategory defined in Definition 5.5 and let $\mathcal{R}$ and $\mathcal{L}''$ be the functors defined in sections three and four respectively. Then* *(i)* $\mathcal{L}''$ *is a functor from* $\mathcal{H}$ *to* $\mathcal{G}rph$*.*

(ii) $\mathcal{L}''$ *is naturally isomorphic to* $\mathcal{L}$ *on* $\mathcal{H}$*.*

(iii) If a directed graph G *satisfies condition (H) then* $\mathcal{L}''G$ *also satisfies (H).*

(iv) If a directed graph G *satisfies condition (H) then* $\mathcal{R}G$ *also satisfies (H).*

Proof.

(i) The proof is a straightforward verification.

(ii) It is sufficient to show that $\mathcal{L}'' \simeq \mathcal{L}'$ on $\mathcal{H}$ because we have $\mathcal{L}' \simeq \mathcal{L}$ on $\mathcal{G}rph$ by Proposition 4.3. We define a natural isomorphism $\phi : \mathcal{L}'' \to \mathcal{L}'$ as follows. Given a directed graph $G = (A, O, \partial_0, \partial_1)$, the components of ϕ_G are two maps. The arrow part is defined by $(\phi_G)_A := id_O : \mathcal{L}''A = O \to \mathcal{L}'A = O$. The object part $(\phi_G)_O : \mathcal{L}''O \to \mathcal{L}'O$ is defined as a map that sends $[(x,y)]_{\sim''}$ to $[(x,y)]_{\sim'}$. This map is well-defined because $(x,y)R''(z,w)$ implies $(x,y)R'(z,w)$.

(iii) For any $[(x,y)]_{\sim''} \in \mathcal{L}''O$ there exists $f \in A$ such that $\partial_0 f = x$ and $\partial_1 f = y$. By the definition of source and target maps for $\mathcal{L}''G$, we obtain $\partial_1^{\mathcal{L}''} x = [(x,y)]_{\sim''} = \partial_0^{\mathcal{L}''} y$.

(iv) Take any $f \in \mathcal{R}O = A$. There exist $g, h \in A$ such that $\partial_0 f = \partial_1 g$ and $\partial_1 f = \partial_0 h$ by condition (H). Because $(g,f), (f,h) \in \mathcal{R}A$, we obtain $\partial_1^{\mathcal{R}}(g,f) = f = \partial_0^{\mathcal{R}}(f,h)$.

□

By Theorem 5.1 and Proposition 5.7, we obtain the following adjunction on $\mathcal{H}$.

Theorem 5.8 $\mathcal{L}''$ *is a left adjoint functor to* $\mathcal{R}$ *on* $\mathcal{H}$*. That is, we have a natural isomorphism*

$$\mathcal{H}(\mathcal{L}''G, G') \simeq \mathcal{H}(G, \mathcal{R}G')$$

for any pair of directed graphs G, G' *in* $\mathcal{H}$*.*

Thus the adjunction in Theorem 5.8 is a restriction of the one in Theorem 5.1 to the subcategory $\mathcal{H}$. Meanwhile, condition (H) implies conditions (ii-a) and (ii-b) in Lemma 5.3. Hence the counit $\epsilon : \mathcal{L}''\mathcal{R} \to \mathcal{I}$ is a natural isomorphism. On the other hand, condition (H) has nothing to do with the proof of the conditions for injectivity and surjectivity of $\eta_G : G \to \mathcal{R}\mathcal{L}''G$ in Lemma 5.2. Therefore, a necessary and sufficient condition for $\eta_G : G \to \mathcal{R}\mathcal{L}''G$ to be an injection is condition (i-a) in Lemma 5.2 and the necessary and sufficient condition for surjectivity is condition (i-b) in Lemma 5.2. Summarizing these facts, we obtain the following theorem.

Theorem 5.9 *The counit $\epsilon : \mathcal{L}''\mathcal{R} \to \mathcal{I}$ is a natural isomorphism on $\mathcal{H}$. The full subcategory of $\mathcal{H}$ whose objects are directed graphs satisfying conditions (i-a) and (i-b) is the largest subcategory of $\mathcal{H}$ on which $\eta : \mathcal{I} \simeq \mathcal{R}\mathcal{L}''$ holds.*

6 Discussion

In this section we discuss the significance of the adjunctions found in the previous section and their consequences for theoretical biology.

As we have seen above, we have to work in the subcategory $\mathcal{H}$ of $\mathcal{G}rph$ in order to define the idea of 'gluing functions' as a functor. Condition (H) means that any directed graph in $\mathcal{H}$ is closed when following arrows either forward or backward. If G is a finite directed graph that satisfies (H) then this implies there exists a cycle in G. Thus we obtain the existence of a cycle in a directed graph as a necessary condition for 'gluing functions' to be defined. Furthermore the existence of a cycle is conserved by both $\mathcal{L}''$ and $\mathcal{R}$. In particular, by Theorem 5.9, the same cycle is recovered under the sequential operations of first 'decomposition into functions' and second 'gluing functions'.

Mathematically, a directed graph G is a line-graph of some directed graph if and only if G satisfies conditions (i-a) and (i-b) (Pultr, 1979). However, here we shall consider them from the point of view of theoretical biology.

Condition (i-a) means there is only one directed interaction between two nodes. It seems that this says any node can only be active or passive but not both. However, if we keep in mind that a directed graph is a syntactic representation of a system, this is not

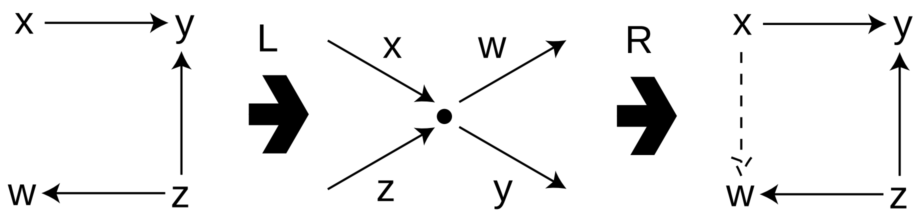

Figure 10: A schematic explanation of condition (i-b). Because x and w are tied by a node in the picture in the center, a new directed edge from x to w is made.

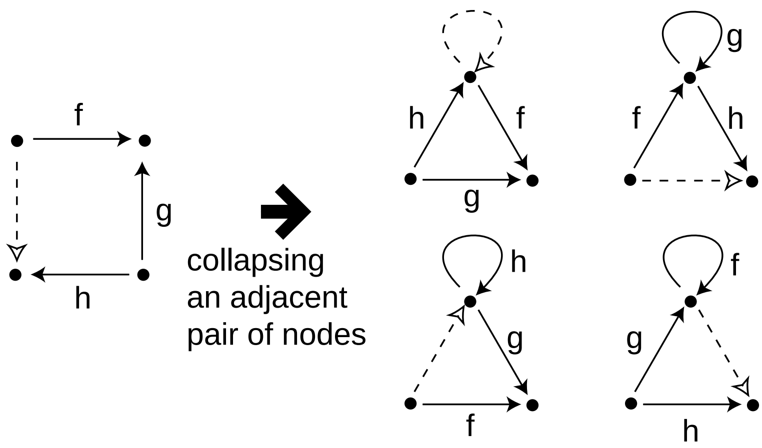

Figure 11: Under condition (i-b) the square graph of Figure 10 becomes one of four diagrams whose shapes are the same (triangle with a loop) if one of directed edges in the square is made into a loop. The four diagrams can be seen as anticipatory diagrams.

such a disappointing condition. A system can still have rich semantic structures. Instead, we can avoid a complicated description of a system by using this constraint.

Condition (i-b) seems mysterious at first sight. However, this is a trivial gluing condition if we make nodes into directed edges. Suppose $x \rightarrow y, z \rightarrow y$ and $z \rightarrow w$. If we operate with $\mathcal{L}$ on a directed graph that consists of four nodes and the described directed edges, we obtain a directed graph with one node and two incoming edges x, z and two outgoing edges y, z. Before the operation of $\mathcal{L}$, there are three links from x to y, from z to y and from z to w. However, After the operation of $\mathcal{L}$, a new link from x to w is generated. Graphically, this is obvious (see Figure 10). Actually we obtain a new directed edge from x to w by operating with $\mathcal{R}$.

Condition (i-b) can be seen from another point of view. It can be interpreted as a diagram for anticipation. Here we use the term anticipation in the sense of making a link between two things from some clues. This becomes visible by collapsing an adjacent pair of nodes in the square graph (Figure 11). All four diagrams on the right hand side of Figure 11 have the same shape. Only the position of the broken arrow is different, depending on which pair of nodes is collapsed. The square diagram is the least graph which unifies the four triangular diagrams. Let us examine the upper right diagram. This diagram says that f and h are 'composable' if there exists a loop g. (Assume that there exists at most one directed edge between two nodes, that is, condition (i-a) also holds.) The role of g is to link the target of f and the source of h. Edges f and h cannot be composed without g. A link between the source of f and the target of h is 'anticipated' by g. In the upper left diagram the broken loop which links the target of h and the source of f is 'anticipated' by the commutative triangle. In the lower left and lower right diagrams the commutative triangle is completed by the linking action of a loop. As observed above, the construction of the commutative triangle involves a linkage between the upper left edge and the upper right edge. We can regard this as a representation of anticipation in the sense given above.

7 Summary and Outlook

In this paper we have developed an adjunction between 'decomposition into functions' and 'gluing functions' in the category of directed graphs. The existence of a cycle and anticipatory diagrams are obtained as implications from the invariant structures of the adjunction. This is a new derivation of these significant conditions for theoretical biology.

We propose two directions for future research. The first is the study of the gluing closure which is derived from the unit of adjunction $\eta : \mathcal{I} \to \mathcal{RL}$. Let X be a set and R be a relation on X. (X, R) can be seen as a directed graph with X as the set of nodes and $R \subset X \times X$ as the set of directed edges. This is a directed graph without multiple directed edges. Let R_2 be a binary relation on $X \times 2$ defined by $(x, 1)R_2(y, 0) \Leftrightarrow xRy$ and let $\sim$ be the equivalence relation on $X \times 2$ generated by R_2. We define the gluing closure of R by

$$\overline{R} := \{(x, y) \in X \times X | (x, 1) \sim (y, 0)\}.$$

In particular, studying the gluing closure on random graphs would provide an insight into the effects of 'gluing functions' from the statistical point of view.

The second direction is the mathematical formalization of Ray Paton's idea. As described in section two he represents a concept by a pair of graphs, the star graph and the tetrahedron graph. We pointed out that the star graph is an extent part and the tetrahedron graph is an intent part. His idea seems to be generalizable as a formal concept analysis (FCA) (Ganter and Wille, 1999) on $\mathcal{G}rph$. Because $\mathcal{G}rph$ can be viewed as a topos (MacLane and Moerdijk, 1992; Vigna, 1997), everything in FCA can be generalized trivially in $\mathcal{G}rph$. However, so called polar operations become uninteresting in such a trivial generalization. Because Paton's idea includes the adjunction described in this paper, a generalization in relation to the adjunction might be needed. We expect that we can obtain a mathematical framework that describes collective concepts (e.g. ecosystem, protein, family, army and so on) because the node set of a graph can include objects belonging to different levels.

In relation to collective concepts, the problem of the coherence of parts to be glued is a significant issue in real biological phenomena (Gunji et al., 2006; Matsuno, 2006). A concept for the foundation of coherence has been proposed: it is called material cause by Gunji et al. (2006) and quantum by Matsuno (2006). However, it seems difficult to treat the problem of coherent gluing in the proposed framework because the process of gluing treated in this paper is a logical process that has no time dependent aspect: the gluing is performed in a single operation. In order to make a link between the proposed framework and these concepts, we must think carefully about what is represented by a graph.

There are at least three different ways to look at a graph. The first way involves a local perspective. A directed edge between two nodes expresses a causal sequence. This is a local structure of a graph and a whole graph is a disjoint sum of such structures. The second way of looking involves a global perspective. A graph represents the time-independent relational structure of a system. The third perspective is a compromise between the first and the second. On the one hand uncoordinated causal sequences proceed in parallel, on the other they are coherently glued as a whole. The framework used in this paper is apparently based on the second perspective. It will not be until the third perspective is formalized in the language of graphs that the problem of the coherent gluing of parts can be treated.

8 Acknowledgements

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A Another Formulation of the Duality

In the main text we formalize 'decomposition into functions' as a functor from $\mathcal{G}rph$ to itself that sends each directed graph to its line-graph. However, we can see the construction of the line-graph from a given directed graph in a different way. It can be formalized as a functor from $\mathcal{G}rph$ to the category of two-dimensional directed graphs $2\mathcal{G}rph$. This functor also has a left adjoint. We construct this adjunction in this appendix.

Let Γ be the category defined by the following diagram.

$$C_2 \underset{t_1}{\overset{s_1}{\leftleftarrows}} C_1 \underset{t_0}{\overset{s_0}{\leftleftarrows}} C_0$$

The category of two-dimensional directed graphs is defined as the presheaf category $2\mathcal{G}rph := \mathcal{S}ets^{\Gamma^{op}}$. We define a functor $\mathcal{R} : \mathcal{G}rph \to 2\mathcal{G}rph$ that sends each directed graph $G = A_1 \underset{\partial_{0,1}}{\overset{\partial_{0,0}}{\rightrightarrows}} A_0$ to

$$\mathcal{R}G = \{(f,g) \in A_1 \times A_1 | \partial_{0,1} f = \partial_{0,0} g\} \underset{\partial^{\mathcal{R}}_{1,1}}{\overset{\partial^{\mathcal{R}}_{1,0}}{\rightrightarrows}} A_1 \underset{\partial^{\mathcal{R}}_{0,1}}{\overset{\partial^{\mathcal{R}}_{0,0}}{\rightrightarrows}} A_0.$$

The source and target maps are defined by $\partial^{\mathcal{R}}_{1,0}(f,g) = f, \partial^{\mathcal{R}}_{1,1}(f,g) = g, \partial^{\mathcal{R}}_{0,i} = \partial_{0,i}$ $(i = 0,1)$. $\mathcal{R} : \mathcal{G}rph \to 2\mathcal{G}rph$ transforms a given directed graph into a line graph.

An inverse functor $\mathcal{L} : 2\mathcal{G}rph \to \mathcal{G}rph$ is defined by sending $G = A_2 \underset{\partial_{1,1}}{\overset{\partial_{1,0}}{\rightrightarrows}} A_1 \underset{\partial_{0,1}}{\overset{\partial_{0,0}}{\rightrightarrows}} A_0$ to

$$A_1 \underset{\partial^{\mathcal{L}}_{0,1}}{\overset{\partial^{\mathcal{L}}_{0,0}}{\rightrightarrows}} A_0/\sim,$$

where $\sim$ is the equivalence relation generated by the relation R on A_0 defined by

$$xRy \Leftrightarrow \exists \alpha \in A_2 \ \exists f,g \in A_1 \ \partial_{1,0}\alpha = f, \partial_{1,1}\alpha = g, \partial_{0,1} f = x, \partial_{0,0} g = y.$$

The source and target maps are $\partial^{\mathcal{L}}_{0,i} f = [\partial_{0,i} f]_\sim$ $(i = 0, 1)$. $\mathcal{L}$ glues zero-dimensional arrows so that two-dimensional arrows represent links between two one-dimensional arrows such that the target of one is tied to the source of the other.

We, therefore, have the following adjunction.

Theorem A.1 *For any two-dimensional directed graph G and directed graph G', we have a natural isomorphism*

$$\mathcal{G}rph(\mathcal{L}G, G') \simeq 2\mathcal{G}rph(G, \mathcal{R}G').$$

Proof. We only describe the construction of the bijection. Put $G = A_2 \underset{\partial_{1,1}}{\overset{\partial_{1,0}}{\rightrightarrows}} A_1 \underset{\partial_{0,1}}{\overset{\partial_{0,0}}{\rightrightarrows}} A_0$ and $G' = A'_1 \underset{\partial'_{0,1}}{\overset{\partial'_{0,0}}{\rightrightarrows}} A'_0$. First we define a map $\varphi_{G,G'} : \mathcal{G}rph(\mathcal{L}G, G') \to 2\mathcal{G}rph(G, \mathcal{R}G')$. Suppose $D : \mathcal{L}G \to G'$ is given. We have two maps $D_1 : A_1 \to A'_1$ and $D_0 : A_0/\sim \to A'_0$. The zero-dimensional and one-dimensional parts of $\varphi_{G,G'}(D)$ are defined by

$$\begin{aligned} \varphi_{G,G'}(D)_0 \quad &: \quad A_0 \to A'_0 : x \mapsto D_0([x]_\sim) \\ \varphi_{G,G'}(D)_1 := D_1 \quad &: \quad A_1 \to A'_1. \end{aligned}$$

The two-dimensional part

$$\varphi_{G,G'}(D)_2 : A_2 \to \{(f, g) \in A'_1 \times A'_1 | \partial'_{0,1} f = \partial'_{0,0} g\}$$

is as follows. Given $\alpha \in A_2$, we have $(\partial_{1,0}\alpha, \partial_{1,1}\alpha) \in A_1 \times A_1$ and $(D_1\partial_{1,0}\alpha, D_1\partial_{1,1}\alpha) \in A'_1 \times A'_1$. We define $\varphi_{G,G'}(D)_2(\alpha) := (D_1\partial_{1,0}\alpha, D_1\partial_{1,1}\alpha)$. We have to check $\partial'_{0,1} D_1 \partial_{1,0}\alpha = \partial'_{0,0} D_1 \partial_{1,1}\alpha$. This result is obtained by the following calculation. We have

$$\partial'_{0,1} D_1 \partial_{1,0}\alpha = D_0 \partial^{\mathcal{L}}_{0,1} \partial_{1,0}\alpha = D_0([\partial_{0,1}\partial_{1,0}\alpha]_\sim)$$

and

$$\partial'_{0,0} D_1 \partial_{1,1}\alpha = D_0 \partial^{\mathcal{L}}_{0,0} \partial_{1,1}\alpha = D_0([\partial_{0,0}\partial_{1,1}\alpha]_\sim).$$

Because $\partial_{0,1}\partial_{1,0}\alpha R \partial_{0,0}\partial_{1,1}\alpha$, the right hand sides are identical.

Next we describe $\varphi^{-1}_{G,G'} : 2\mathcal{G}rph(G, \mathcal{R}G') \to \mathcal{G}rph(\mathcal{L}G, G')$. For any $\hat{D} : G \to \mathcal{R}G'$ we have three maps $\hat{D}_2 : A_2 \to \{(f, g) \in A'_1 \times A'_1 | \partial'_{0,1} f = \partial'_{0,0} g\}$, $\hat{D}_1 : A_1 \to A'_1$ and $\hat{D}_0 : A_0 \to A'_0$. The one-dimensional part of $\varphi^{-1}_{G,G'}(\hat{D})$ is defined by

$$\varphi^{-1}_{G,G'}(\hat{D})_1 := D_1 : A_1 \to A'_1.$$

The zero-dimensional part

$$\varphi^{-1}_{G,G'}(\hat{D})_0 : A_0/\sim \longrightarrow A'_0$$

is defined by $\varphi^{-1}_{G,G'}(\hat{D})_0([x]_\sim) := \hat{D}_0(x)$ for $x \in A_0$. In order to check the well-definedness of $\varphi^{-1}_{G,G'}$, it is sufficient to show that if xRy then $\hat{D}_0(x) = \hat{D}_0(y)$. Suppose xRy then there exist $\alpha \in A_2$ and $f, g \in A_1$ such that $\partial_{1,0}\alpha = f, \partial_{1,1}\alpha = g, \partial_{0,1}f = x$ and $\partial_{0,0}g = y$. We have

$$\hat{D}_0(x) = \hat{D}_0\partial_{0,1}f = \partial'_{0,1}\hat{D}_1 f = \partial'_{0,1}\hat{D}_1\partial_{1,0}\alpha = \partial'_{0,1}\partial^{\mathcal{R}}_{1,0}\hat{D}_2\alpha$$

and

$$\hat{D}_0(y) = \hat{D}_0\partial_{0,0}g = \partial'_{0,0}\hat{D}_1 g = \partial'_{0,0}\hat{D}_1\partial_{1,1}\alpha = \partial'_{0,0}\partial^{\mathcal{R}}_{1,1}\hat{D}_2\alpha.$$

By the definition of the codomain of $\hat{D}_2$ the right hand sides must be identical. □

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