A Generalization of Formal Model of Internal Measurement: A Construction on One-Dimensional Maps

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A generalization of formal model of internal measurement proposed by one of the authors (Gunji, *et al.* (1997). *Physica D 110*, 289-312) is addressed. An interface between defining a fixed point and holding an adjunction is constructed motivated by the problem of the gap between parts and wholeness in complex systems. We define a construction of the generalized procedure of formal model of internal measurement on families of one-dimensional maps. We introduce two-dimensional time evolutionary systems by examining computation of time evolutionary steps of one-dimensional maps in terms of a negotiation process between a collection of parts and wholeness. We discuss a power law behavior or fractals in some concrete one-dimensional maps.

Key words: Complex Systems, Internal Measurement, Adjunction, Fixed Point, One-Dimensional Maps, Fractals.

1 INTRODUCTION

One of the most important problems in complex systems research is how to address relationships between the collection of parts and wholeness in a

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system. A major approach to this problem is modeling systems by computer simulations such as cellular automata [1], coupled map lattices [2] or random Boolean networks [3]. In these models, first one collects individual parts of a system that cannot be divided anymore. Next local interactions between the individual parts are defined. Finally one finds some global phenomena appeared through huge number of iterations of the local interactions with the aid of computer simulations. These global phenomena are analyzed in the name of self-organization or emergence. It is said that such constructed and observed globalness can be understood as a whole that cannot be resolved by the collection of parts [2]. However, what are "self" or "emergence" constructed and understood here? It seems that we can get the totally opposite conclusion from the fact that we can construct wholeness by computer simulations. That is, a whole can be reduced to a collection of parts by the huge number of repetitions of computations. Although this speculation is naive, it entails the difficulty addressing the gap between parts and a whole in the realm of computer simulations since computer simulations always start from collections of parts [4]. Consequently the question how we can internalize wholeness into each parts in a system arises. In this paper, we concern a construction of an interface between parts and wholeness in each computational step of one-dimensional maps.

Some authors start from accepting wholeness that cannot be resolved to a collection of parts. For example, Pattee [5] distinguished semantic processes such as measurement from formal processes such as computation. He addressed that it is difficult to distinct the former from the latter in computer simulations and functions of semantic processes are limited. Rosen [6] defined organisms which are distinct from machines in terms of the closure of efficient causations. He showed that the (M,R) systems which are models of organisms cannot be simulated by Turing machines [7]. Obviously, the distinctions between formal processes and semantic processes or between machines and organisms are parallel to the distinction between a collection of parts and wholeness.

An approach to the problem of the gap between parts and wholeness called internal measurement [8, 9] is comparable to that of Pattee or Rosen except for one point. However, the excepted point is quite essential. That is, the distinction between parts and a whole can hold, but at the same time, it is destined to be relative in the framework of internal measurement theory. The idea of internal measurement is similar to a line of thought proposed by Wittgenstein, which is called "language game" [10]. A language game is a dynamically changing network of words in which they define or are defined each other indefinitely. The meaning of each word constraints its use on one hand, the use of a word is latently open to indefinite environment on the other hand. Here we can also see a parallelism between the distinction between the meaning of a word and the use of a word and the distinction between a collection parts and wholeness. It is easy to see that the use of a word refers to the whole environment in which the meaning of the word is relevant. Wittgenstein clung neither the meaning of a word nor the use of a word. He focused on a series of performances over the meaning of a word or the use of a word. In other words, he found a process that resolves a gap between parts and wholeness but at the same time generates a new gap between them.

One of the authors proposed formal model of internal measurement in previous works [11, 12]. The gap between parts and wholeness is replaced by a interface between defining a map as a fixed point and using a map in this model. A "language game" over defining a map and using a map is formally constructed and results in a dynamically changing interface between them. In this paper, we concern a generalization of the existing formal model of internal measurement.

The organization of this paper is as follows. First we review the existing formal model of internal measurement in section two. Next we discuss how to generalize the existing model in terms of an interface between fixed points and an adjunction in section three. Finally we apply the generalized procedure of formal model of internal measurement to one-dimensional maps in section four. We examine what kinds of interfaces between parts and wholeness can be constructed by applications to concrete one-dimensional maps.

2 A REVIEW OF FORMAL MODEL OF INTERNAL MEASUREMENT

In this section we review the existing formal model of internal measurement [11, 12]. We start from the distinction between objects and observers. Consider a system consisting of objects and observers. We concern a description of the objects including its observers. Descriptions that include both objects and observers are said to be needed for living systems or social systems [13, 14]. However, it is also said that an essential difficulty caused by having different two levels, that is, objects and observers, in a description is hard to remove [15]. The idea of internal measurement addresses this difficulty from different angle. This idea does not attach our minds to giving a simultaneous description of both objects and observers but leads us to a description of a "language game" on the system.

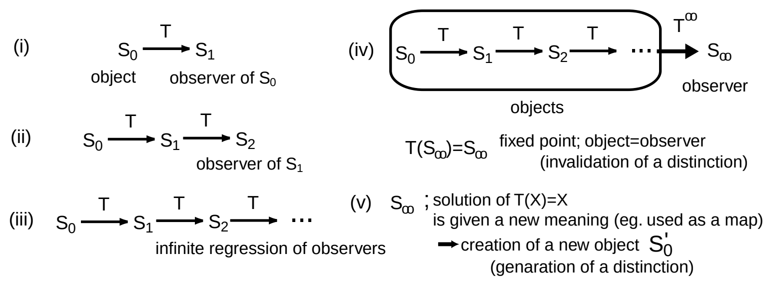

FIGURE 1
A sketch of the "language game" that alternates making a distinction and invalidating the distinction between objects and observers. See text for details.

Let S_0 be an object and S_1 be an observer. Assume that S_1 observes S_0 by a measurement process T. We express this situation by $S_0 \xrightarrow{T} S_1$ (Figure 1(i)). Here we are only interested in S_1's measurement on S_0, so we write $S_1 = T(S_0)$. If we distinguish the observer S_1 from the measurement process T, S_1 can be also an object and can be observed by another observer S_2 by the measurement process T. Thus we get $S_1 \xrightarrow{T} S_2$ and $S_2 = T(S_1)$ (Figure 1(ii)). By the same way, an observer S_n that observes an object S_{n-1} by the measurement process T with $S_n = T(S_{n-1})$ is constructed for $n = 1, 2, 3, \cdots$. As a result we have an infinite regression of observers: $S_0 \xrightarrow{T} S_1 \xrightarrow{T} S_2 \xrightarrow{T} \cdots \xrightarrow{T} S_{n-1} \xrightarrow{T} S_n \xrightarrow{T} \cdots$ (Figure 1(iii)). Once we derive the infinite sequence of observers, we are destined to consider an observer S_∞ that observes each S_n since we intend to construct a description that includes any level of objects and observers. How can we get the observer S_∞? If n goes to infinity in $S_n = T(S_{n-1})$ formally, we get $S_\infty = T(S_\infty)$ (Figure 1(iv)). We can regard this equation as the definition of S_∞. If S_∞ exists, then S_∞ is derived as a fixed point of T. S_∞ has the status of both an object and an observer. Hence S_∞ has a paradoxical status in terms of the position in which we make a distinction between objects and observers. However, once we recognize S_∞ as a fixed point of T, we can say that a system that contains both objects and observers can be described as S_∞ which paradoxically shows indistinguishability between objects and observers.

We first mentioned that we do not describe a system but describe a "lan-

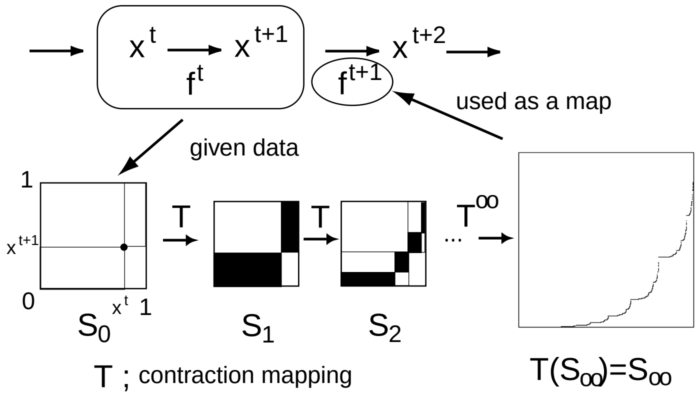

FIGURE 2
A simple example of formal model of internal measurement. Contraction mapping T defined by a observed datum at time t (x^t, x^{t+1}) is regarded as a measurement process. A function S_∞ is obtained as a fixed point of T. S_∞ is used as the time evolution rule at the next time step $t+1$.

guage game" on the system. It's now time to concern this point. We focus on the aspect that S_∞ is open to a use in indefinite environment. For example, as we will see below, we can get S_∞ as a subset of $X \times X$ for a set X for specific T. If we regard S_∞ as a map from X to X and apply it to a point in X, we can obtain a new object $S_0^{'}$ (Figure 1(v)). Although we get a conclusion that the distinction between objects and observers is invalidated by the fixed point of T, the distinction is recovered by using the fixed point in another way. Thus we get a form of "language game" that alternates making a distinction and invalidating the distinction between objects and observers.

Now we give a simple example, one-dimensional time evolutionary system (Figure 2). We concern a system that takes its state in $X = [0, 1]$, the interval of real numbers from 0 to 1. We assume that the system changes its state in discrete time steps. Let x^t be the state at time t and f^t be the time evolution rule from time t to $t+1$. If we know the rule f^t in advance, we can obtain the state at time $t+1$ by applying it to x^t, that is, $x^{t+1} = f^t(x^t)$. Here we concern the reverse direction. Given a state pair (x^t, x^{t+1}), the question how we can know the rule f^t arises. Of course, if we do not have any condition, there exist infinitely many rules which send x^t to x^{t+1}. Therefore in order to get f^t uniquely, we must define a meta-rule for obtaining the time evolution rule.

We define a meta-rule by a measurement process for the state pair (x^t, x^{t+1}) in the product of state space $X \times X$. The measurement process is expressed as a coarse graining of $X \times X$. Let T_1, T_2 be maps from $X \times X$ to $X \times X$ defined as follows. For $(x, y) \in X \times X$, define $T_1(x, y) = (x^t x, x^{t+1} y), T_2(x, y) = ((1 - x^t)x + x^t, (1 - x^{t+1})y + x^{t+1})$. Put $T(A) = T_1(A) \cup T_2(A)$ for a subset A of $X \times X$. We regard this contraction mapping T as a measurement process. Consider the following sequence: $S_0 = X \times X, S_1 = T(S_0) = T(X \times X), \cdots, S_n = T(S_{n-1}) = T^{n+1}(S_0), \cdots$. The set S_∞ which satisfies $T(S_\infty) = S_\infty$ is a nowhere differentiable continuous function from X to X. Thus the fixed point S_∞ of T can be used as a map from X to X. We regard this map as the time evolution rule at time $t + 1$, that is, f^{t+1}. Consequently, we get a new state pair $(x^{t+1}, f^{t+1}(x^{t+1})) = (x^{t+1}, x^{t+2})$.

Seeing the above coarse graining process, one might recall an approach that is called symbolic dynamics. A symbolic dynamics constructed by a Markov partition of a dynamical system concerns how much information about given dynamical system can be preserved [16]. In contrast, the coarse graining process proposed here entails information loss about the time evolution rule at the previous time step. In this sense, the measurement process defined by the coarse graining can be regarded as an abstract representation for incompleteness of measurement in the real world [17].

There are some applications of formal model of internal measurement for complex systems or biological modeling: an interactive system consisting of many elements [11], a model of fish schooling [17, 18] or a model of punctuated equilibrium in thoroughbred evolution [19]. As we mentioned in introduction, this paper does not concern applications but a generalization. Developing a generalization can be not only helpful to understand the essence of the formal model of internal measurement but also benefit for considering wider applications. In next section, we re-organize formal model of internal measurement based on an adjunction and make a path for a generalization.

3 TOWARD A GENERALIZATION

We concern how to generalize the existing formal model of internal measurement in this section. First we see that the operations of obtaining fixed points or using fixed points can be positioned in a more general and formal framework. Formally, obtaining a fixed point P can be described by defining the fixed point as an object that satisfies $T(P) = P$ for an operator T. In contrast, using an object cannot be described formally in general since we cannot define how to use an object definitely in indefinite environment. Therefore we consider a restricted case here; a formal way of describing how to use a map.

We concern maps from a set A to a set B. Then an adjunction $\mathrm{Hom}(\{*\} \times A, B) \simeq \mathrm{Hom}(\{*\}, B^A)$ holds [20], where $\{*\}$ is a singleton set, $\mathrm{Hom}(Y, Z)$ is the set of maps from a set Y to a set Z, B^A is the set of maps from B to A and $\simeq$ is a bijection defined by the following map; $\phi : \mathrm{Hom}(\{*\} \times A, B) \rightarrow \mathrm{Hom}(\{*\}, B^A)$ with $\phi(f) = \hat{f}$ for a map $f : \{*\} \times A \rightarrow B$, where $\hat{f}(*) = f^{'}$ and $f^{'}(x) = f(*, x)$ for $x \in A$. In choosing an element of $\mathrm{Hom}(\{*\}, B^A)$ it goes that we get a map from A to B as an operation that transform inputs to outputs (Figure 3. right hand side). On the other hand, each element in $\mathrm{Hom}(\{*\} \times A, B)$ defines a map from A to B as a collection of input-output pairs $(x, y),\ x \in A, y \in B$ (Figure 3. left hand side). The adjunction says that defining a map as an operation is equivalent to defining the map as a collection of input-output pairs. In other words, a function as a whole is equivalent to a function as a collection of parts. How can we interpret a use of a map in this framework? Given a map f from A to B, we define an element of $\mathrm{Hom}(\{*\} \times A, B)$ f_* as $f_*(*, x) = f(x)$ for $x \in A$ and an element of $\mathrm{Hom}(\{*\}, B^A)$ $\hat{f}$ as $\hat{f}(*) = f$. Then we have $f_*(*, x) = \mathrm{ev}(\hat{f}(*), x)$ for any $x \in A$, where ev is a map from $B^A \times A$ to B defined by $\mathrm{ev}(f, x) = f(x)$ for $f \in B^A$ and $x \in A$. The map ev is called evaluation map in category theory. By definition, $f_*(*, x) = \mathrm{ev}(\hat{f}(*), x)$ is equivalent to $f(x) = \mathrm{ev}(f, x)$. Hence using a map f, that is, applying $f \in B^A$ to $a \in A$ means applying the evaluation map to the pair (f, x). On the other hand, the existence of evaluation map ev is equivalent to holding the adjunction $\mathrm{Hom}(\{*\} \times A, B) \simeq \mathrm{Hom}(\{*\}, B^A)$ [20]. Therefore we interpret the use of a map $f \in B^A$ as a change of view points for the map f from the right hand side of the adjunction to the left hand side.

Now we can see the interface between obtaining a fixed point and using a fixed point as an interface between obtaining a fixed point and holding an adjunction. We re-examine the example in section two in terms of alteration between obtaining a fixed point and holding an adjunction $\mathrm{Hom}(\{*\} \times X, X) \simeq \mathrm{Hom}(\{*\}, X^X)$, where $X = [0, 1]$. We summarize the procedure in four steps:

(i) An input-output pair (x^t, x^{t+1}) is given as a partial information about f^t as an element of $\mathrm{Hom}(\{*\} \times X, X)$.

(ii) A contraction map T that entails incompleteness of measurement is constructed.

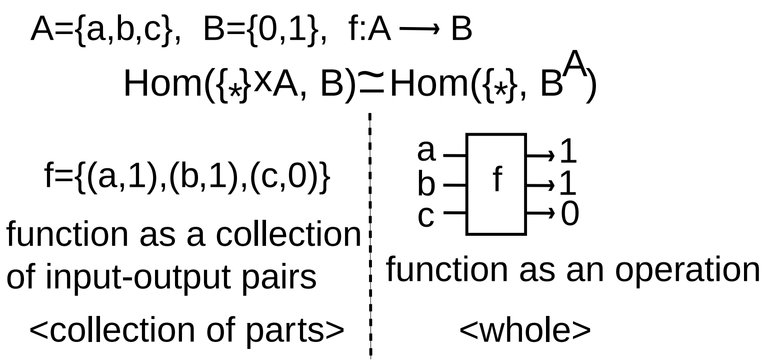

FIGURE 3
An explanation of adjunction. We here set $A = \{a, b, c\}, B = \{0, 1\}$ and $f(a) = 1, f(b) = 1, f(c) = 0$. The adjunction $\mathrm{Hom}(\{*\} \times A, B) \simeq \mathrm{Hom}(\{*\}, B^A)$ implies that the function f as a collection of input-output pairs $\{(a, 1), (b, 1), (c, 0)\}$ is equivalent to the function f regarded as an operation which sends a to 1, b to 1 and c to 0.

(iii) A subset S_∞ of $X \times X$ is obtained as a fixed point of T (from a part to a whole).

(iv) S_∞ is regarded as an element of X^X and $\hat{S_\infty} \in \mathrm{Hom}(\{*\}, X^X)$ is defined.

(v) $\hat{S_\infty}(*)$ is used as the time evolution rule at the next time step f^{t+1} and a new input-output pair (x^{t+1}, x^{t+2}) is obtained as a part of S_∞ regarded as an element of $\mathrm{Hom}(\{*\} \times X, X)$(from a whole to a part).

Repetition of this procedure constructs the time evolution of the system.

We can address the procedure in formal model of internal measurement in terms of two relations of a whole to its parts. The adjunction $\mathrm{Hom}(\{*\} \times A, B) \simeq \mathrm{Hom}(\{*\}, B^A)$ means that the following two viewpoints to a map $f : A \to B$ is equivalent; (i) f is the collection of state pairs $(a, f(a)), a \in A$ (left hand side, the collection of parts). (ii) f is an operation (right hand side, a whole). Or rather, the whole is constructed so that it is equivalent to the collection of its parts since the definition of B^A is an object C that satisfies $\mathrm{Hom}(\{*\} \times A, B) \simeq \mathrm{Hom}(\{*\}, C)$ in category theory. Therefore the whole in the adjunction is a whole that can be resolved to its parts. On the other hand, a fixed point of an operator T is a point where the distinction between object (parts) and observers (whole) is vanished. A whole that invalidates the distinction between parts and whole is expressed as a fixed point. In this sense, wholeness embodied by a fixed point cannot be resolved to its parts. To sum up, an adjunction represents wholeness that can be resolved to its parts and a fixed point represents wholeness that cannot be resolved to its parts. These two wholeness are connected in formal model of internal measurement.

In next section we address a construction of formal model of internal measurement on one-dimensional maps as a generalization of the example in section two.

4 A CONSTRUCTION ON ONE-DIMENSIONAL MAPS

In the previous section we find that an interface between obtaining fixed points and using fixed points in the existing formal model of internal measurement can be generalized to an interface between obtaining a fixed point and holding an adjunction. In this section we concern an example of generalized formal model of internal measurement. This construction is defined on one-dimensional maps. First we examine the construction on the family of Bernoulli maps.

4.1 General construction exemplified by Bernoulli maps

Let X and A be real intervals. We concern a family of maps f_a from X to X parameterized by $a \in A$. Such a family of maps can be seen as an element f of $\mathrm{Hom}(A \times X, X)$ by setting $f(a, x) := f_a(x)$ for $a \in A, x \in X$. In the realm of sets and maps, we have an adjunction $\mathrm{Hom}(A \times X, X) \simeq \mathrm{Hom}(A, X^X)$. In this adjunction a map $f : A \times X \to X$ is sent to a map in $\mathrm{Hom}(A, X^X)$ which sends $a \in A$ to $f_a \in X^X$. We apply the generalized procedure of formal model of internal measurement to this adjunction. We exemplify the construction by an example; the family of Bernoulli maps,

$$f(a, x) = \begin{cases} x/a & (0 \leq x \leq a), \\ (x-a)/(1-a) & (a < x \leq 1). \end{cases}$$

We set $X = [0, 1]$ and $A = (0, 1)$. First assume that $a \in A$ is arbitrarily fixed. Given a time evolution $x^{t+1} = f_a(x^t)$ from time t to time $t + 1$ by f_a, we regard that the input-output pair $((a, x^t), x^{t+1})$ is obtained as partial information of $f \in \mathrm{Hom}(X \times A, X)$. We shall define a contraction mapping T based on the datum and get an element of $\mathrm{Hom}(A, X^X)$ as a fixed point of T. If $a < x^t < 1$ then we set

$$T_1(x, y) = (ax, y),$$

$$\begin{aligned} T_2(x,y) &= ((x^t - a)x + a, x^{t+1}y), \\ T_3(x,y) &= ((1-x^t)x + x^t, (1-x^{t+1})y + x^{t+1}). \end{aligned}$$

We define $T(Y) = \bigcup_{i=1}^{3} T_i(Y)$ for a subset Y of $X \times X$. When $0 < a < x^t$ we set

$$\begin{aligned} T_1(x,y) &= (x^t x, x^{t+1}y), \\ T_2(x,y) &= ((a-x^t)x + x^t, (1-x^{t+1})y + x^{t+1}), \\ T_3(x,y) &= ((1-a)x + a, y). \end{aligned}$$

The definition of the map T which define a coarse graining of the product of phase space X has arbitrariness but here we obey the following procedure (CCM).

(CCM): construction of contraction mapping First we divide the graph of f_a into maximal monotone curves and cover each curve by a rectangle defined by two terminal points of the curve. Second the rectangle L which contain the point (x^t, x^{t+1}) is divided into four smaller rectangles by two lines $x = x^t$ and $y = x^{t+1}$ and the two of them which cover the monotone curve covered by L are chosen. These two rectangles and the rectangles that cover maximal monotone curves other than L consist of the ultimate coarse graining.

The procedure (CCM) for construction of contraction mapping is also adopted in other examples below. The set $S^a_\infty := \bigcap_{n=0}^{\infty} T^n(X \times X)$ is a fixed point of T, that is, we have $T(S^a_\infty) = S^a_\infty$. S^a_∞ is a subset of $X \times X$, however, it is not an element of X^X since for $x \in X$ that defines an edge along y-axis of a block in $T^n(X \times X)$ there are multiple elements $y \in X$ such that $(x,y) \in S^a_\infty$. However, here we regard S^a_∞ as an element of X^X virtually and concern the condition that the adjunction $\mathrm{Hom}(A \times X, X) \simeq \mathrm{Hom}(A, X^X)$ holds. A statement that mathematically equivalent to the adjunction can be expressed that for any map $f : A \times X \to X$ there uniquely exists a map $\hat{f} : A \to X^X$ such that $f(a,x) = \mathrm{ev}(\hat{f}(a), x)$ holds for any $a \in A, x \in X$. We mimic this condition by requiring the equation $f(a^{'}, x^{t+1}) = \mathrm{ev}(S^a_\infty, x^{t+1})$ holds. $a^{'}$ must be chosen so that this equation holds since $\hat{f}(a) \neq S^a_\infty$ in general. If we write $a = a^t$ and $a^{'} = a^{t+1}$, the equation $f(a^{t+1}, x^{t+1}) = \mathrm{ev}(S^{a^t}_\infty, x^{t+1})$ can be regarded as a definition of a time evolution rule for parameter a. Thus we get a two-dimensional time evolutionary system $(a^{t+1}, x^{t+1}) = F(a^t, x^t)$. The whole procedure of the time evolution is summarized in Figure 4. In the case of Bernoulli maps, the

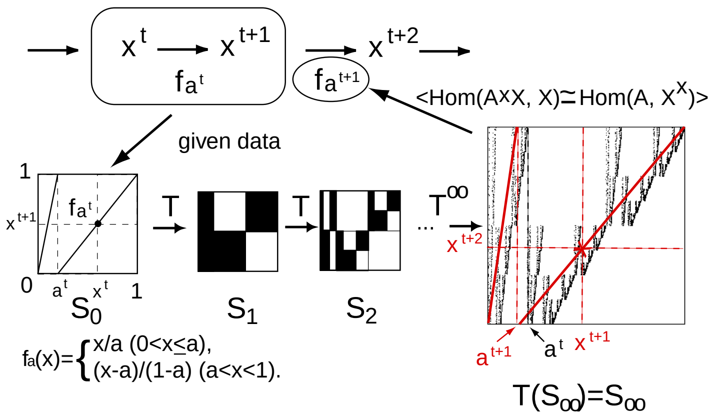

FIGURE 4

The time evolution procedure of the Bernoulli system as a formal model of internal measurement. A subset S_∞ of $X \times X$ is obtained as a fixed point of contraction mapping T. The use of S_∞ as the time evolution rule for at the next time step $t+1$ entails a condition mimicking $\mathrm{Hom}(A \times X, X) \simeq \mathrm{Hom}(A, X^X)$ that leads to define a time evolution rule for parameter a.

equation $f(a^{t+1}, x^{t+1}) = \mathrm{ev}(S_\infty^{a^t}, x^{t+1})$ becomes $\frac{x^{t+1} - a^{t+1}}{1 - a^{t+1}} = S_\infty^{a^t}(x^{t+1})$ or $a^{t+1} = \frac{x^{t+1} - S_\infty^{a^t}(x^{t+1})}{1 - S_\infty^{a^t}(x^{t+1})}$ if $S_\infty^{a^t}(x^{t+1}) < x^{t+1}$. In the same way we have $a^{t+1} = \frac{S_\infty^{a^t}(x^{t+1})}{x^{t+1}}$ if $S_\infty^{a^t}(x^{t+1}) > x^{t+1}$.

In general, the condition for existence of a^{t+1} is divided into the following three cases for each (a^t, x^t):

(i) a^{t+1} exists uniquely.

(ii) There are multiple values for a^{t+1}.

(iii) a^{t+1} does not exist (in the case a map $S_\infty^{a^t}$ cannot be defined on the point x^{t+1} or a^{t+1} cannot be defined in the set A though $S_\infty^{a^t}$ is defined on the point x^{t+1}).

Given a family of one-dimensional maps f, we call a point $(a^t, x^t) \in A \times X$ that satisfies the condition (i) or (ii) a defined point of f and the set of defined points of f is called the defined set of f. A point that satisfies the condition (iii) is called an undefined point of f and the set of undefined points of f is called the undefined set of f.

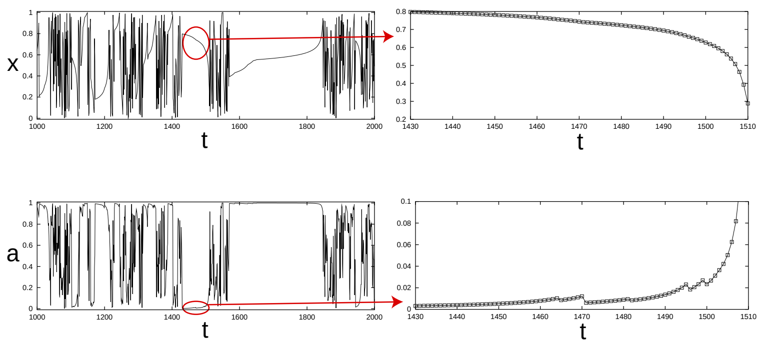

FIGURE 5
A time sequence of the Bernoulli system starting from randomly chosen initial condition from $t = 1000$ to $t = 2000$ in both x-coordinate (upper left) and a-coordinate (lower left) is depicted. A laminar phase from $t = 1430$ to $t = 1510$ is enlarged at right hand side.

In the case of family of Bernoulli maps f, a^{t+1} is uniquely determined for any defined point (a^t, x^t). We can prove that the undefined set of f has null two-dimensional Lebesgue measure. The proof depends on the fact that the two-dimensional Lebesgue measure of the set of null point of a real-coefficient polynomial with two variables is zero. Therefore we can repeat the time evolution rule defined above arbitrary times in numerical simulations. We shall ignore this null measure set in the following discussion. The numerical simulations are performed as follows: $x^{t+2} = z$ where z is chosen from the set $S_N^{a^t}(x^{t+1}) := \{y \in X | (x^{t+1}, y) \in S_N^{a^t}\}$ arbitrarily, $S_N^{a^t} := \bigcap_{n=0}^{N} T^n(X \times X)$ and N is the minimum number such that $\sup S_N^{a^t}(x^{t+1}) - \inf S_N^{a^t}(x^{t+1}) < \epsilon$. We set $\epsilon = 10^{-10}$ here.

A time sequence of the Bernoulli system as a formal model of internal measurement is shown in Figure 5. The dynamics of parameter a shows an intermittent behavior. It shows laminar phases near 0 or 1 and burst phases otherwise. In laminar phases, x increases or decreases monotonously on one hand, the dynamics of a shows a small scale zigzag motion as shown in Figure 5 on the other hand. Indeed, we can prove that if $x^t > a^t$ then $x^{t+1} < x^t$ and if $x^t < a^t$ then $x^{t+1} > x^t$ by the definition of x^{t+1}. The time evolution rule $(a^{t+1}, x^{t+1}) = F(a^t, x^t)$ commutes the transformation R which sends (a, x) to $(1 - a, 1 - x)$. That is, if (a^t, x^t) is mapped to (a^{t+1}, x^{t+1}) then $(1 - a^t, 1 - x^t)$ is mapped to $(1 - a^{t+1}, 1 - x^{t+1})$. So we investigate be-

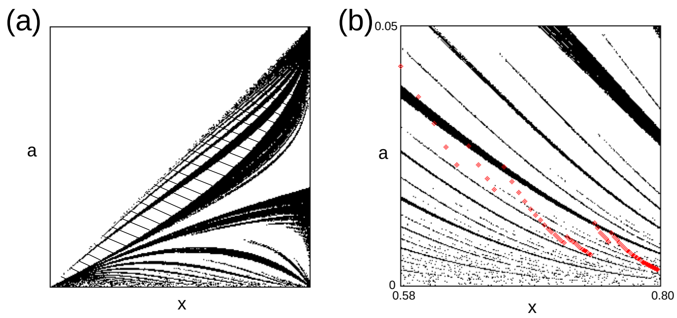

FIGURE 6
The structure of time evolution of the Bernoulli system depicted in x-a plane. (a) The shaded region corresponds to the set of points which escape from the region below the diagonal line (the region D). The black region and the white region below the diagonal line correspond to the set $\{(x^t, a^t)|a^{t+1} < a^t\}$ and the set $\{(x^t, a^t)|a^{t+1} > a^t\}$, respectively. (b) The enlarged time sequence in Figure 5 is depicted on x-a plane as a sequence of red points.

haviors in the region $D := \{(a, x)|a < x\}$ in what follows. As noted just above, x decreases monotonously in the region D. The necessary and sufficient condition for escape from the region D can be determined. That is, we have $x^{t+1} < a^{t+1}$ when $x^t > a^t$ if and only if there exists a natural number k such that $(a^t)^k x^t < x^{t+1} < (a^t)^k$. The condition is equivalent to the condition that (a^t, x^t) is in the set $\bigcup_{k=1}^{\infty}\{(a, x)|\frac{a}{1-a^k+a^{k+1}} < x < a + a^k - a^{k+1}\}$ since we have $x^{t+1} = \frac{x^t - a^t}{1-a^t}$ in the region D. This escape region is shown in Figure 6(a) as the shaded region. The black region in Figure 6(a) shows the set $\{(x^t, a^t)|a^{t+1} < a^t\}$ and the white region below the diagonal line shows the set $\{(x^t, a^t)|a^{t+1} > a^t\}$. The enlarged time sequence in Figure 5 is plotted on x-a plane in Figure 6(b). This picture is suggestive of how the small scale zigzag motion along a-coordinate arises. However, its detail mechanism is an outstanding problem for future works.

Figure 7 shows the frequency distribution of residence time for the region D. The distribution can scale as a power law with exponent -2.20. It could be expected that the small scale zigzag motion prolongs the residence time for the region D. The relationship between the small scale zigzag motion and the power law is also an outstanding issue.

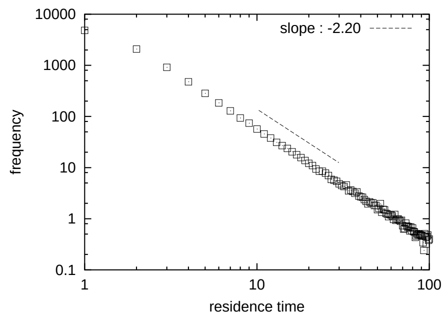

FIGURE 7
The frequency distribution of residence time for the region D (the region below the diagonal line in Figure 6(a)). The frequency is averaged over 100 different orbits of length 10^5. The distribution can scale as a power law with exponent -2.20.

4.2 Fractal structure of the defined sets

In the above example of the Bernoulli system, the undefined set has null Lebesgue measure. Next we concern two examples whose undefined sets have positive measures. The first example is a family of logistic maps,

$$f_a(x) = (3+a)x(1-x),$$

where $0 \leq x \leq 1$ and $0 < a < 1$. Given (a^t, x^t), a contraction mapping T is defined by obeying the procedure (CCM) as follows. If $0 < x^t < \frac{1}{2}$, we set

$$\begin{aligned} T_1(x,y) &= (x^t x, x^{t+1} y), \\ T_2(x,y) &= ((\frac{1}{2} - x^t)x + x^t, (\frac{3+a^t}{4} - x^{t+1})y + x^{t+1}), \\ T_3(x,y) &= (\frac{1}{2}x + \frac{1}{2}, \frac{3+a^t}{4}y). \end{aligned}$$

We define $T(Y) = \bigcup_{i=1}^{3} T_i(Y)$ for a subset Y of $X \times X$. The other case is defined similarly. The contraction mapping T defines the logistic system as a formal model of internal measurement. a^t does not exist if a point (x^t, x^{t+1}) is outside of the region bounded by the two curves $y = 3x(1-x)$ and $y = 4x(1-x)$. If (x^t, x^{t+1}) is inside of the region then a new logistic curve is drawn so that it contains the point (x^t, x^{t+1}). a^t is defined as the new parameter of this new curve. The defined set of the logistic system is shown

FIGURE 8
The defined set of the logistic system as a formal model of internal measurement. The defined points are depicted as black dots.

in Figure 8. The magnified drawings of the defined set near the center of Figure 8 are shown in Figure 9(a). It seems that the fine structure of the defined set has a finite scale near the center. The magnified drawings of the defined set near the upper left of Figure 8 are also shown in Figure 9(b). In contrast to near the center, it seems that the defined set has a infinitely fine structure near the upper left. Therefore it may be relevant that we calculate the fractal (box-counting) dimension of the boundary of the defined set.

Let K be the boundary of the defined set. The fractal (box-counting) dimension of K is defined as a non-negative real number

$$d = \lim_{\delta \to 0} \frac{\ln N(\delta)}{\ln \delta^{-1}}$$

if the limit in right hand side exists, where $N(\delta)$ is the minimum number of squares δ on a side needed to cover K. Here we calculate the fractal dimension of the boundary of the defined set by finding the uncertainty exponent [21, 22] of the defined set. Let $f(\epsilon)$ be the fraction of points (x, a) that are uncertain for ϵ-perturbations whether they are contained in the defined set or not. If $f(\epsilon) \sim \epsilon^{\alpha}$ holds then we obtain $\alpha = 2 - d$. For a smooth boundary it can be easily shown that $\alpha = 1$. On the other hand, we can expect that $0 < \alpha < 1$ for a fractal boundary. This relation can be understood intuitively as follows.

By the definition of the fractal dimension, we have $N(\epsilon) \sim \epsilon^{-d}$ for small

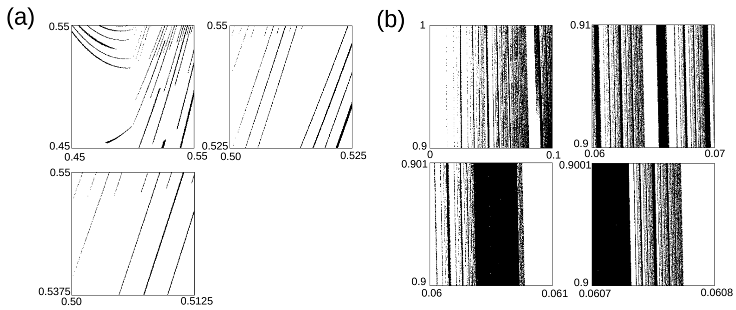

FIGURE 9
The magnified drawings of the defined set of the logistic system shown in Figure 8. (a) Near the center. (b) Near the upper left. It seems that the fine structure has a finite scale near the center on one hand, it has arbitrarily small scales near the upper left on the other hand.

ϵ. $f(\epsilon)$ is proportional to the area of the set $K(\epsilon)$ of points whose distances from K are within ϵ. Therefore $f(\epsilon)$ is of the order of $\epsilon^2 N(\epsilon)$ since the area of $K(\epsilon)$ will be of the order of the total area of all the $N(\epsilon)$ squares ϵ on a side needed to cover K. Hence we have $\epsilon^\alpha \sim f(\epsilon) \sim \epsilon^2 N(\epsilon) \sim \epsilon^{2-d}$ if $f(\epsilon)$ is proportional to ϵ^α.

We calculate the uncertainty exponent α for the boundary of the defined set K of the logistic system numerically. When α is obtained then the fractal dimension of K can be calculated as $d = 2 - \alpha$. The perturbation method adopted here is the most robust one among the methods proposed by [23]. For given a point (x, a), a number m of perturbed points on a circle of radius ϵ, centered at the point (x, a) are randomly selected. If all m perturbed points are either defined points or undefined points simultaneously, then the unperturbed point (x, a) is called certain. If not so, the unperturbed point (x, a) is called uncertain. We choose $m = 50$ here. The uncertainty exponent of α for the boundary of the defined set K of the logistic system can be estimated as $\alpha \approx 0.17$ (Figure 10). Hence we obtain the fractal dimension of K as $d \approx 2 - 0.17 = 1.83$.

The second example is a family of tent maps,

$$f(a, x) = \begin{cases} x/a & (0 \leq x \leq a), \\ (1-x)/(1-a) & (a < x \leq 1), \end{cases}$$

where $0 \leq x \leq 1$ and $0 < a < 1$. Given (a^t, x^t), a contraction mapping

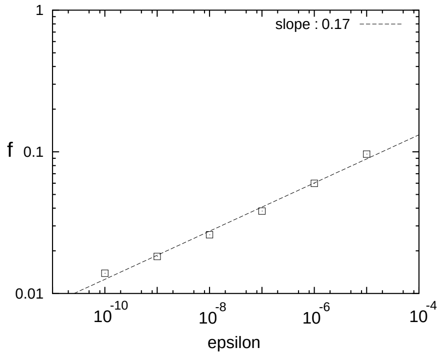

FIGURE 10
The fraction $f(\epsilon)$ of the uncertain points for ϵ-perturbations corresponding to the defined set of the logistic system is shown as a function of ϵ. We have $f(\epsilon) \sim \epsilon^{0.17}$ approximately.

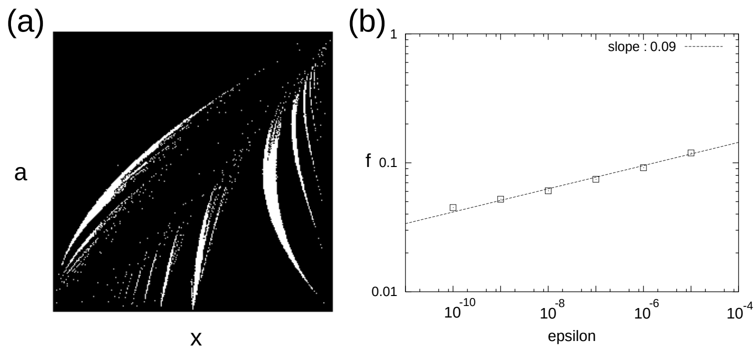

FIGURE 11
(a) The defined set of the tent system as a formal model of internal measurement. (b) The fraction $f(\epsilon)$ of the uncertain points for ϵ-perturbations corresponding to the defined set of the tent system. We have $f(\epsilon) \sim \epsilon^{0.09}$ approximately.

T is defined by obeying the procedure (CCM) in a similar fashion to the other examples. For the obtained tent system as a formal model of internal measurement, the defined set is shown in Figure 11(a). Note that if both $x^{t+1} > x^t$ and $x^{t+1} > 1 - x^t$ hold we have two candidate for the value of a^t. However, whether one of the two candidate is chosen is not relevant to the defined set. The uncertainty exponent for the boundary of the defined set of the tent system can be calculated as in the case of the logistic system (Figure 11(b)). The uncertainty exponent can be estimated as $\alpha \approx 0.09$. The corresponding fractal dimension is $d \approx 2 - 0.09 = 1.91$.

In this subsection we estimate the fractal dimensions of the boundaries of the defined sets for two specific examples of formal model of internal measurement. However, the calculation of uncertainty exponent may provide a basic tool for studying fractal structure of the defined set of any family of one-dimensional maps as a formal model of internal measurement.

5 CONCLUDING REMARKS

In this paper, we construct an alternating procedure between obtaining a map as a fixed point and holding an adjunction motivated by the problem of the gap between parts and wholeness in complex systems. Switching back and forth between a collection of parts (an adjunction) and wholeness (a fixed point) can be regarded as a metaphor for evolutionary processes in the broadest sense that an evolution entails pragmatic negotiations between syntax and semantics [24]. In this context, the relationship among an adjunction, a fixed point and the whole procedure of formal model of internal measurement can be comparable to the one among syntax, semantics and pragmatics. Here the pragmatics is linked to the persistence of a system itself.

The study of one-dimensional maps as formal models of internal measurement can lead to two different directions of investigations. The one is geometric studies of the procedure of formal model of internal measurement through a class of families of one-dimensional maps. In this direction, questions such as whether some unified dynamical properties or geometric properties of defined sets are found or not will be investigated. The other direction is focusing on the given family of one-dimensional maps in terms of "response" to the procedure of formal model of internal measurement. Such studies might provide new perspectives on the given family of one-dimensional maps. Neither of both directions are examined in earnest in this paper. However, we believe that the proposed numerical simulation method or the notion of defined sets provide a momentum for beginning a systematic investigation.

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