Local Cellular Automata as an Abstract Model of Self-Organizing Process

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Abstract. In this paper we concern an abstract model of self-organizing process called local cellular automata (LCA) proposed by us recently. The circular organization of living systems is addressed. A consistent circularity is defined by a closure operation on complete lattice. An inconsistent circularity is defined by a quasi-closure called weak closure implied by an internal perspective. Each cell in a LCA receives data about the time developments of its neighbors at one step before. It constructs a (in general incomplete) look-up table by taking closure (or weak closure) of the received data on an appropriate lattice. It applies obtained rule to its own present state and changes the state. In the former half of the paper, the theory which is the basis for LCA based on set lattice is reformulated in terms of complete lattice. In the latter half, we restrict cells' information receiving ability and define restricted local cellular automata (RLCA). The space-time patterns of RLCA are estimated by the variance of input-entropy over a span of time steps. The difference between closure driven RLCA and weak closure driven RLCA is discussed.

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1 INTRODUCTION

Living systems self-organize [8, 10, 11]. Self-organization means something constructs itself by itself. The former self can be seen a self as an operand and the latter can be seen a self as an operator. Both self cannot be separated in living systems. They are dependent on each other. It is a problem to formalize this circular organization of living systems. One of convincing attempts to address this problem is Rosen's (M,R) system [12]. An (M,R) system consists of abstracted three biological functions - metabolism, repair and self-replication. They are mutually dependent on each other (in Rosen's term 'entailed'). (M,R) system is closed under dependency relations in these biological functions. It is, therefore separated from its environment in terms of the biological functions (Of course, transmissions of materials and energy from/to the environment are allowed). Although this gives a form of the circular organization of living systems, such a description of living systems is too static. Real living systems are more dynamic. They interact with their environments, adapt to changes in the environments and learn from the interactions. Living systems are open to their changes resulting from interactions with their environments. It is not enough to focus on only the circular organization of living systems in order to address such dynamic aspects of living systems.

In (M,R) system a inseparability between an operator and an operand (circularity) is constructed consistently. However, in terms of internal perspective [3, 4, 5], the circularity can imply a kind of indefiniteness. So the circularity must be inconsistent in some sense. In our previous work [9], we proposed a definition of the indefiniteness and called it *autonomous indefiniteness*. We discussed how a complex behavior emerges when each element of a system is subject to autonomous indefiniteness. The system we proposed is an abstract model of self-organizing process called elementary local cellular automaton (ELCA). In this model, a 'consistent' circularity is defined by a closure operation on set lattice and an 'inconsistent' circularity that can imply indefiniteness is defined by modifying the closure operation by the partial universal quantifier [6, 7]. The partial universal quantifier defines an incomplete wholeness that can cause an inconsistent inseparability between the closure operation and the domain of the closure operation. The derived quasi-closure operation is called weak closure operation. We defined two systems, the one in which each cell possesses only the consistent circularity and the other in which each cell possesses the inconsistent circularity. The former system is called closure driven elementary local cellular automaton (CD-ELCA) and the latter is called weak closure driven elementary local cellular automaton (WCD-ELCA). In both systems, first each cell (which has two possible states, 0 or 1) receives data about the time developments of its nearest neighbors at one step before. For example, if the array of states of the nearest neighbors at present is (110) and that within the second nearest sites at the last time step is (11010) then the cell recives three pairs ((110),1), ((101),1) and ((010),0). Next it construct a look-up table for the possible eight triplets by taking closure (or weak closure) of the received data on a set lattice. This look-up table is incomplete in general. Finally it applies the obtained (incomplete) rule to the triplet at present time step and changes its state. We showed that CD-ELCA falls into ordered state under the periodic boundary condition. In contrast, WCD-ELCA shows generation and destruction of clusters of various sizes and the frequency distribution of cluster size can scale as power law with exponent -2.0.

In this paper, we concern two distinct ways which extend ELCA. The one enables us to define many-valued local cellular automata including two-valued ones. The other is a modification on rule construction at each cell in ELCA. As pointed out in our previous paper, we need to extend the theory of weak closure operation on set lattice to the framework of complete lattice in order to define many-valued local cellular automata. This can be easily done. We can translate the theory in set lattice into that in complete lattice directly. We discuss this issue in section 2. For the latter issue, which is concerned in section 3, we introduce restrictions into the information receiving ability of cells in ELCA. Given one of 256 look-up tables of elementary cellular automata, each cell can receive only the data that match the given table. Then we obtain 256 kinds of local cellular automata (LCA) for both closure driven case and weak closure driven case. We call the obtained LCA restricted local cellular automata (RLCA). The space-time patterns of RLCA are estimated by the variance of input-entropy over a span of time step [14]. They are classified into ordered, chaotic and complex ones by plotting the time average of input-entropy against the variance of it. The difference in spectrum of space-time patterns between closure driven RLCA and weak closure driven RLCA is shown.

2 CLOSURE AND WEAK CLOSURE IN COMPLETE LATTICE

The result in this section is an easy generalization of that in [9]. A proof of any statement is direct translation from that in set lattice to that in complete lattice. However, we give a proof for each statement in what follows for reader's convenience.

2.1 Closure

Let $(P, \leq)$ be a partially ordered set. Recall a map c from P to P is called *a closure operator on* P [1] if it satisfies following conditions for any $x, y \in P$: (i) $x \leq c(x)$, (ii) $x \leq y \Rightarrow c(x) \leq c(y)$, (iii) $c(c(x)) = c(x)$. A point x in P is called *closed* if $c(x) = x$ holds. We denote the set of all closed points in P by P_c.

Let L be a complete lattice with 0(the least element of L) and 1(the greatest element of L). That is, L is a poset in which every subset has both a join(least upper bound) and a meet(greatest lower bound). We write $\leq$ for the partial order on L. In what follows, we treat only meets. We write $\bigwedge T$ for the meet of a subset T. A subset S of L is called a *complete meet-semilattice* if every subset T of S has its meet in S. Then $1 = \bigwedge \emptyset$ is an element of any complete meet-semilattice.

The next theorem 1 is an analog for one in [1](theorem 7.3, p.147).

Theorem 1. Let c be a closure operator on a complete lattice with 0 and 1. Then L_c is a complete meet-semilattice of L. Conversely, for any complete meet-semilattice S of L if we define a map c_S from L to L by $c_S(a) = \bigwedge\{s \in S | a \leq s\}$ for $a \in L$, then c_S is a closure operator on L.

Proof. Let c be a closure operator on L. Consider a subset T of L_c. For any $t \in T$, we have $\bigwedge T \leq t$. Therefore, $c(\bigwedge T) \leq c(t) = t$ holds. This shows that $c(\bigwedge T)$ is a lower bound for T. Let $a \in L$ satisfy $a \leq t$ for any $t \in T$. Then we have $a \leq \bigwedge T \leq c(\bigwedge T)$. Hence $c(\bigwedge T)$ is the meet for T and this means $c(\bigwedge T) = \bigwedge T$.
Conversely, let S be a complete meet-semilattice of L. By the definition of c_S, it follows that $a \leq c_S(a)$ and if $a \leq b$ then $c_S(a) \leq c_S(b)$ for any $a, b \in L$. Since S is a complete meet-semilattice, $c_S(a) \in S$ and $c_S(c_S(a)) = \bigwedge\{s \in S | c_S(a) \leq s\} \leq c_S(a)$ for any $a \in L$. ∎

There exists one-to-one correspondence between the set of all closure operators on a complete lattice L with 0 and 1 and the set of all complete meet-semilattices of L. Actually, it is a consequence of theorem 1 as we show below.

First we prove that $c_{L_c} = c$ holds for a closure operator c on L. It is enough to show that $c(a) = \bigwedge\{s \in L_c | a \leq s\}$ for any $a \in L$. $c(a)$ is a lower bound for $\{s \in L_c | a \leq s\}$ since if $a \leq s$ for $s \in L_c$ then $c(a) \leq c(s) = s$ holds. Let t be a lower bound for $\{s \in L_c | a \leq s\}$. We have $c(a) \in L_c$ by $c(c(a)) = c(a)$. Hence $t \leq c(a)$ holds by $a \leq c(a)$ and this shows $c(a)$ is the meet.

Second we prove that $L_{c_S} = S$ holds for a complete meet-semilattice S of L. Since we have $c_S(s) = \bigwedge\{s' \in S | s \leq s'\} = s$ for $s \in S$, we get $s \in L_{c_S}$. Hence $S \subset L_{c_S}$ holds. On the other hand, since we have $t = c_S(t) = \bigwedge\{s \in S | t \leq s\}$ for $t \in L_{c_S}$ and S is a complete meet-semilattice, $t \in S$ holds. So we get $L_{c_S} \subset S$.

Next proposition 2 defines the closure of given subset S of a complete lattice L.

Proposition 2. Let L be a complete lattice with 0 and 1 and S be a subset of L. We define a map c_S from L to L by $c_S(a) = \bigwedge\{s \in S | a \leq s\}$ for $a \in L$. Then $\overline{S} := \{a \in L | c_S(a) = a\}$ is a complete meet-semilattice (therefore $c_{\overline{S}}$ is a closure operator on L).

Proof. For simplicity, we write c instead of c_S. Let T be a subset of $\overline{S}$. We have to show that $\bigwedge T \in \overline{S}$. By the definition of c, we have $\bigwedge T \leq c(\bigwedge T)$. The reverse inequality is shown as follows.

In the case $T = \emptyset$, we have $c(\bigwedge T) = \leq 1 = \bigwedge T$. Suppose $T \neq \emptyset$. Since we have $t \in \overline{S}$ for any $t \in T$, $t = c(t)$ holds. So we get $\bigwedge T = \bigwedge c(T) = \bigwedge_{t \in T} \bigwedge\{s \in S | t \leq s\}$. On the other hand, we have $c(\bigwedge T) = \bigwedge\{s \in S | \bigwedge T \leq s\} = \bigwedge\{s \in S | \bigwedge_{t \in T} \bigwedge\{s' \in S | t \leq s'\} \leq s\}$. For any $s \in S$ with $t \leq s$ for some $t \in T$, $\bigwedge_{t \in T} \bigwedge\{s' \in S | t \leq s'\} \leq s$ holds. Hence we get $c(\bigwedge T) \leq \bigwedge_{t \in T} \bigwedge\{s \in S | t \leq s\} = \bigwedge T$. ∎

We call $\overline{S}$ in proposition 2 *the closure of S in L*.

Given a subset S of a complete lattice L with 0 and 1, we have $S \subset \overline{S} = c_{\overline{S}}(\overline{S})$ by proposition 2 and the one to one correspondence between closure operators and meet-semilattices. Here, S and $\overline{S}$ can be considered to be operands and c_S and $c_{\overline{S}}$ are operators. All elements of S are fixed points of c_S and $\overline{S}$ is invariant by $c_{\overline{S}}$. Thereby, we refer to these facts as *consistent circularity*.

2.2 Weak Closure

Let G, M be nonempty sets and I be a binary relation on the product $G \times M$, i.e, I is a subset of $G \times M$. For any subset A of G, we define $\forall_p a \in A \overset{\text{def}}{\Leftrightarrow} a \in A \cap ((A')^c)^+$, where $A' := \{m \in M | \forall g \in A\ gIm\}$, $(A')^c := M \setminus A'$ and $((A')^c)^+ := \{g \in G | \exists m \in (A')^c\ s.t.\ gIm\}$. The symbol $\forall_p$ is called *the partial universal quantifier* [6, 7].

Originally G is a set of objects and M is a set of attributes in terms of formal concept analysis [2]. The triplet (G, M, I) defines a formal context. Given a subset A of G, A' is the set of all attributes that are common in all objects in A. $(A')^c$ is the set of attributes which are not possessed by at least one object in A. Finally, $((A')^c)^+$ is the set of objects which have at least one attribute which is not possessed by at least one object in A. Thus $a \in A \cap ((A')^c)^+$ means that a possesses at least one attribute that is not common in A. In other words, those which have only attributes common in A are discarded by $\forall_p$.

Let L be a complete lattice with 0 and 1. We consider the partial universal quantifier in the case $G = M = L$ and $I = \leq$. In this case we can omit the operation $^+$ in the definition of $\forall_p$.

Lemma 3. For any subset S of L, we have $((S')^c)^+ = (S')^c$.

Proof. Let $s \in ((S')^c)^+$. By the definition, there exists $t \in (S')^c$ such that $s \leq t$. Now suppose $s \in S'$. Then $a \leq s \leq t$ holds for any $a \in S$ so t is an element of S'. But this is impossible since we have $t \in (S')^c$. Hence $s \notin S'$ must hold and we get the inclusion $((S')^c)^+ \subset (S')^c$. The reverse inclusion immediately follows from the definition. ∎

Let L be a complete lattice with 0 and 1 and S be a subset of L. We define a map wc_S from L to L by $wc_S(a) = \bigwedge_{\forall_p s \in S_a} s$ for each $a \in L$, where $S_a = \{s \in S | a \leq s\}$. We call $\widetilde{S} := \{a \in L | wc_S(a) = a\}$ *the weak closure of S in L*.

The next theorem 4 determines the relation between the closure and the weak closure.

Theorem 4. Let L be a complete lattice with 0 and 1 and S be a subset of L.

(i) If $1 \in S$ then we have $\widetilde{S} = \overline{S}$.

(ii) If $1 \notin S$ then we have $\widetilde{S} = \overline{S} \setminus M$, where M is the set of all maximal element of S with respect to naturally induced order by L.

In order to prove theorem 4, we need some lemmas.

Lemma 5. For each $a \in L$, $S_a \neq \emptyset$ and $S_a \cap ({S_a}')^c = \emptyset$ hold if and only if S_a is a singleton set. If at least one of the two conditions is satisfied then the unique element of S_a is a maximal element of S.

Proof. Suppose $S_a \neq \emptyset$ and $S_a \cap ({S_a}')^c = \emptyset$. For any $s \in S_a$ $s \in {S_a}'$ holds since $S_a \subset {S_a}'$. Hence $t \leq s$ for any fixed $s, t \in S_a$. If we interchange s and t, we get $t \leq s$. This means that $s = t$ for any $s, t \in S_a$. Since we assume that $S_a \neq \emptyset$, S_a is a singleton set.

Conversely, suppose S_a is a singleton set. Since the unique point in S_a belogs to the set ${S_a}'$, $S_a \cap ({S_a}')^c = \emptyset$ holds. The remaining fact we have to show is that the unique element of S_a is a maximal element of S. Put $S_a = \{s\}$ and assume that s is not an element of S. There exists $t \in S$ such that $s \leq t$ and $s \neq t$. Then $t \in S_a$ since $a \leq s \leq t$. But this is impossible. ∎

Lemma 6. If $1 \in S$ then $S_a \cap ({S_a}')^c = S_a \setminus \{1\}$ for any $a \in S$.

Proof. Suppose $1 \in S$. $1 \in S_a$ and $1 \in {S_a}'$ hold. Then ${S_a}' = \{1\}$ holde since $1 \leq s$ for any $s \in {S_a}'$. Hence we get $S_a \cap ({S_a}')^c = S_a \setminus \{1\}$. ∎

Lemma 7. If $1 \notin S$ then

$$S_a \cap ({S_a}')^c = \begin{cases} S_a \setminus \{m\}, & \text{if } S_a \text{ has the maximum element } m. \\ S_a, & \text{otherwise.} \end{cases}$$

Proof. Assume that S_a has the maximum element m. Then $m \notin ({S_a}')^c$ holds since $m \in {S_a}'$. For any other $s \in S_a$ with $s \neq m$, $s \in ({S_a}')^c$ holds since $s \leq m$. Hence we get $S_a \cap ({S_a}')^c = S_a \setminus \{m\}$.

Next assume that S_a has no maximum element. Then there exists no $t \in S_a$ such that $s \leq t$ for any $s \in S_a$. So $S_a \cap {S_a}' = \emptyset$ and this means $S_a \subset ({S_a}')^c$. ∎

Proof of theorem 4. We devide the proof into four parts: (i) $\widetilde{S} \subset \overline{S}$. (ii) $1 \notin S \Rightarrow \widetilde{S} \subset \overline{S} \setminus M$. (iii) $1 \in S \Rightarrow \widetilde{S} \supset \overline{S}$. (iv) $1 \notin S \Rightarrow \widetilde{S} \supset \overline{S} \setminus M$.

(i) Let $a \in \widetilde{S}$. We prove $c_S(a) \leq a$. But we have

$$c_S(a) = \bigwedge_{s \in S_a} s, \quad wc_S(a) = \bigwedge_{s \in S_a \cap ({S_a}')^c} s$$

and $S_a \cap ({S_a}')^c \subset S_a$, so we get $c_S(a) \leq wc_S(a) = a$.

(ii) Let $1 \notin S$. Considering the result in case (i), it is enough to show that if $wc_S(a) = a$ for $a \in L$ then a is not a maximal element of S. Suppose $a \in L$ is maximal in S. $S_a \cap ({S_a}')^c = \emptyset$ holds by lemma 5. Then we have

$$a = wc_S(a) = \bigwedge_{s \in S_a \cap ({S_a}')^c} s = \bigwedge \emptyset = 1.$$

But this is impossible since $1 \notin S$ by assumption.

(iii) Let $1 \in S$. Suppose $a \in \overline{S}$. We show that $\bigwedge_{s \in S_a} s = \bigwedge_{s \in S_a \cap ({S_a}')^c} s$. We have $S_a \cap ({S_a}')^c = S_a \setminus \{1\}$ by lemma 6. Hence

$$\bigwedge_{s \in S_a \cap ({S_a}')^c} s = \bigwedge_{s \in S_a \setminus \{1\}} s = (\bigwedge_{s \in S_a \setminus \{1\}} s) \wedge 1 = \bigwedge_{s \in S_a} s.$$

(iv) Let $1 \notin S$ and $a \in \overline{S} \setminus M$.

If $a \in S$ then a is not a maximal element of S. Since we assume $1 \notin S$, there exists $s \in S$ such that $a \leq s, a \neq s, s \neq 1$. Suppose $a \in (S_a)'$ holds. Then we have $s \leq a$ but this is impossible since $a \neq s$. Hence we have $a \notin (S_a)'$ and this implies $a \in S_a \cap ({S_a}')^c$. So $wc_S(a) = \bigwedge_{s \in S_a \cap ({S_a}')^c} s = a$.

Next we consider the case $a \notin S$. We prove $\bigwedge_{s \in S_a} s = \bigwedge_{s \in S_a \cap ({S_a}')^c} s$. There is nothing to prove in the case $S_a = \emptyset$. So we assume $S_a \neq \emptyset$. Suppose $S_a \cap ({S_a}')^c = \emptyset$. By lemma 5, There exists $s \in S$ such that

$S_a = \{s\}$. Then we have $a = c_S(a) = s \in S$ but this is impossible since we assume $a \notin S$. Therefore we have $S_a \cap ({S_a}')^c \neq \emptyset$. When S_a does not have the maximum element, $S_a \cap ({S_a}')^c = S_a$ holds by lemma 7. On the other hand, if S_a has the maximum element m, $S_a \cap ({S_a}')^c = S_a \setminus \{m\}$ holds. Since we have $S_a \cap ({S_a}')^c \neq \emptyset$,

$$\bigwedge_{s \in S_a \cap ({S_a}')^c} s = \bigwedge_{s \in S_a \setminus \{m\}} s = \Big(\bigwedge_{s \in S_a \setminus \{m\}} s\Big) \wedge m = \bigwedge_{s \in S_a} s.$$

∎

In general, the weak closure $\widetilde{S}$ of S is not the closure of S as one can see by theorem 4. However, $\widetilde{S}$ is a complete meet-semilattice of L.

Theorem 8. Let L be a complete lattice with 0 and 1. Then the weak closure $\widetilde{S}$ for any subset S of L is a complete meet-semilattice of L.

Proof. If $1 \in S$ then $\widetilde{S} = \overline{S}$ by theorem 4 so $\widetilde{S}$ is a complete meet-semilattice. Let $1 \notin S$. We write M for the set of all maximal elements of S. By theorem 4, we have $\widetilde{S} = \overline{S} \setminus M$. Take a subset T of $\widetilde{S}$. Since $\overline{S}$ is a complete meet-semilattice of L, $\bigwedge T \in \overline{S}$ holds. If $T = \emptyset$ then $\bigwedge T = \bigwedge \emptyset = 1 \in \widetilde{S}$. We show that $\bigwedge T \notin M$ when $T \neq \emptyset$. We write m for $\bigwedge T$. Suppose $m \in M$. We have $m \leq t$ for any $t \in T$. If there exist $t \in T$ such that $t \in S$ then either $m = t$ or $t = 1$ holds by the maximality of m. $m = t$ must hold by the assumption. But this is impossible since we have $m \notin \widetilde{S}$. Hence $t \notin S$ for any $t \in T$. We have $t = wc_S(t) = \bigwedge_{s \in S_t \cap ({S_t}')^c} s$ since we assume $T \subset \widetilde{S}$. Suppose there exists $t \in T$ such that $S_t \cap ({S_t}')^c \neq \emptyset$. Then there exists $s \in S_t \cap ({S_t}')^c \subset S$ such that $m \leq t \leq s$. By the maximality of m, it follows that $m = s$ and $t = m$. But this is impossible since $t \notin S$. Hence we have $S_t \cap ({S_t}')^c = \emptyset$ for any $t \in T$. This means $t = 1$ for any $t \in T$. So we get $1 = \bigwedge T = m$ but this is impossible since $m \notin \widetilde{S}$. Therefore $m \notin M$ holds. ∎

In contrast to the closure, we have $S \not\subset \widetilde{S} = wc_{\widetilde{S}}(\widetilde{S})$ by theorem 4 in general. $\widetilde{S}$ is invariant by $wc_{\widetilde{S}}$ on one hand, there exist some elements of S that are not fixed points of wc_S on the other hand. Thereby, we refer to these facts as *inconsistent circularity*.

Given a function on a subset S of a complete lattice, we define the extensions of the function on $\overline{S}, \widetilde{S}$. Let L, M be complete lattices with 0 and 1 and f be a map from $S \subset L$ to M. *The extension of f on $\overline{S}$* is defined as a map $\overline{f}$ from $\overline{S}$ to M which sends each $a \in \overline{S}$ to $\overline{f}(a) = \bigwedge_{s \in S_a} f(s)$. In the same way, *the extension of f on $\widetilde{S}$* is a map $\widetilde{f}$ from $\widetilde{S}$ to M defined by $\widetilde{f}(a) = \bigwedge_{\forall_p s \in S_a} f(s)$ for each $a \in \widetilde{S}$. Here we use the term 'extension'. However, The values of $\overline{f}$ (or $\widetilde{f}$) do not coincide with those of f on S since we define $\overline{f}$ and $\widetilde{f}$ so that they become order-preserving maps.

Proposition 9. Let L, M be complete lattices with 0 and 1. Given a subset S of L and a map f from S to M. Then $\overline{f}$ and $\widetilde{f}$ are order-preserving maps from $\overline{S}$ and $\widetilde{S}$ to M, respectively.

Proof. Let $a \leq b \in \overline{S}$. We have $S_b \subset S_a$, so $\overline{f}(a) = \bigwedge_{s \in S_a} f(s) \leq \bigwedge_{s \in S_b} f(s) = \overline{f}(b)$. We also have ${S_a}' \subset {S_b}'$ by $S_b \subset S_a$. Therefore we have $({S_b}')^c \subset ({S_a}')^c$ and $\widetilde{f}(a) = \bigwedge_{s \in S_a \cap ({S_a}')^c} f(s) \leq \bigwedge_{s \in S_b \cap ({S_b}')^c} f(s) = \widetilde{f}(b)$. ∎

Proposition 10. Let L, M be complete lattices with 0 and 1. Given a subset S of L and a map f from S to M.

(i) If $1 \in S$ then $\overline{f}(a) = \widetilde{f}(a) \wedge f(1)$ for each $a \in \overline{S} = \widetilde{S}$.

(ii) If $1 \notin S$ then

$$\overline{f}(a) = \begin{cases} \widetilde{f}(a) \wedge f(m), & \text{if } S_a \text{ has the maximum element } m. \\ \widetilde{f}(a), & \text{otherwise.} \end{cases}$$

for each $a \in \widetilde{S} \subset \overline{S}$.

Proof. (i) follows from lemma 6 and (ii) follows from lemma 7 immediately. ∎

3 RESTRICTED LOCAL CELLULAR AUTOMATA

A local cellular automaton is defined as follows [9]. We concern only one-dimensional and nearest neighbor interaction case. Each cell in the automaton takes its value in a complete lattice V. In this paper, V is always $\{0, 1\}$.

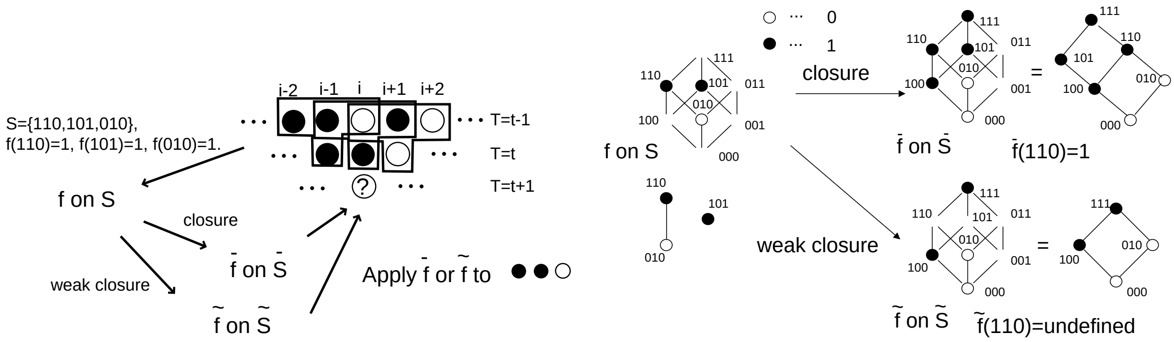

FIGURE 1. The time developmental procedures at site i and time t in both CD- and WCD-LCA are shown (Left). Circles filled in black and white on triplets consisting of 0 and 1 denote 1 and 0, respectively. The function f is an abbreviation for f_i^t, and so on. Functions $f, \overline{f}$ and $\widetilde{f}$ are represented by Hasse diagrams with colored circles(Right).

The state of a cell at time t and site i is represented by $a_i^t \in V$, where t is a non-negative integer, $1 \leq i \leq N$ and N is the size of the one-dimensional lattice. Fix a cell at time t and site i. It determines its state at time $t+1$ (i.e. a_i^{t+1}) by following procedure. First it receives information about the last time development at its neighbors. That is, three pairs of triplet and state value $((a_{i+k-1}^{t-1}, a_{i+k}^{t-1}, a_{i+k+1}^{t-1}), a_{i+k}^t)$ for $k = -1, 0, 1$. Put $S_i^t = \{(a_{i+k-1}^{t-1}, a_{i+k}^{t-1}, a_{i+k+1}^{t-1}) | k = -1, 0, 1\}$. Next it defines a map f from $S_i^t \subset V \times V \times V$ to V by $f_i^t(a_{i+k-1}^{t-1}, a_{i+k}^{t-1}, a_{i+k+1}^{t-1}) = a_{i+k}^t$ with $k = -1, 0, 1$. If there are two distinct values of f_i^t for a triplet then the value of f_i^t for the triplet is arbitrarily chosen from the two values. The map f_i^t is extended to $\overline{S_i^t}$ (or $\widetilde{S_i^t}$) written by $\overline{f_i^t}$ (resp. $\widetilde{f_i^t}$). Finally, it applies the obtained map $\overline{f_i^t}$ (resp. $\widetilde{f_i^t}$) to the triplet $(a_{i-1}^t, a_i^t, a_{i+1}^t)$ and get the state value of the cell at site i and time step $t+1$ written by a_i^{t+1}. If the triplet $(a_{i-1}^t, a_i^t, a_{i+1}^t)$ is not contained in the set $\overline{S_i^t}$ (resp. $\widetilde{S_i^t}$) then the value of a_i^{t+1} is arbitrarily chosen from the set V. If the rule construction of each cell is given by the closure operation, then the local cellular automaton is called closure driven local cellular automaton (CD-LCA). If the rule construction of each cell is given by the weak closure operation, then the local cellular automaton is called weak closure driven local cellular automaton (WCD-LCA). For example, if

$$
\begin{aligned}
(a_{i-2}^{t-1}, a_{i-1}^{t-1}, a_i^{t-1}, a_{i+1}^{t-1}, a_{i+2}^{t-1}) &= (1,1,0,1,0), \\
(a_{i-1}^t, a_i^t, a_{i+1}^t) &= (1,1,0),
\end{aligned}
$$

then $S_i^t = \{(1,1,0), (1,0,1), (0,1,0)\}$, $f_i^t(1,1,0) = f_i^t(1,0,1) = 1$ and $f_i^t(0,1,0) = 0$ (figure 1). By proposition 2, $\overline{S_i^t} = \{(0,0,0), (0,1,0), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$. By the definition of $\overline{f_i^t}$, $\overline{f_i^t}(0,0,0) = \overline{f_i^t}(0,1,0) = 0$ and $\overline{f_i^t}(1,0,0) = \overline{f_i^t}(1,0,1) = \overline{f_i^t}(1,1,0) = \overline{f_i^t}(1,1,1) = 1$. Therefore $a_i^{t+1} = \overline{f_i^t}(1,1,0) = 1$ in the CD-LCA. On the other hand, since $a_i^{t+1} = \widetilde{f_i^t}(1,1,0)$ is not defined, the value is chosen from the set $\{0,1\}$ randomly (in probability 0.5 for each value 0 and 1) in the WCD-LCA. Actually, $\widetilde{S_i^t} = \{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$ by theorem 4 since the set of all maximal elements in S_i^t is $\{(1,0,1), (1,1,0)\}$. Therefore $\widetilde{f_i^t}$ is not defined on $(1,1,0)$ though $f_i^t(1,1,0)$ is defined as 1.

Now we define restricted local cellular automata (RLCA). They are modified LCA in which each cell has restricted information receiving ability. Given a look-up table of elementary cellular automaton, each cell can receive only the pairs of triplet and state value which match the given rule table. The set S_i^t, therefore becomes smaller than that of ordinary LCA in general. For example, consider the case, $(a_{i-2}^{t-1}, a_{i-1}^{t-1}, a_i^{t-1}, a_{i+1}^{t-1}, a_{i+2}^{t-1}) = (1,1,0,1,0)$, $(a_{i-1}^t, a_i^t, a_{i+1}^t) = (1,1,0)$ and the restriction rule table is given by following table ;

000	001	010	011	100	101	110	111
0	1	0	0	1	1	0	0

Then the pairs $((1,0,1),1)$ and $((0,1,0),0)$ match the rule but the pair $((1,1,0),1)$ does not match the rule since 110 is bound to 0 in the table. Therefore $S_i^t = \{(1,0,1), (0,1,0)\}$ in this circumstance. The remaining rule construction procedure in RLCA is the same as that in ordinary LCA(figure 2). We get 256 closure driven RLCA (CD-RLCA) and the same number of weak closure driven RLCA (WCD-RLCA). We number the restriction rules as well as ordinary ECA's rule numbering [13]. The numbers are called *restriction rule numbers* (RRN's). For example, RRN $= 2 + 16 + 32 = 50$ for the above example.

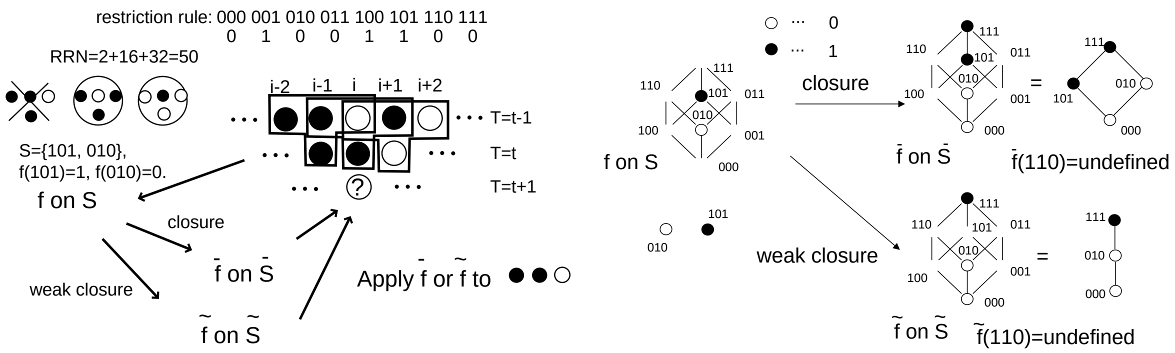

FIGURE 2. The time developmental procedures at site i and time t in both CD- and WCD-RLCA with RRN=50 are shown (Left). Coressponding $f, \overline{f}$ and $\widetilde{f}$ are also shown(Right).

Now we state some easily derivable facts on RLCA from the definition. If the difference of RRN's between two CD-RLCA is just 128 then the two systems have the same time development rule. This is because whether S_i^t contains $(1,1,1)$ or not $(1,1,1)$ always belongs to $\overline{S_i^t}$. This fact is not true for WCD-RLCA in general. If $(a_{i-1}^t, a_i^t, a_{i+1}^t) = (1,1,1)$ then $a_i^{t+1} = 1$ for any tuple of five $(a_{i-2}^{t-1}, a_{i-1}^{t-1}, a_i^{t-1}, a_{i+1}^{t-1}, a_{i+2}^{t-1})$ independent on restriction rules in both closure and weak closure driven systems. So the homogeneous state in which all cells' states are 1 is a fixed point of any RLCA with the periodic boundary condition. If RRN is less than 128 then a cell sitting at site i and time t in the system cannot receive $((1,1,1),1)$. Suppose that the cell receives $((1,1,1),1)$. Then S_i^t does not contain $(1,1,1)$ and $\overline{S_i^t} \neq \widetilde{S_i^t}$ by theorem 4. So we can expect that the difference between CD-RLCA and WCD-RLCA is strengthen if the RRN is less than 128.

We estimate the space-time patterns of RLCA with the periodic boundary condition and disordered initial configurations by the variance of input-entropy over a span of time steps. According to Wuensche [14], we can classify the space-time patterns into ordered, chaotic and complex ones by plotting the time average of input-entropy against the variance of it. The input-entropy at time t is given by

$$E^t = -\sum_{i=1}^{8} \left(\frac{Q_i^t}{N} \times \log \frac{Q_i^t}{N}\right)$$

where N is the system size and Q_i^t is the amount of ith triplet at time t. The classification can be described as follows.

Ordered: Lower average entropy and low variance.

Complex: High variance of input-entropy.

Chaotic: Higher average entropy and low variance.

For each RRN both the closure system and the weak closure system are classified into one of the above classes respectively. So each RRN can be mapped to one of nine possible types in the following table.

c \ wc	ordered	complex	chaotic
ordered	○	○	○
complex	×	○	○
chaotic	×	○	○

'C' and 'wc' are abbreviations of closure system and weak closure system respectively. The circle at first low and second column means that there exists an example of RRN with which the closure system shows ordered dynamics and the weak closure system shows complex one. The cross at second low and first column means that there is no example of RRN with which the closure system shows complex dynamics and the weak closure system shows ordered one, and so on. Since weak closure systems are less deterministic than closure ones, it is plausible that there is no RRN such that (c,wc)=(complex, ordered) or (chaotic, ordered).

Figure 3 shows examples of RLCA corresponding to existing seven types of RRN. The graphs along right hand sides of space-time patterns are time series of input-entropy at each time step. Note that there are some RRN like RRN=18

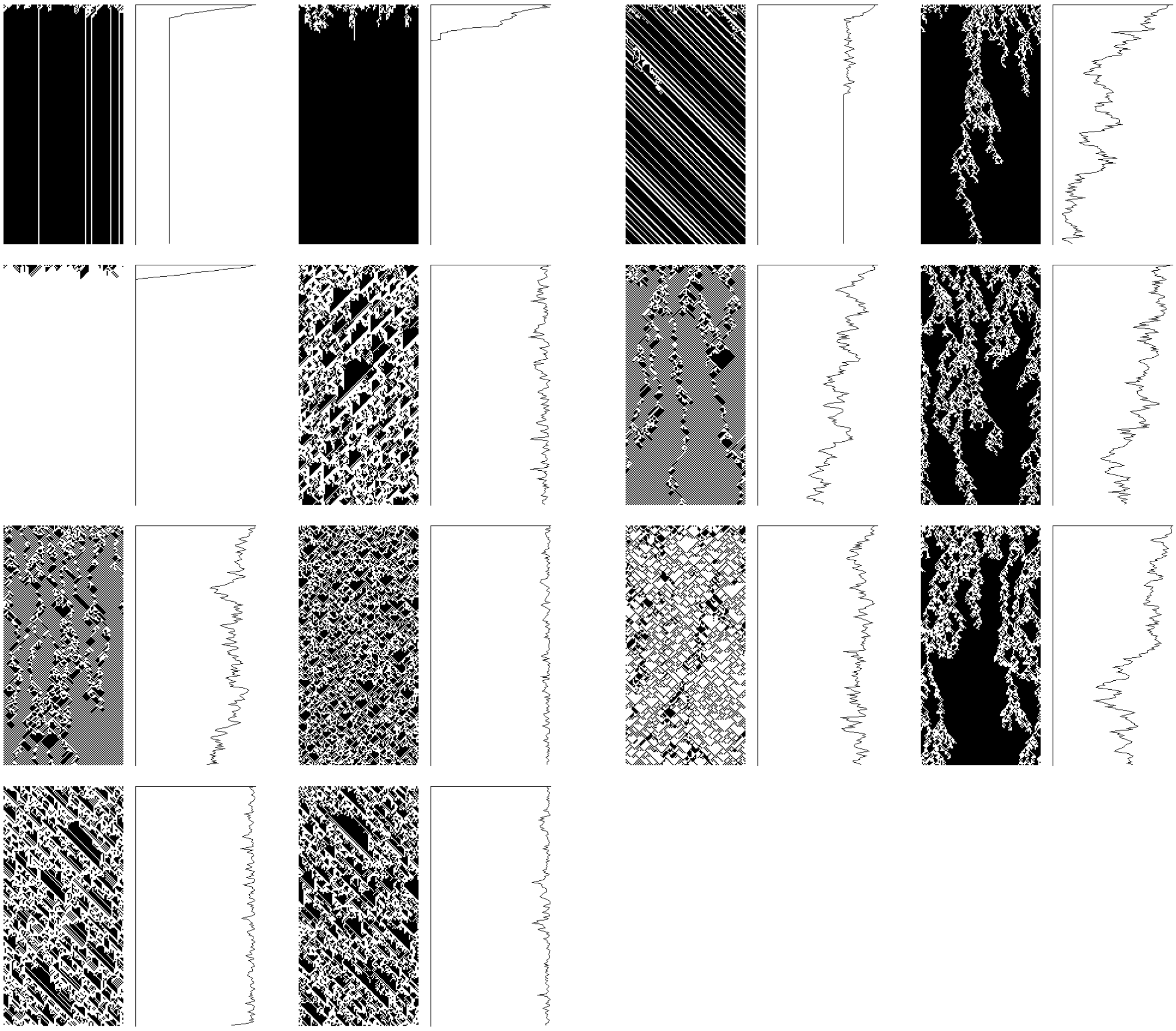

FIGURE 3. Examples of RLCA corresponding to existing seven types of RRN. The system size is 100 and the first 200 time steps from disordered initial configurations are shown. The graphs along right hand sides of space-time patterns are time series of input-entropy at each time step. The upper left two are closure system (left) and weak closure system (right) with RRN=217, which is an example of (c, wc)=(ordered, ordered). (c,wc)=(ordered, complex) for the upper right two with RRN=47, (c,wc)=(ordered, chaotic) for the left two in the second row with RRN=224, (c,wc)=(complex, complex) for the right two in the second row with RRN=51, (c,wc)=(complex, chaotic) for the left two in the third row with RRN=179, (c,wc)=(chaotic, complex) for the right two in the third row with RRN=18 and (c,wc)=(chaotic, chaotic) for the bottom two with RRN=169.

with which closure systems belong to the chaotic class and weak closure systems belong to the complex class though weak closure systems are less deterministic than corresponding closure systems in terms of time development rules.

The variance of input-entropy is plotted against the average of input-entropy in figure 4. Only a few CD-RLCA shows higher variance among closure sytems (0.05-0.1) on one hand, many WCD-RLCA shows high variance (more than 0.1) on the other hand. For example, there are 96 WCD-RLCA in which the variance exceeds 0.1 among weak closure systems with which RRN is less than 128. They visually show complex space-time patterns with localized structures as one can see in some examples in figure 3. The positions of examples in figure 3 are also shown in figure 4. We must tune some unknown parameters in order to find complex space-time patterns in CD-RLCA. Contrary to this, we can easily find complex space-time patterns without any parameter tuning in WCD-RLCA. WCD-RLCA with lower average entropy (less than around 2.5) are relaxed to the homogeneous state in which all cells are black from disordered initial configurations. Complex space-time patterns appear in their relaxation process. Thereby the spread both in variance and average entropy in figure 4 reflects the difference in transient lengths of weak closure systems.

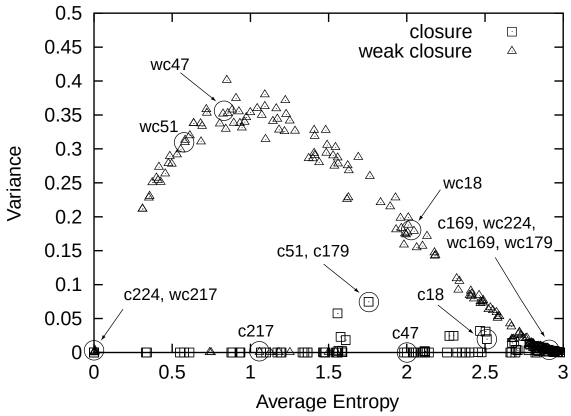

FIGURE 4. The variance of input-entropy vs the average of input-entropy. Data for all 256 CD-RLCA (squares) and WCD-RLCA (triangles) are plotted. The system size is 150. First 50 time steps from disordered initial configurations are discarded and the input-entropy is calculated from next 400 time steps. In the calculation, the 400 steps are divided into 80 windows consisting of 5 successive time steps and the input-entropy is calculated for each window. The variance and the average of input-entropy are averaged over 100 different initial disordered configurations. Positions of examples in figure 3 are indicated. For example, 'wc18' means WCD-RLCA with RNN=18.

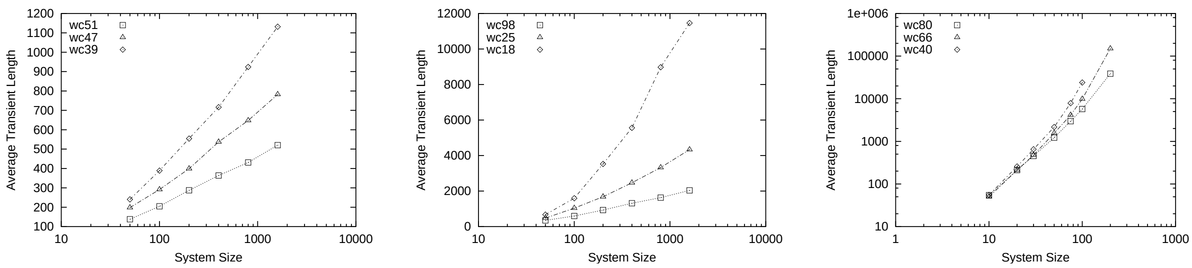

FIGURE 5. Average transient length vs system size. The transient length are averaged over 500 different initial disordered configurations. Left: WCD-RLCA with average entropy$= 0.5-1.5$. Middle: WCD-RLCA with average entropy$= 1.5-2.5$. Right: WCD-RLCA with average entropy≈ 2.5.

The dependency of the transient length on the system size is shown in figure 5 for some complex WCD-ELCA. The higher the average entropy is, the longer the transient length is. However in lower average entropy range (less than around 2.5) the transient length increases only the order of logarithm of the system size.

4 CONCLUDING REMARKS

In this paper two ways of extending elementary local cellular automaton are shown. One is the formalization in complete lattice which enables us to define many-valued LCA. The study of many-valued LCA by computer simulation is remained for a future work. The other is a modification on the rule construction procedure at each cell in LCA and define restricted local cellular automata(RLCA). In RLCA a cell's information receiving ability is restricted by one of 256 look-up table of elementary cellular automata. The difference in space-time patterns between CD-RLCA and WCD-RLCA is addressed in terms of the variance of input-entropy. Only a few CD-RLCA show complex space-time patterns on one hand, many WCD-RLCA show complex space-time patterns on the other hand. The result suggests that an inconsistent circularity in the form of the weak closure operation is one of significant factors for the emergence of complex space-time patterns with localized structures.

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