Autonomous indefiniteness versus external indefiniteness: Theory of weak topped $\cap$-structure and its application to elementary local cellular automaton

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Abstract

We propose a theory to formalize the indefinite features of living systems in the framework of weak topped $\bigcap$-structure. This theory contains two notions of indefiniteness, one is called external indefiniteness and the other is called autonomous indefiniteness. The former is defined as the outside of fixed points of the closure operator and the latter is defined as the difference between the set of fixed points of the weakened closure operator and a given set which defines the closure. This theory is then applied to elementary local cellular automaton (ELCA) in which the time development of its cell is driven by observing the dynamics of its nearest neighbors at the previous time step, followed by taking the closure (or the weak closure) in the appropriate space. The behavior of ELCA is characterized by its algebraic and statistical properties. In particular, we show that Self-Organized Criticality(SOC)-like behavior appears in ELCA driven by the weakened closure operator.

Key words: indefiniteness; closure operator; weak topped $\bigcap$-structure; local cellular automaton; self-organized criticality; open limit

1 Introduction

In theoretical studies of living systems such as autopoiesis and (M,R)-systems, the characteristic of circular organization has been the main focus in the past decades [1, 2, 3]. Circularity means that every biochemical processes in a biological network (such as the metabolic network) must be implemented by biochemical materials within the system itself. On the other hand, by referring to the Aristotelian categories of causation [4], Rosen pointed out that in general one cannot realize the efficient causality (operator) in terms of the material causality (operand). The Gödel's incompleteness theorem is an example of this assertion. Another simple example is the discordance $X \not\simeq \mathrm{Hom}(X, X)$ for some set X which has at least two points, where $\mathrm{Hom}(X, X)$ denotes the set of all endomaps on X.

However, if one attempts to describe the functions and relations in living systems, it is necessary to consider the situation when $X \simeq \mathrm{Hom}(X, X)$ holds, since biochemical processes in a living system evolve under finite speed of observation propagation [5, 6, 7, 8]. On the other hand, the condition $X \simeq \mathrm{Hom}(X, X)$ implies contradiction in the naive set theory. In order to resolve this problem, some attempts have been done in terms of internal perspective. For example, we can consider the self-reference problem (which corresponds to $X \simeq \mathrm{Hom}(X, X)$) and the frame problem at the same time. Both of these problems have no superficial relation to each other, but we can focus on the fact that they invalidate to each other's premise [9]. Formulations by alternate change between recursive definition and the domain equation [10], or by disequilibration process between tree- and loop-program [11, 12] have been proposed. Such studies address the indefinite boundary of the domain of circularity rather than circularity itself.

Indefiniteness which cannot be controlled from the inside of a system is usually treated as heat bath. The inside and outside of a system is clearly separated and the heat bath is described by probability distribution. In other words, the separation of the inside from the outside of a system is characterized by the difference of their logical status such as deterministic/non-deterministic. As mentioned above, it has been proposed that the indefinite boundary of living systems is not such kind of indefiniteness, but is induced from the inseparability between operand and operator resulting from the internal perspective (such as $X \simeq \mathrm{Hom}(X, X)$). Therefore, there are two notions of indefiniteness. Indefiniteness resulting from the separation of the inside and outside is called external while indefiniteness resulting from inseparability of operand and operator will be called autonomous. The main purpose of this paper is to formalize these notions of indefiniteness in terms of lattice theory and closure operation. We also discuss how they work in a local interactive system, elementary local cellular automaton which will be defined later.

The inside of a system, where all operations are closed and rigorously defined, can be constructed by collecting fixed points of an appropriate operator of the system (in terms of category theory, this operator is called a monad [13]). A closure operator defined on a set lattice is a simple example of monad [14]. Given a subset of a set lattice, an algebraic structure can be induced by defining a closure operator on this set lattice, followed by collecting the fixed points of the closure operator. This algebraic structure is called topped $\bigcap$-structure and it is closed under intersection. Therefore such a closure operator divides the whole world (the set lattice) into two parts definitely. The first part of the world can be reached by taking the intersection of elements in a given subset. On the other hand, the second part which contains the rest, cannot be reached by taking the intersection of elements of a given subset. This implies that the second part is indefinite if one stands inside the given subset. As a result, we can use a closure operator as a tool to separate the inside from the outside (or definiteness/indefiniteness).

Next we introduce the partial universal quantifier to mix up the two parts of the world separated by the closure operator. Originally, the partial universal quantifier is defined to formalize phenomenological computation [15, 16]. Phenomenological computation is not an ideal computation separated from the real world (i.e. syntax and semantics are invariant under computation, which are separated from each other. There also exist some correspondences between them such as completeness). In phenomenological computation, although syntax is invariant, semantics varies ad hoc (the so-called local semantics) with some consistency between objects and attributes (in category theory, this is called adjunction) at each computational step. Moreover, phenomenological computation is based on the notion of weak wholeness that is formalized in the context of formal concept analysis [17]. A formal concept is defined by a triplet which consists of a set of objects, a set of attributes and a binary relation between these two sets. Universal quantifier bounds all elements of a given subset of the set of objects, while partial universal quantifier only bounds the elements which have at least one extra attribute in addition to the one possessed by all elements of a given set of objects. Therefore, partial universal quantifier weakens the notion of wholeness.

In addition, if an operation contains the universal quantifier in its definition, we can weaken the operator by replacing it by the partial universal quantifier. The partial universal quantifier interferes with the operand of the operation. Indeed, in following sections we will show that in general the collection of fixed points of a weakened closure operator does not contain a given subset of the set lattice which is used to define the closure operator. In such case, the separation of the definite part (which contains the given subset) and the indefinite part (which cannot be reached from the definite part) does not hold any longer. Therefore, we can formalize the inseparability between operand and operator, and this inseparability induces another kind of indefiniteness which we call autonomous indefiniteness.

Finally, we implement the above ideas on elementary local cellular automaton (ELCA) in order to test the validity of the notion of indefiniteness proposed in this paper. Cellular automaton (CA) is a local interactive system in which both space and time are discrete [18, 19, 20]. A rule of CA is usually given in a priori and it is fixed over all cells at all time steps during a single run of CA. However, in this paper we will adopt different form of local interaction from the original CA in order to implement the closure operator. Similar to elementary cellular automaton, ELCA consists of many cells that are arranged on a one-dimensional discrete lattice with periodic (or random) boundary condition. Each cell can have value 0 or 1 as its state at each time step. The time development of a cell in ELCA is then driven by observing those of its nearest neighbors at the previous time step, followed by taking the closure (or the weak closure) in the appropriate space. We will also discuss the algebraic properties which characterize computations driven by both the ordinary closure operator and the weakened one. Moreover, we will discuss the statistical quantities of ELCA's which exhibit the difference between them.

2 Closure and weak topped $\bigcap$-structure

2.1 Closure and partial universal quantifier

A closure operator defined on an ordered set is an example of a monad in category theory [13]. A monad is an endofunctor T on some category $\mathcal{C}$ satisfying some commutative diagrams which correspond to the associative law and the identity law. If a monad is defined on a category $\mathcal{C}$, then an algebraic structure (such as monoid, module or semi-lattice, etc.) is realized as a T-algebra. Therefore, a monad can be considered as a tool to construct the mathematical structure which is closed under some definite operations. Here, we focus on a closure operator on the power set lattice for some set X ordered by the set inclusion. The general definition of a closure operator is as follows.

Definition 2.1.1 *Let $P = (P, \leq)$ be an ordered set. A map $c : P \longrightarrow P$ is called a* **closure operator** *on P if the following conditions are satisfied for all $x, y \in P$.*

(i) $x \leq c(x)$,

(ii) $x \leq y \Rightarrow c(x) \leq c(y)$,

(iii) $c(c(x)) = c(x)$.

We call $x \in P$ closed if $c(x) = x$. We denote the set of all closed elements in P by P_c. When $P = (\mathcal{P}(X), \subset)$ for some set X, we refer to a closure operator on P as a closure operator on X.

A topped $\bigcap$-structure on a set X is an algebraic structure on $\mathcal{P}(X)$ that is closely related to a closure operator on X. First, we give the definition and discuss the relationship between them later.

Definition 2.1.2 *Let X be a set and suppose $\mathcal{L} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X. $\mathcal{L}$ is called a* **topped $\bigcap$-structure** *on X if the following conditions are satisfied.*

(i) $X \in \mathcal{L}$.

(ii) For all non-empty families $\{A_i\}_{i \in I} \subset \mathcal{L}$, $\bigcap_{i \in I} A_i \in \mathcal{L}$.

We can define supremum and infimum in $\mathcal{L}$ as follows. $\mathcal{L}$ can be regarded as a complete lattice with the following definitions.

$\bigwedge_{i \in I} A_i := \bigcap_{i \in I} A_i$,

$\bigvee_{i \in I} A_i := \bigcap \{B \in \mathcal{L} | \bigcup_{i \in I} A_i \subset B\}$.

Now we describe the theorem that determine the relationship between a closure operator on a set X and a topped $\bigcap$-structure on X.

Theorem 2.1.3 *Let C be a closure operator on X, then*

$$\mathcal{L}_C := \{A \in \mathcal{P}(X) | C(A) = A\}.$$

is a topped $\bigcap$-structure on X.

Conversely, for arbitrary topped $\bigcap$-structure $\mathcal{L}$ on X, if we define a map from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ as

$$C_{\mathcal{L}}(A) := \bigcap\{B \in \mathcal{L} | A \subset B\},$$

then $C_{\mathcal{L}}$ is a closure operator on X.

Proof. Omitted. See [14]. ∎

From theorem 2.1.3, one can easily see that there exists a bijection between the set of all topped $\bigcap$-structures on X and the set of all closure operators on X. That is,

$$C_{(\mathcal{L}_C)} = C, \qquad \mathcal{L}_{(C_{\mathcal{L}})} = \mathcal{L}.$$

One can define the closure of a given subset $\mathcal{L}$ of $\mathcal{P}(X)$ by extending $\mathcal{L}$ so that it is closed under intersection. Precise description is as follows.

Proposition 2.1.4 *Let X be a set and suppose $\mathcal{L} \subset \mathcal{P}(X)$ is an ordered set by set inclusion. We define a map $C : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ as*

$$C(A) := \bigcap\{B \in \mathcal{L} | A \subset B\},$$

then

$$\overline{\mathcal{L}} := \{A \in \mathcal{P}(X) | C(A) = A\}$$

is a topped $\bigcap$-structure on X (and therefore $C_{\overline{\mathcal{L}}}$ is a closure operator on X). $\overline{\mathcal{L}}$ is called the **closure** *of $\mathcal{L}$.*

Proof. If $X \in \mathcal{L}$, then $C(X) = \bigcap\{B \in \mathcal{L} | X \subset B\} = X$. If $X \notin \mathcal{L}$, then $C(X) = \bigcap \emptyset = X$. In both case, $X \in \overline{\mathcal{L}}$ holds.

Next we will show that $\bigcap_{i \in I} A_i \in \overline{\mathcal{L}}$ holds for all non-empty families $\{A_i\}_{i \in I} \subset \overline{\mathcal{L}}$. It is enough to prove that $C(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} A_i$. $C(\bigcap_{i \in I} A_i) \supset \bigcap_{i \in I} A_i$ follows immediately from the definition of C. The reverse inclusion is shown as follows.

Since $A_i \in \overline{\mathcal{L}}$, we have $A_i = C(A_i)$. So

$$\begin{aligned} \bigcap_{i \in I} A_i &= \bigcap_{i \in I} C(A_i) \\ &= \bigcap_{i \in I} \bigcap \{B_i \in \mathcal{L} | A_i \subset B_i\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} C(\bigcap_{i \in I} A_i) &= \bigcap\{B \in \mathcal{L} | \bigcap_{i \in I} A_i \subset B\} \\ &= \bigcap\{B \in \mathcal{L} | \bigcap_{i \in I} C(A_i) \subset B\} \\ &= \bigcap\{B \in \mathcal{L} | \bigcap_{i \in I} \bigcap\{B_i \in \mathcal{L} | A_i \subset B_i\} \subset B\}. \end{aligned}$$

Since

$$\bigcap_{i' \in I} \bigcap \{B_{i'} \in \mathcal{L} | A_{i'} \subset B_{i'}\} \subset B_i$$

holds for all B_i with $A_i \subset B_i$, it follows that

$$C(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} \bigcap \{B_i \in \mathcal{L} | A_i \subset B_i\} = \bigcap_{i \in I} A_i.$$

∎

From the above discussion, the characters of the closure operator can be summarized in the following two aspects.

(i) It constructs a mathematical structure that is closed under some operations by collecting closed elements (i.e. fixed points of the closure operator).

(ii) It separates the structure from the whole world $\mathcal{P}(X)$ in the case of topped $\bigcap$-structure and ignores the rest of the world $(\mathcal{P}(X) \setminus \overline{\mathcal{L}})$ as long as one only concerns the intersection operation.

Therefore, if one only concerns the closure operator, the whole world $\mathcal{P}(X)$ can be clearly separated into the set of fixed points of the closure operator and its compliment. Moreover, if one stands inside the set of fixed points of the closure operator, the inside then can be regarded as the definite part of the whole world and the complement is indefinite in the sense that it cannot be reached by taking the intersection from the inside. We shall call such indefiniteness, which is defined as the outside of the definite part of the world, **external indefiniteness**.

Next we introduce the partial universal quantifier. The definition of partial universal quantifier $\forall_p$ is as follows [15].

Definition 2.1.5 *Let $M, G \neq \emptyset$ be sets and $I \subset G \times M$ (i.e. I is a binary relation on $G \times M$). For all $A \subset G$, we define*

$$\forall_p a \in A \overset{\text{def}}{\Leftrightarrow} a \in A \cap ((A^{'})^c)^+,$$

where $A^{'} := \{m \in M | \forall g \in A \ gIm\}$, $(A^{'})^c := M \setminus A^{'}$ and $((A^{'})^c)^+ := \{g \in G | \exists m \in (A^{'})^c \ s.t. \ gIm\}$.

Here, G is regarded as a set of objects and M is regarded as a set of attributes. The triplet (G, M, I) defines a formal context in terms of formal concept analysis [17]. Given $A \subset G$, the operation $A^{'}$ collects all attributes of all objects in A. $(A^{'})^c$ collects attributes which are not possessed by at least one object contained in A. Finally, $((A^{'})^c)^+$ collects objects which have at least one attribute which is not possessed by at least one object in A. In other words, objects which have no individuality in A cannot be in the domain where the partial universal quantifier bounds.

2.2 Weak topped $\bigcap$-structure

In this section, we consider the partial universal quantifier at $(G, M, I) = (\mathcal{P}(X), \mathcal{P}(X), \subset)$ for some set X. In this case, we can omit the operation $^+$ in the definition of the partial universal quantifier.

Lemma 2.2.1 *We have* $((\mathcal{A}^{'})^c)^+ = (\mathcal{A}^{'})^c$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$.

Proof. Let $E \in ((\mathcal{A}^{'})^c)^+$. From the definition, there exists $B \in (\mathcal{A}^{'})^c$ such that $E \subset B$. Assume that $E \in \mathcal{A}^{'}$, then $A \subset E \subset B$ for all $A \in \mathcal{A}$. It follows that $B \in \mathcal{A}^{'}$, but this contradicts $B \in (\mathcal{A}^{'})^c$. Therefore, $E \notin \mathcal{A}^{'}$ and this shows $((\mathcal{A}^{'})^c)^+ \subset (\mathcal{A}^{'})^c$.

The reverse inclusion is trivial. ∎

Now we define the weak closure on $\mathcal{P}(X)$ by the partial universal quantifier.

Definition 2.2.2 *Let X be a set and $\mathcal{L} \subset \mathcal{P}(X)$. We define a map $C_p : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ as follows. For $\mathcal{A} \in \mathcal{P}(X)$, define $\mathcal{B}_A := \{B \in \mathcal{L} | A \subset B\}$ and*

$$C_p(A) := \bigcap_{\forall_p B \in \mathcal{B}_A} B.$$

We define **weak topped $\bigcap$-structure** *relative to $\mathcal{L}$ on X (or the* **weak closure** *of $\mathcal{L}$) as*

$$\widetilde{\mathcal{L}} := \{A \in \mathcal{P}(X) | C_p(A) = A\}.$$

Next theorem characterizes the relation between the closure and the weak closure completely.

Theorem 2.2.3 *(i) If $X \in \mathcal{L}$, then $\widetilde{\mathcal{L}} = \overline{\mathcal{L}}$.*

(ii) If $X \notin \mathcal{L}$, then $\widetilde{\mathcal{L}} = \overline{\mathcal{L}} \setminus \mathcal{M}$, where $\mathcal{M}$ is the set of all maximal elements in $\mathcal{L}$.

In order to prove theorem 2.2.3, we need some lemmas.

Lemma 2.2.4 *For $A \in \mathcal{P}(X)$,*

$$\mathcal{B}_A \neq \emptyset \text{ and } \mathcal{B}_A \cap (\mathcal{B}_A^{'})^c = \emptyset \Leftrightarrow \mathcal{B}_A \text{ is a singleton set.}$$

If either side of the above statement holds, then the element of $\mathcal{B}_A$ is maximal in $\mathcal{L}$.

Proof. $(\Rightarrow)$ Assume $\mathcal{B}_A \neq \emptyset$ and $\mathcal{B}_A \cap (\mathcal{B}_A^{'})^c = \emptyset$. Since

$$\mathcal{B}_A \cap (\mathcal{B}_A^{'})^c = \emptyset \Leftrightarrow \mathcal{B}_A \subset \mathcal{B}_A^{'},$$

$B \in \mathcal{B}_A^{'}$ holds for all $B \in \mathcal{B}_A$. This implies $B^* \subset B$ holds for all $B, B^* \in \mathcal{B}_A$. If we interchange B with B^*, we have $B \subset B^*$. We then conclude that $B = B^*$ and that $\mathcal{B}_A$ is a singleton set because $\mathcal{B}_A \neq \emptyset$.

Define $\mathcal{B}_A = \{B\}$ and assume that B is not maximal in $\mathcal{L}$. Then there exists $B^* \in \mathcal{L}$ such that $B \subset B^*$ and $B \neq B^*$. Since $A \subset B$, $B^* \in \mathcal{B}_A$ holds. But this shows that $\mathcal{B}_A$ has at least two distinct elements, which contradicts to the fact that $\mathcal{B}_A$ is a singleton set. Therefore B is a maximal element in $\mathcal{L}$.

($\Leftarrow$) Trivial. ▮

Lemma 2.2.5 *If $X \in \mathcal{L}$, then $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A \setminus \{X\}$.*

Proof. Let $X \in \mathcal{L}$, then we have $X \in \mathcal{B}_A$ and $X \in \mathcal{B}'_A$. Since $X \in \mathcal{B}_A$, $X \subset E$ for all $E \in \mathcal{B}'_A$. Therefore we have $\mathcal{B}'_A = \{X\}$. This implies $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A \setminus \{X\}$. ▮

Lemma 2.2.6 *If $X \notin \mathcal{L}$, then*

$$\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \begin{cases} \mathcal{B}_A \setminus \{M\}, & \text{if } \mathcal{B}_A \text{ has the maximum element } M. \\ \mathcal{B}_A, & \text{otherwise.} \end{cases}$$

Proof. First assume that $\mathcal{B}_A$ has the maximum element M. Since $M \in \mathcal{B}'_A$, $M \notin (\mathcal{B}'_A)^c$. Since an element $B \in \mathcal{B}_A$ distinct from M satisfies $B \subset M$ and $B \neq M$, we have $B \in (\mathcal{B}'_A)^c$. Therefore $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A \setminus \{M\}$.

If $\mathcal{B}_A$ has no maximum element, then there exists no $M \in \mathcal{B}_A$ such that $B \subset M$ for all $B \in \mathcal{B}_A$. Hence $\mathcal{B}_A \cap \mathcal{B}'_A = \emptyset$ (i.e. $\mathcal{B}_A \subset (\mathcal{B}'_A)^c$). Therefore, $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A$ holds. ▮

Now we prove theorem 2.2.3.

Proof of Theorem 2.2.3 We divide the proof into four parts.

(i) $\widetilde{\mathcal{L}} \subset \overline{\mathcal{L}}$.

(ii) $X \notin \mathcal{L} \Rightarrow \widetilde{\mathcal{L}} \subset \overline{\mathcal{L}} \setminus \mathcal{M}$.

(iii) $X \in \mathcal{L} \Rightarrow \widetilde{\mathcal{L}} \supset \overline{\mathcal{L}}$.

(iv) $X \notin \mathcal{L} \Rightarrow \widetilde{\mathcal{L}} \supset \overline{\mathcal{L}} \setminus \mathcal{M}$.

(i) Assume that $A \in \widetilde{\mathcal{L}}$, we will show that $C(A) \subset A$. Since

$$C(A) = \bigcap_{B \in \mathcal{B}_A} B, \quad C_p(A) = \bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B$$

and $\mathcal{B}_A \supset \mathcal{B}_A \cap (\mathcal{B}'_A)^c$, so $A = C_p(A) \supset C(A)$ holds.

(ii) By considering the result of (i), it is enough to show that if $C_p(A) = A$ then A is not maximal in $\mathcal{L}$. Assume that A is a maximal element in $\mathcal{L}$, then $\mathcal{B}_A = \{A\}$ holds. By lemma 2.2.4, $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \emptyset$. Therefore, we have

$$A = C_p(A) = \bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B = \bigcap \emptyset = X.$$

On the other hand, since $A \in \mathcal{L}$ and $X \notin \mathcal{L}$, so we have a contradiction. Hence, A is not maximal in $\mathcal{L}$.

(iii) Assume that $A \in \overline{\mathcal{L}}$. To show $A \in \widetilde{\mathcal{L}}$, it is sufficient to show that $\bigcap_{B \in \mathcal{B}_A} B = \bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B$. By lemma 2.2.5, $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A \setminus \{X\}$. Therefore, we have

$$\bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B = \bigcap_{B \in \mathcal{B}_A \setminus \{X\}} B = \bigcap_{B \in \mathcal{B}_A \setminus \{X\}} B \cap X = \bigcap_{B \in \mathcal{B}_A} B$$

(iv) Assume that $A \in \overline{\mathcal{L}} \setminus \mathcal{M}$. We divide the proof into two cases, (a) when $A \notin \mathcal{L}$ and (b) when $A \in \mathcal{L}$.

(a) To prove $A \in \widetilde{\mathcal{L}}$, it is enough to show that $\bigcap_{B \in \mathcal{B}_A} B = \bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B$. The proof is trivial if $\mathcal{B}_A = \emptyset$, so we assume $\mathcal{B}_A \neq \emptyset$. If $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \emptyset$, then $\mathcal{B}_A = \{B\}$ holds by lemma 2.2.4. Therefore, we have $A = C(A) = B \in \mathcal{L}$ because $A \in \overline{\mathcal{L}}$. But this contradicts to $A \notin \mathcal{L}$. Hence, $\mathcal{B}_A \cap (\mathcal{B}'_A)^c \neq \emptyset$.

If $\mathcal{B}_A$ has no maximum element, then $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A$ by lemma 2.2.6. If $\mathcal{B}_A$ has the maximum element M, then $\mathcal{B}_A \cap (\mathcal{B}'_A)^c = \mathcal{B}_A \setminus \{M\}$. Since $\mathcal{B}_A \cap (\mathcal{B}'_A)^c \neq \emptyset$, we have

$$\bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B = \bigcap_{B \in \mathcal{B}_A \setminus \{M\}} B = \bigcap_{B \in \mathcal{B}_A \setminus \{M\}} B \cap M = \bigcap_{B \in \mathcal{B}_A} B.$$

(b) Since A is not maximal in $\mathcal{L}$ and $X \notin \mathcal{L}$, there exists $A^* \in \mathcal{L}$ such that $A \subset A^*$, $A \neq A^*$ and $A^* \neq X$. Note that $A, A^* \in \mathcal{B}_A$. If $A \in \mathcal{B}'_A$ then $A^* \subset A$ and this contradicts to $A \subset A^*$ and $A \neq A^*$, so $A \notin \mathcal{B}'_A$ holds. Therefore, we have $A \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c$ and it follows that

$$C_p(A) = \bigcap_{B \in \mathcal{B}_A \cap (\mathcal{B}'_A)^c} B = A.$$

This implies $A \in \widetilde{\mathcal{L}}$. ∎

From theorem 2.2.3, $\widetilde{\mathcal{L}}$ is not the closure of $\mathcal{L}$ if $X \notin \mathcal{L}$, this is because $\mathcal{L}$ is not a subset of $\widetilde{\mathcal{L}}$ in this case. But we can prove that $\widetilde{\mathcal{L}}$ is itself a topped $\bigcap$-structure on X.

Theorem 2.2.7 *$\widetilde{\mathcal{L}}$ is a topped $\bigcap$-structure on X.*

Proof. It is sufficient to prove the case when $X \notin \mathcal{L}$ since $\overline{\mathcal{L}}$ is a topped $\bigcap$-structure on X. $X \in \widetilde{\mathcal{L}}$ holds because $\widetilde{\mathcal{L}} = \overline{\mathcal{L}} \setminus \mathcal{M}$, $X \notin \mathcal{M}$ and $X \in \overline{\mathcal{L}}$, where $\mathcal{M}$ is the set of all maximal elements in $\mathcal{L}$.

Let $\emptyset \neq \{A_i\}_{i \in I} \subset \widetilde{\mathcal{L}}$. Since $\overline{\mathcal{L}}$ is a topped $\bigcap$-structure on X and $A_i \in \widetilde{\mathcal{L}} \subset \overline{\mathcal{L}}$ for all $i \in I$, $\bigcap_{i \in I} A_i \in \overline{\mathcal{L}}$ holds. Therefore it is enough to prove that $\bigcap_{i \in I} A_i \notin \mathcal{M}$ in order to show that $\bigcap_{i \in I} A_i \in \widetilde{\mathcal{L}}$.

Suppose $\bigcap_{i \in I} A_i \in \mathcal{M}$ holds. Put $M := \bigcap_{i \in I} A_i$. We then have $M \subset A_i$ for all $i \in I$. If there exists some $i \in I$ such that $A_i \in \mathcal{L}$, then $A_i = M$ or $A_i = X$ by the maximality of M. Since we assume $X \notin \mathcal{L}$, so $A_i = M$ holds. But this contradicts to $M \notin \widetilde{\mathcal{L}}$ and $A_i \in \widetilde{\mathcal{L}}$. So we have $A_i \notin \mathcal{L}$ for all $i \in I$. Since $A_i \in \widetilde{\mathcal{L}}$, $A_i = C_p(A_i) = \bigcap_{B \in \mathcal{B}_{A_i} \cap (\mathcal{B}'_{A_i})^c} B$ holds. Assume that there exists some $i \in I$ such that $\mathcal{B}_{A_i} \cap (\mathcal{B}'_{A_i})^c \neq \emptyset$, then there exists $B \in \mathcal{B}_{A_i} \cap (\mathcal{B}'_{A_i})^c \subset \mathcal{L}$ such that $M \subset A_i \subset B$.

By the maximality of M, $M = B$ holds and this implies $A_i = M$. However, this contradicts to $M \in \mathcal{L}$ and $A_i \notin \mathcal{L}$, and so $\mathcal{B}_{A_i} \cap (\mathcal{B}'_{A_i})^c = \emptyset$ holds for all $i \in I$. We therefore have $A_i = X$ for all $i \in I$. However, this implies $X = \bigcap_{i \in I} A_i = M$ and contradicts to $M \notin \widetilde{\mathcal{L}}$. So it follows that $\bigcap_{i \in I} A_i \notin \mathcal{M}$. ∎

We note that both the closure and weak closure construct algebraic structures that are closed under intersection, but the latter destroys the closure relation between $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ if $X \notin \mathcal{L}$. In taking the weak closure, the maximal elements of $\mathcal{L}$ are dropped from $\widetilde{\mathcal{L}}$ if $X \notin \mathcal{L}$. However, these dropped elements of $\mathcal{L}$ are necessary to construct $\widetilde{\mathcal{L}}$ in general. This exhibits the inseparability between the operator and the operand. If one stands in $\widetilde{\mathcal{L}}$, one cannot reach the dropped elements by taking the intersection of elements in $\widetilde{\mathcal{L}}$ because $\widetilde{\mathcal{L}}$ is closed under intersection, so the dropped elements seem to be indefinite from the inside of $\widetilde{\mathcal{L}}$. Nevertheless, these are elements of $\mathcal{L}$, and therefore such indefiniteness can be clearly distinguished from external indefiniteness since $\widetilde{\mathcal{L}} \subset \overline{\mathcal{L}}$ and external indefiniteness is defined by the set $\mathcal{P}(X) \setminus \overline{\mathcal{L}}$. As a result, the inseparability between operator and operand yields another kind of indefiniteness that is different from external indefiniteness. We call such indefiniteness **autonomous indefiniteness**.

Next we define the extension of a map on $\mathcal{L}$ to $\overline{\mathcal{L}}$ (or $\widetilde{\mathcal{L}}$).

Definition 2.2.8 *Let X, Y be sets and $\mathcal{L} \subset \mathcal{P}(X)$. For each map $f : \mathcal{L} \longrightarrow \mathcal{P}(Y)$, we define the extension of f on $\overline{\mathcal{L}}$ (or $\widetilde{\mathcal{L}}$) as follows.*

(i) The extension of f on $\overline{\mathcal{L}}$ is a map $\overline{f} : \overline{\mathcal{L}} \longrightarrow \mathcal{P}(Y)$ such that for $A \in \overline{\mathcal{L}}$,

$$\overline{f}(A) := \bigcap_{B \in \mathcal{B}_A} f(B).$$

Note that $A = C(A) = \bigcap_{B \in \mathcal{B}_A} B$ for $A \in \overline{\mathcal{L}}$.

(ii) The extension of f on $\widetilde{\mathcal{L}}$ is a map $\widetilde{f} : \widetilde{\mathcal{L}} \longrightarrow \mathcal{P}(Y)$ such that for $A \in \widetilde{\mathcal{L}}$,

$$\widetilde{f}(A) := \bigcap_{\forall_p B \in \mathcal{B}_A} f(B).$$

Note that $A = C_p(A) = \bigcap_{\forall_p B \in \mathcal{B}_A} B$ for $A \in \widetilde{\mathcal{L}}$.

Here we use the term "extension" but in general f does not coincide with $\overline{f}$ (or $\widetilde{f}$) on $\mathcal{L}$ since we define $\overline{f}$ and $\widetilde{f}$ so that they are order-preserving.

Proposition 2.2.9 *$\overline{f}$ and $\widetilde{f}$ are order-preserving maps on $\overline{\mathcal{L}}$ and $\widetilde{\mathcal{L}}$, respectively.*

Proof. First we consider $\overline{f}$. Let $A_1 \subset A_2 \in \overline{\mathcal{L}}$, then we have $\mathcal{B}_{A_1} \supset \mathcal{B}_{A_2}$. Therefore,

$$\overline{f}(A_1) = \bigcap_{B_1 \in \mathcal{B}_{A_1}} f(B_1) \subset \bigcap_{B_2 \in \mathcal{B}_{A_2}} f(B_2) = \overline{f}(A_2).$$

Next we consider $\widetilde{f}$. Assume $A_1 \subset A_2 \in \widetilde{\mathcal{L}}$, then $\mathcal{B}_{A_1} \supset \mathcal{B}_{A_2}$, $\mathcal{B}'_{A_1} \subset \mathcal{B}'_{A_2}$ and $(\mathcal{B}'_{A_1})^c \supset (\mathcal{B}'_{A_2})^c$ hold. Therefore,

$$\widetilde{f}(A_1) = \bigcap_{B_1 \in \mathcal{B}_{A_1} \cap (\mathcal{B}'_{A_1})^c} f(B_1) \subset \bigcap_{B_2 \in \mathcal{B}_{A_2} \cap (\mathcal{B}'_{A_2})^c} f(B_2) = \widetilde{f}(A_2).$$

∎

Finally the relation between $\overline{f}$ and $\widetilde{f}$ is characterized as follows.

Proposition 2.2.10 *(i) If $X \in \mathcal{L}$, then for $A \in \overline{\mathcal{L}} = \widetilde{\mathcal{L}}$, $\overline{f}(A) = \widetilde{f}(A) \cap f(X)$.*

(ii) If $X \notin \mathcal{L}$, then for $A \in \widetilde{\mathcal{L}} \subset \overline{\mathcal{L}}$

$$\overline{f}(A) = \begin{cases} \widetilde{f}(A) \cap f(M), & \text{if } \mathcal{B}_A \text{ has maximum element } M, \\ \widetilde{f}(A), & \text{otherwise.} \end{cases}$$

Proof. (i) immediately follows by lemma 2.2.5 and (ii) also follows by lemma 2.2.6. ∎

In this section, we define external indefiniteness as the outside of the set of fixed points of the closure operator. We also define autonomous indefiniteness as the set in which its elements are fixed points of the closure operator but are not fixed points of the weakened closure operator.

In the next section, in order to examine the validity of the notion of external and autonomous indefiniteness, we implement such a formulation on elementary local cellular automaton (ELCA) whose rules are locally constructed by the closure (or the weak closure) operator. We also discuss how the dynamics of ELCA are different between the cases above rule construction is driven by the ordinary closure and by the weak closure, both algebraically and statistically.

3 An application to elementary local cellular automaton

3.1 Definitions and algebraic properties

Usually, a rule of ECA is fixed during one run of ECA. But here, we adopt the formulation in which the rule is locally constructed site by site and step by step. We call such a cellular automaton as local cellular automaton (LCA). In elementary local cellular automaton (ELCA), the rule at a cell is constructed by the closure (or the weak closure) based on the time developments of the cell itself and its nearest neighbors at the previous time step. At first, the time development at each cell is defined as follows.

Definition 3.1.1 *Let $\Omega = \{0,1\}$ and suppose $(\mathbf{u}^*, \mathbf{b}) \in \Omega^5 \times \Omega^3$ is given. Here, Ω^n denotes the n-fold product $\Omega \times \cdots \times \Omega$. We also use the notation*

$$\begin{aligned} \mathbf{u}^* &= (u_1, u_2, u_3, u_4, u_5) \in \Omega^5, \\ \mathbf{u}_i &= (u_i, u_{i+1}, u_{i+2}) \in \Omega^3, \quad i = 1, 2, 3, \\ \mathbf{b} &= (b_1, b_2, b_3) \in \Omega^3, \end{aligned}$$

where b_2 denotes the state of a cell considered at the present time step and b_1 and b_3 are the states of the nearest cells of b_2. u_3 denotes the state of the cell at the previous time step, u_2 and u_4 are the state of the nearest cells of u_3. u_1 and u_5 are the states of the second-nearest cells of u_3.

We regard Ω as an ordered set by $0 \leq 1$. Also Ω^3 can be regarded as an ordered set by imposing the coordinate-wise order. We introduce the naturally induced lattice structures into Ω and Ω^3 from the partial orders. Since $\Omega^3 \simeq \mathcal{P}(X)$ holds as a lattice for any three points set $X = \{a, b, c\}$ and the infimum $\wedge$ is equivalent to the intersection $\cap$, we can define $\overline{\mathcal{L}}, \widetilde{\mathcal{L}}$ for $\mathcal{L} := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subset \Omega^3$ by proposition 2.1.4 and definition 2.2.2.

In addition, we define a map $f : \mathcal{L} \longrightarrow \Omega$ by

$$f(\mathbf{u}_i) = b_i, \quad i = 1, 2, 3,$$

where $\Omega \simeq \{\emptyset, Y\}$ and $\{\emptyset, Y\}$ denotes the power set of the singleton set Y. If there exists $i \neq j$ such that $\mathbf{u}_i = \mathbf{u}_j, b_i \neq b_j$, we choose 0 or 1 arbitrarily as the value of $f(\mathbf{u}_i)$. We will show later in this section that this choice does not affect the actual time development of cellular automaton (see proposition 3.1.4 below). Let $\overline{f}$ and $\widetilde{f}$ be extensions of f on $\overline{\mathcal{L}}$ and $\widetilde{\mathcal{L}}$, respectively. In order to construct a rule for the cell at the present time step, we choose maps $\overline{g}, \widetilde{g} : \Omega^3 \longrightarrow \Omega$ which satisfy for

$$\overline{g}|_{\overline{\mathcal{L}}} = \overline{f}, \quad \widetilde{g}|_{\widetilde{\mathcal{L}}} = \widetilde{f}.$$

Also we define $s := \overline{g}(\mathbf{b})$ (or $s := \widetilde{g}(\mathbf{b})$) as the state of the cell at the next time step.

The time development of whole system is described as follows.

Definition 3.1.2 *Let N be a natural number that represents the system size, we define a map τ^t : $\Omega^{N+4} \times \Omega^{N+2} \longrightarrow \Omega^N$ as follows:*

We take any $(a_i^{t-1})_{i=1}^{N+4} \times (a_i^t)_{i=1}^{N+2} \in \Omega^{N+4} \times \Omega^{N+2}$ (where t and i represents the time and the coordinate of the cell, respectively) with $1 \leq i \leq N$. By using the notation in definition 3.1.1, we put

$$\mathbf{u}^* = (a_i^{t-1}, a_{i+1}^{t-1}, a_{i+2}^{t-1}, a_{i+3}^{t-1}, a_{i+4}^{t-1}) \in \Omega^5, \quad \mathbf{b} = (a_i^t, a_{i+1}^t, a_{i+2}^t) \in \Omega^3, \quad s = a_i^{t+1}.$$

We also define

$$\tau^t((a_i^{t-1})_{i=1}^{N+4} \times (a_i^t)_{i=1}^{N+2}) := (a_i^{t+1})_{i=1}^N.$$

If $s = \overline{g}(\mathbf{b})$ for all i, then we define $\overline{\tau}^t := \tau^t$ and if $s = \widetilde{g}(\mathbf{b})$ for all i, then we define $\widetilde{\tau}^t := \tau^t$. $\overline{\tau}^t$ and $\widetilde{\tau}^t$ are called the **closure driven elementary local cellular automaton (CD-ELCA)** *and the* **weak closure driven elementary local cellular automaton (WCD-ELCA)**, *respectively.*

It is easy to see that the value of s can be determined by the pair $(\mathbf{u}^*, \mathbf{b})$, so we assign a number to each pair $(\mathbf{u}^*, \mathbf{b})$. Given $\mathbf{u}^* = (u_1, u_2, u_3, u_4, u_5) \in \Omega^5$ and $\mathbf{b} = (b_1, b_2, b_3) \in \Omega^3$, the table

$$\begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \\ & b_1 & b_2 & b_3 & \end{array}$$

is called the **local arrangement** and the **local arrangement number** is defined as follows,

$$RAN(\mathbf{u}^*, \mathbf{b}) := \sum_{k=1}^{5} u_k 2^{k-1} + \sum_{k=1}^{3} b_k 2^{k+4}.$$

From the above definition one can show that there exists 256 different local arrangements.

Now we give some examples of the local time development of ELCA.

Example 1. $RAN(\mathbf{u}^*, \mathbf{b}) = 0$. The local arrangement is

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0. & \end{matrix}$$

Thus $\mathcal{L} = \{(0,0,0)\}$ and $f : \mathcal{L} \longrightarrow \Omega$ is defined as $f((0,0,0)) = 0$. Moreover, we have $\overline{\mathcal{L}} = \{(0,0,0),(1,1,1)\}$, $\overline{f}((0,0,0)) = 0$ and $\overline{f}((1,1,1)) = 0$. Since $\mathbf{b} = (0,0,0)$, we have $s = \widetilde{f}(\mathbf{b}) = 0$. On the other hand, $(0,0,0)$ is a maximal element in $\mathcal{L}$, so we have $\widetilde{\mathcal{L}} = \{(1,1,1)\}$ and $\widetilde{f}((1,1,1)) = 0$. Since $\mathbf{b} = (0,0,0)$ is not in the domain of $\widetilde{f}$ and $\widetilde{f}(\mathbf{b})$ is indefinite, $s = \widetilde{g}(\mathbf{b})$ can be chosen from Ω arbitrarily.

Example 2. $RAN(\mathbf{u}^*, \mathbf{b}) = 43$. The local arrangement is

$$\begin{matrix} 1 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 0. & \end{matrix}$$

Thus, $\mathcal{L} = \{(1,1,0),(1,0,1),(0,1,0)\}$ and $f : \mathcal{L} \longrightarrow \Omega$ is defined by $f((1,1,0)) = 1$, $f((1,0,1)) = 0$ and $f((0,1,0)) = 0$. $\overline{\mathcal{L}}$ is equal to the set of all intersections of an arbitrary subset of $\mathcal{L}$ according to proposition 2.1.4. On the other hand, $(1,1,1)$ is the intersection of $\emptyset(\subset \mathcal{L})$ and each element of $\mathcal{L}$ is the intersection of itself. Therefore, we have $\mathcal{L} \cup \{(1,1,1)\} \subset \overline{\mathcal{L}}$. The remaining subsets of $\mathcal{L}$ just contains two points and $\mathcal{L}$ itself. Since

$$(1,1,0) \wedge (1,0,1) = (1,0,0),$$
$$(1,0,1) \wedge (0,1,0) = (0,0,0),$$
$$(0,1,0) \wedge (1,1,0) = (0,1,0),$$

and $(1,1,0) \wedge (1,0,1) \wedge (0,1,0) = (0,0,0)$, so $(1,0,0)$ and $(0,0,0)$ are also in $\overline{\mathcal{L}}$. Hence we have $\overline{\mathcal{L}} = \{(0,0,0),(1,0,0),(0,1,0),(1,0,1),(1,1,0),(1,1,1)\}$. Since the set of all maximal elements of $\mathcal{L}$ is $\{(1,1,0),(1,0,1)\}$, we have $\widetilde{\mathcal{L}} = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$.

Next we calculate the values of $\overline{f}(\mathbf{b})$ and $\widetilde{f}(\mathbf{b})$. We have $\mathbf{b} = (1,0,0)$ and $\mathcal{B}_{\mathbf{b}} := \{\mathbf{u} \in \mathcal{L} | \mathbf{b} \leq \mathbf{u}\} = \{(1,1,0),(1,0,1)\}$, so

$$\overline{f}(\mathbf{b}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}}} f(\mathbf{u}) = f((1,1,0)) \wedge f((1,0,1)) = 1 \wedge 0 = 0.$$

On the other hand, since $(1,1,1) \notin \mathcal{L}$ and $\mathcal{B}_{\mathbf{b}}$ does not have the maximum element, we have $\mathcal{B}_{\mathbf{b}} \cap (\mathcal{B}'_{\mathbf{b}})^c = \mathcal{B}_{\mathbf{b}}$ by lemma 2.2.6. Thus (see figure 1a),

$$\widetilde{f}(\mathbf{b}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}} \cap (\mathcal{B}'_{\mathbf{b}})^c} f(\mathbf{u}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}}} f(\mathbf{u}) = 0.$$

Example 3. $RAN(\mathbf{u}^*, \mathbf{b}) = 71$. The local arrangement is

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 0. & \end{matrix}$$

Since $\mathcal{L} = \{(1,1,1),(1,1,0),(1,0,0)\}$ is closed under intersection, $\mathbf{b} = (0,1,0) \notin \overline{\mathcal{L}}$. Hence both $\overline{f}(\mathbf{b})$ and $\widetilde{f}(\mathbf{b})$ are indefinite.

Example 4. $RAN(\mathbf{u}^*, \mathbf{b}) = 107$. The local arrangement is

$$\begin{matrix} 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0. & \end{matrix}$$

In this case, $\overline{f}(\mathbf{b}) = 1$ but $\widetilde{f}(\mathbf{b})$ is indefinite because $\mathbf{b} = (1,1,0)$ is a maximal element in $\mathcal{L} = \{(1,1,0),(1,0,1),(0,1,0)\}$ (see figure 1b).

As we have seen in the above examples, there are three types of local time development,

(i) definite by both the closure and the weak closure,

(ii) definite by the closure and indefinite by the weak closure, and

(iii) indefinite by both the closure and the weak closure.

In all 256 local arrangements, the number of type (i), type (ii) and type (iii) are 93, 49 and 114, respectively. Note that type (ii) is a realization of autonomous indefiniteness and type (iii) is a realization of external indefiniteness.

Next we prove a series of algebraic properties which characterize the dynamics of ELCA.

Lemma 3.1.3 *By using the notation in definition 3.1.1, if* $\mathbf{b} \in \overline{\mathcal{L}}$ *then for all* $i \in \{1,2,3\}$,

$$\mathbf{u}_i = (1,1,1) \Rightarrow b_i = 1.$$

Proof. Since $(b_1, b_2, b_3) = \mathbf{b} = \bigwedge_{\mathbf{b} \le \mathbf{u}_j \in \mathcal{L}} \mathbf{u}_j = (\wedge_{\mathbf{b} \le \mathbf{u}_j \in \mathcal{L}} u_j, \wedge_{\mathbf{b} \le \mathbf{u}_j \in \mathcal{L}} u_{j+1}, \wedge_{\mathbf{b} \le \mathbf{u}_j \in \mathcal{L}} u_{j+2})$, we have $1 = u_i \wedge u_{i+1} \wedge u_{i+2} \le b_i$. ∎

The next proposition guarantees that the conflict in the definition of f (see definition 3.1.1) does not affect the actual time development of ELCA.

Proposition 3.1.4 *If there exists* $i \neq j \in \{1,2,3\}$ *such that* $b_i \neq b_j$ *and* $\mathbf{u}_i = \mathbf{u}_j$, *then* $\mathbf{b} \notin \overline{\mathcal{L}}$.

Proof. The proof is too long to describe here, so we move it to the appendix. ∎

Proposition 3.1.5 *If* $\mathbf{b} = (1,1,1)$ *then* $\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b}) = 1$.

Proof. If $(1,1,1) \in \mathcal{L}$ then,

$$\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b}) = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} f(\mathbf{u}) = f(\mathbf{b}) = 1.$$

On the other hand, if $(1,1,1) \notin \mathcal{L}$, there exists no $\mathbf{u} \in \mathcal{L}$ such that $\mathbf{b} = (1,1,1) \leq \mathbf{u}$. Thus

$$\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b}) = \wedge \emptyset = 1. \blacksquare$$

From proposition 3.1.5, it is expected that both CD-ELCA and WCD-ELCA are inclined to form space-time clusters in which all cells have value 1. Indeed, the actual time developments of both CD-ELCA and WCD-ELCA show such a feature (see figure 2 and 3). With the periodic boundary condition, the behavior of CD-ELCA is simple and similar to that of the ordinary class 2 ECA (see figure 2a). After short transient time development, almost all runs converge to fixed points which are not homogeneous except the one which falls into the periodic orbits. The periods of these periodic orbits depend on the system size. Moreover, all local time developments are definite at the fixed points. On the other hand, the behavior of WCD-ELCA with periodic boundary condition seems to be highly complex (see figure 3a). One can see the creation and destruction of value 1 clusters of various sizes. Local time developments in the clusters and on their boundaries are definite, while those in the other regions are mixture of definite and indefinite zones. Thus the co-existence of definite and indefinite local time developments is an important factor for the creation and destruction of clusters.

In figure 2b and 3b, we show the time development of CD-ELCA and WCD-ELCA with the random boundary condition. The behavior of CD-ELCA differs from that with the periodic boundary. The perturbations at the boundary propagate through the whole space at maximum speed (one site per time step). If the perturbations flowing from both the left and right side of the boundary collide with each other, they disappear at the same time. Therefore the behavior of CD-ELCA is originally stable similar to the class 2 ECA. However, this stability is so fragile since perturbations flowing from the boundary propagate throughout the whole space. On the other hand, the behavior of WCD-ELCA with the random boundary is very similar to that with periodic boundary. This can be explained by the fact that some of the local arrangements which transmit perturbation flows become indefinite by the weak closure.

Proposition 3.1.6 *Let $\mathbf{b} \in \mathcal{L}$ be maximal in $\mathcal{L}$, then there exists $i \in \{1,2,3\}$ such that $u_{2i-1} = b_i = s$, where $s = \overline{f}(\mathbf{b})$.*

Proof. Since $\mathbf{b} \in \mathcal{L}$, there exists $i \in \{1,2,3\}$ such that $\mathbf{b} = \mathbf{u}_i$. That is, $(b_1, b_2, b_3) = (u_i, u_{i+1}, u_{i+2})$. Then we have $b_j = u_{i+j-1}$, hence $b_i = u_{2i-1}$ holds. On the other hand, $\overline{f}(\mathbf{b}) = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} f(\mathbf{u})$ and so $\mathbf{b}$ is maximal in $\mathcal{L}$. Therefore, we have $\overline{f}(\mathbf{b}) = f(\mathbf{b}) = f(\mathbf{u}_i) = b_i$. $\blacksquare$

We refer the property of local arrangements in proposition 3.1.6 as flow retaining property. For example, the local arrangement with $RAN(\mathbf{u}^*, \mathbf{b}) = 43$ in Example 2 has flow retaining property both in CD-ELCA and in WCD-ELCA because $u_5 = b_3 = \overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b}) = 0$. On the other hand, the local arrangement with $RAN(\mathbf{u}^*, \mathbf{b}) = 107$ in Example 4 has flow retaining property in CD-ELCA but it is broken in WCD-ELCA because we have $u_1 = b_1 = \overline{f}(\mathbf{b}) = 1$ while $\widetilde{f}(\mathbf{b})$ is indefinite. Since maximal elements in $\mathcal{L}$ become indefinite by the weak closure, this proposition says that some of the flow retaining local arrangements are indefinite in WCD-ELCA even though they are definite in CD-ELCA. On the relation between the flow retaining property of a local arrangement and the order-preserving property of f on $\mathcal{L}$, we can prove the following proposition and theorem. Since proposition 3.1.7 and theorem 3.1.8 below are less related to the main discussion in this paper, so the proofs of them are given in the appendix.

Proposition 3.1.7 *Let $\mathbf{b} \in \mathcal{L}$ and assume that f is an order-preserving map on $\mathcal{L}$. Then there exists $i \in \{1, 2, 3\}$ such that $u_{2i-1} = b_i = s$, where $s = \overline{f}(\mathbf{b})$.*

Theorem 3.1.8 *If $\mathbf{b} \in \overline{\mathcal{L}}$, then f is an order-preserving map on $\mathcal{L}$.*

Corollary 3.1.9 *Let $\mathbf{b} \in \mathcal{L}$, then the following statements are equivalent.*

(i) f is an order-preserving map on $\mathcal{L}$.

(ii) there exists $i \in \{1, 2, 3\}$ such that $u_{2i-1} = b_i = s$, where $s = \overline{f}(\mathbf{b})$.

Proof. This follows immediately from proposition 3.1.7 and theorem 3.1.8. ∎

Thus, whether f can have algebraically favorable property (order-preserving) is closely related to the flow retaining property of its local arrangements.

Finally we turn to the question if $\widetilde{f}(\mathbf{b})$ is equal to $\overline{f}(\mathbf{b})$ when $\mathbf{b} \in \overline{\mathcal{L}}$ and $\mathbf{b}$ is not maximal in $\mathcal{L}$. We will show that the answer to this question is true.

Proposition 3.1.10 *Assume that $\mathbf{b}$ is not maximal in $\mathcal{L}$ and $\mathbf{b} \in \overline{\mathcal{L}}$, then $\mathbf{b} \in \widetilde{\mathcal{L}}$ holds by theorem 2.2.3. Also we have $\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b})$.*

Proof. Define $\mathcal{B}_{\mathbf{b}} := \{\mathbf{u} \in \mathcal{L} | \mathbf{b} \le \mathbf{u}\}$, then we have

$$\overline{f}(\mathbf{b}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}}} f(\mathbf{u}), \quad \widetilde{f}(\mathbf{b}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}} \cap (\mathcal{B}'_{\mathbf{b}})^c} f(\mathbf{u}).$$

If $\mathcal{B}_{\mathbf{b}}$ does not have the maximum element, then $\mathcal{B}_{\mathbf{b}} \cap (\mathcal{B}'_{\mathbf{b}})^c = \mathcal{B}_{\mathbf{b}}$ and $\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b})$ by lemma 2.2.6. If $\mathcal{B}_{\mathbf{b}}$ has the maximum element $\mathbf{m}$, then $\mathcal{B}_{\mathbf{b}} \cap (\mathcal{B}'_{\mathbf{b}})^c = \mathcal{B}_{\mathbf{b}} \setminus \{\mathbf{m}\}$ by lemma 2.2.6. Therefore, we have $\overline{f}(\mathbf{b}) = \widetilde{f}(\mathbf{b}) \wedge f(\mathbf{m})$. When $\widetilde{f}(\mathbf{b}) = 0$, there is nothing to be proved. For $\widetilde{f}(\mathbf{b}) = 1$, it is enough to show that $f(\mathbf{m}) = 1$.

Suppose $\mathcal{B}_{\mathbf{b}} = \{\mathbf{m}\}$, then $\mathbf{b} = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}} \setminus \{\mathbf{m}\}} \mathbf{u} = \bigwedge \emptyset = (1, 1, 1)$ since $\mathbf{b} \in \widetilde{\mathcal{L}}$. Because $\mathbf{m} \in \mathcal{L}$, there exists some $i \in \{1, 2, 3\}$ such that $\mathbf{m} = \mathbf{u}_i$. Since $f(\mathbf{u}_i) = b_i = 1$ for all $i \in \{1, 2, 3\}$, we have $f(\mathbf{m}) = 1$.

Let $\mathcal{B}_{\mathbf{b}} \supset \{\mathbf{m}\}$, $\mathcal{B}_{\mathbf{b}} \neq \{\mathbf{m}\}$ and assume that $f(\mathbf{m}) = 0$. Since $\mathbf{u} \le \mathbf{m}$ for all $\mathbf{u} \in \mathcal{B}_{\mathbf{b}} \setminus \{\mathbf{m}\}$, we have $f(\mathbf{u}) \le f(\mathbf{m}) = 0$ by theorem 3.1.8. Hence, $\widetilde{f}(\mathbf{b}) = \bigwedge_{\mathbf{u} \in \mathcal{B}_{\mathbf{b}} \setminus \{\mathbf{m}\}} f(\mathbf{u}) = 0$. However this contradicts to $\widetilde{f}(\mathbf{b}) = 1$, so $f(\mathbf{m}) = 1$ holds. ∎

Therefore, the difference between CD-ELCA and WCD-ELCA only appears in the type (ii) 49 local arrangements, in which the time developments are definite in CD-ELCA but indefinite in WCD-ELCA.

We have shown that the difference between the behavior of WCD-ELCA and CD-ELCA, which depends on whether some of the flow retaining local arrangements become indefinite or not. The case of CD-ELCA is easily affected by the perturbations at the boundary because the flow retaining local arrangements preserve perturbation flows. In the case of WCD-ELCA, the flow retaining property becomes indefinite at some local arrangements by autonomous indefiniteness. On the other hand, the perturbations at the boundary are negated by indefinite time developments. Such mechanism leads to similarity in behaviors between the periodic and random boundary condition.

3.2 Some statistical properties

We have seen that the behavior of WCD-ELCA is characterized by creation and destruction of value 1 clusters of various sizes. Figure 4 shows that the frequency distribution of cluster size scales as power law with exponent -2.0. Thus, the behavior of WCD-ELCA shows SOC-like behavior [21, 22, 23] in the sense that the critical state is realized without any parameter adjustment.

Figure 5 shows the frequency distribution of the rank of local arrangements with random boundary condition. In CD-ELCA, the time developments of the local arrangements are indefinite (i.e. those of type (iii)). They appears only in ranks lower than 68. Figure 3a and figure 5a suggest that the frequency distribution of high ranks and low ranks follows different exponential distribution laws, and the latter is mainly generated by the random boundary condition. Hence the role of external indefiniteness (which is realized by the type (iii) local arrangements) in CD-ELCA is random noise. On the other hand, in the case of WCD-ELCA, all type (i), type (ii) and type (iii) local arrangements appear in high ranks (there are 10 type (iii) local arrangements at ranks higher than 68) and they seem to follow the same distribution law over wide range (from 1 to around 200 in rank). This distribution law also scales as power law and its exponent is equal to -1.0. This suggests that both autonomous and external indefiniteness in WCD-ELCA have different meaning from the random noise.

4 Concluding remarks

The theory proposed in section 2 can be applied not only to ELCA but also to LCA which has more neighbors, higher dimension with more states in its cells. For many-valued LCA, the theory in section 2.2 should be described in the language of complete lattice. However their behaviors are expected to be different from that of ELCA. For example, consider the case of two dimensional, five-neighbor and two-valued LCA (we refer to this LCA as the (2,5,2)-type LCA). Figure 6 shows its behaviors driven by the closure and the weak closure. In contrast to ELCA, even in the case when time development is driven by the closure with periodic boundary, the behavior of the (2,5,2)-type LCA is highly changeable and complex. However, in the case of the weak closure, the behavior of the (2,5,2)-type LCA seems to be highly random. The reason of such difference can be understood as follows: The power set lattice on which the closure or the weak closure are taken is Ω^5 with 32 elements. If all elements in the set corresponding to $\mathcal{L}$ in ELCA (we also define this set to be $\mathcal{L}$) are coatoms of Ω^5 (elements with only one coordinate equals to 0. There are 5 coatoms in Ω^5) and they are different from each other, the closure of $\mathcal{L}$ is then Ω^5 itself. Hence the number of coatoms of Ω^5 in $\mathcal{L}$ is an important factor to determine the size of the closure of $\mathcal{L}$. For simplicity, we consider here the case when the set of all coatoms is equal to $\mathcal{L}$. The set $\mathcal{L}$ is constructed by observing its 18 neighbors in the (2,5,2)-type LCA. Therefore there are totally 2^{18} patterns in constructing $\mathcal{L}$. In these 2^{18} patterns, it can be shown easily that only 65 patterns make $\mathcal{L}$ equal to the set of all coatoms of Ω^5. Here we call the ratio like $65/2^{18}$ coatom ratio. On the other hand, the coatom ratio of ELCA is $9/2^8$, which is approximately one hundred times greater than that of the (2,5,2)-type LCA. For other types of LCA, for example, the coatom ratio of the (1,5,2)-type LCA is $25/2^{14}$, which is approximately one twentieth of that of LCA. The behavoir of the (1,5,2)-type LCA is shown in figure 7, which is more regular than the (2,5,2)-type LCA but more random than ELCA. These results suggest that coatom ratio somehow reflects the stability of LCA.

The above argument also suggests that the SOC-like behavior does not necessarily characterize the behavior of LCA driven by the weak closure because there exists LCA driven by the closure whose coatom ratio is extremely small. However, if we restrict our concern to ELCA, then we can show that the weak closure drives local interactions in the system and results in behaviors which seems to be SOC-like. In this case, both external and autonomous indefiniteness have different meaning from random noise if only the statistical properties of usage of local arrangements are considered.

In traditional quantitative modeling approaches to life such as those based on cellular automaton [20, 24], coupled nonlinear oscillators [25] and network models [26], conceptual thoughts on life have been given mainly on the level of interpretation of analytical results and computer experiments, and a conceptual approach on the level of constructing models is still absent at this moment. In contrast, our modeling is based on formal conceptual construction on indefiniteness, and the resulting computer experiments show similar features to those derived from traditional approaches. The results in this paper are simple and primitive, but these results provide a foothold to conceptual approaches to life with quantitative estimation.

Finally, we derive the relation between the theory proposed here and the notion of open limit, which is recently proposed by Gunji and Haruna [27]. Open limit is a key concept of internal perspective. It is well known that scientific theories are usually described by using different kinds of limit such as universal and existential quantifiers. However, the notion of life conflicts with the notion of limit (e.g. Taking the limit of life with respect to time results in death since all living systems have finite time lifespan, however, one of the most essential features of life is anticipation like thinking what will happen tomorrow). Therefore a mathematical tool that can describe life with both the notion of finiteness and infinity is necessary, and such tool is called open limit. Open limit is formally defined in lattice theory but here we list the informal conditions which must be satisfied by open limit. Finally we examine whether the weak closure $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following conditions or not.

(i) Open limit is a meaningless symbol satisfying the structure of limit for any meaningful symbols.

(ii) Open limit can be replaced by meaningful symbols.

(iii) If consistency among meaningful symbols is lost, open limit as meaningless symbol is introduced again in order to recover consistency.

First note that the closure operation is a limit operation because it constructs closed structure algebraically by collecting its fixed points. We have seen that $\widetilde{\mathcal{L}}$ is a topped $\bigcap$-structure, so the weak closure $\widetilde{\mathcal{L}}$ has the structure of limit but is not a closure of $\mathcal{L}$ in general. This implies $\widetilde{\mathcal{L}}$ does not have the meaning of limit of $\mathcal{L}$. Therefore, we see that condition (i) is satisfied by the weak closure. However, conditions (ii) and (iii) are ignored in the general theory of weak closure because we do not define the operations that correspond to the replacement mentioned in (ii) and the recovery of consistency mentioned in (iii). On the other hand, in their application to ELCA, these two conditions (ii) and (iii) are realized by arbitrary determination of values on the autonomous indefinite elements in $\mathcal{P}(X)$. Hence the weak closure exemplifies some aspects of the notion of open limit.

5 Acknowledgements

We greatly acknowledge Dr. C. B. Li by carefully reading the manuscript and correcting the English.

6 Appendix

Proof of theorem 3.1.4. Assume that $i \neq j, b_i \neq b_j$ and $\mathbf{u}_i = \mathbf{u}_j$. By symmetry, it is sufficient to prove the following two cases. (i): $i = 1, j = 2$ and (ii): $i = 1, j = 3$.

(i) Suppose $\mathbf{b} \in \overline{\mathcal{L}}$. Since $\mathbf{u}_1 = \mathbf{u}_2$, $u_1 = u_2 = u_3 = u_4$ holds. Define $u := u_1$. Then the local arrangement is identified as

$$\begin{array}{ccccc} u & u & u & u & u_5 \\ & b_1 & b_2 & b_3. & \end{array}$$

The case $u = 1$ contradicts to lemma 3.1.3 because either b_1 or b_2 is equal to 0. So $u = 0$ holds. Then we have $\mathbf{b} \not\leq \mathbf{u}_i$ for $i = 1, 2, 3$. Since $\mathbf{b} \in \overline{\mathcal{L}}$, we have

$$\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \bigwedge \emptyset = (1, 1, 1).$$

But this contradicts to $b_1 \neq b_2$. Hence $\mathbf{b} \notin \overline{\mathcal{L}}$ holds.

(ii) Suppose $\mathbf{b} \in \overline{\mathcal{L}}$. Since $\mathbf{u}_1 = \mathbf{u}_3$, $u_1 = u_3 = u_5$ and $u_2 = u_4$ hold. Define $u := u_1$ and $v := u_2$. Then the local arrangement identified as

$$\begin{matrix} u & v & u & v & u \\ & b_1 & b_2 & b_3. & \end{matrix}$$

Now we divide the proof into the following four parts. (a): $u = 1, v = 1$, (b): $u = 1, v = 0$, (c): $u = 0, v = 1$ and (d): $u = 0, v = 0$.

(a) This case contradicts to lemma 3.1.3 since either b_1 or b_3 is equal to 0.

(b) The local arrangement is identified as

$$\begin{matrix} 1 & 0 & 1 & 0 & 1 \\ & b_1 & b_2 & b_3. & \end{matrix}$$

Since either b_1 or b_3 is equal to 1, $\mathbf{b} \not\leq \mathbf{u}_2 = (0,1,0)$. If $b_2 = 1$ holds, then $\mathbf{b} = (b_1, 1, b_3) \not\leq (1,0,1) = \mathbf{u}_1 = \mathbf{u}_3$. This implies $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \wedge\emptyset = (1,1,1)$ and contradicts to $b_1 \neq b_3$. Therefore, $b_2 = 0$ holds. Thus $\mathbf{b} = (b_1, 0, b_3) \leq (1,0,1) = \mathbf{u}_1 = \mathbf{u}_3$ and $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \mathbf{u}_1 \wedge \mathbf{u}_3$. That is, $(b_1, b_2, b_3) = (u_1 \wedge u_3, u_2 \wedge u_4, u_3 \wedge u_5)$. Since $u_1 = u_3 = u_5 = 1$, $b_1 = b_3 = 1$ holds. But this contradicts to $b_1 \neq b_3$.

(c) The local arrangement is identified as

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 \\ & b_1 & b_2 & b_3. & \end{matrix}$$

Since either b_1 or b_3 is equal to 1, $\mathbf{b} \not\leq \mathbf{u}_1$ and $\mathbf{b} \not\leq \mathbf{u}_3$ hold. If $b_2 = 1$, then $\mathbf{b} = (b_1, 1, b_3) \not\leq (1,0,1) = \mathbf{u}_2$. Therefore, $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \bigwedge \emptyset = (1,1,1)$. But this contradicts to $b_1 \neq b_3$. Thus $b_2 = 0$ must hold. Then $\mathbf{b} = (b_1, 0, b_3) \leq (1,0,1) = \mathbf{u}_2$ and $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \mathbf{u}_2$. That is, $(b_1, b_2, b_3) = (1,0,1)$. But this contradicts to $b_1 \neq b_3$.

(d) The local arrangement is identified as

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ & b_1 & b_2 & b_3. & \end{matrix}$$

Because either b_1 or b_3 is equal to 1, $\mathbf{b} \not\leq \mathbf{u}_i$ for $i = 1,2,3$. Therefore, $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \bigwedge \emptyset = (1,1,1)$. But this contradicts to $b_1 \neq b_3$. ∎

Proof of proposition 3.1.7. When $\mathbf{b} \in \mathcal{L}$, it can be proved that there exists $i \in \{1,2,3\}$ such that $\mathbf{b} = \mathbf{u}_i$ and $u_{2i-1} = b_i$ by the same manner in the proof of proposition 3.1.6. By assumption, f is an order-preserving map on $\mathcal{L}$, so for $\mathbf{u} \in \mathcal{L}$ if $\mathbf{b} \leq \mathbf{u}$ then $f(\mathbf{b}) \leq f(\mathbf{u})$. Therefore, $\overline{f}(\mathbf{b}) = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} f(\mathbf{u}) = f(\mathbf{b}) = f(\mathbf{u}_i) = b_i$. ∎

Proof of theorem 3.1.8. By symmetry, it is enough to prove the following three cases.

(i) If $\mathbf{u}_1 \leq \mathbf{u}_2$ and $\mathbf{u}_1 \neq \mathbf{u}_2$, then $f(\mathbf{u}_1) \leq f(\mathbf{u}_2)$ i.e. $b_1 \leq b_2$.

(ii) If $\mathbf{u}_1 \geq \mathbf{u}_2$ and $\mathbf{u}_1 \neq \mathbf{u}_2$, then $f(\mathbf{u}_1) \geq f(\mathbf{u}_2)$ i.e. $b_1 \geq b_2$.

(iii) If $\mathbf{u}_1 \leq \mathbf{u}_3$ and $\mathbf{u}_1 \neq \mathbf{u}_3$, then $f(\mathbf{u}_1) \leq f(\mathbf{u}_3)$ i.e. $b_1 \leq b_3$.

(i) Suppose that $\mathbf{u}_1 \leq \mathbf{u}_2$ and $\mathbf{u}_1 \neq \mathbf{u}_2$. Then $u_1 \leq u_2 \leq u_3 \leq u_4$ and at least one inequality holds strictly. Assume that $b_1 > b_2$. That is, $b_1 = 1, b_2 = 0$. Suppose $\mathbf{b} \not\leq \mathbf{u}_i$ for all $i \in \{1, 2, 3\}$, then $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \bigwedge \emptyset = (1, 1, 1)$. But this contradicts to $b_2 = 0$. Therefore, there exists some $i \in \{1, 2, 3\}$ such that $\mathbf{b} \leq \mathbf{u}_i$.

Assume that $\mathbf{b} \leq \mathbf{u}_1$. That is $(b_1, b_2, b_3) \leq (u_1, u_2, u_3)$, then we have $b_1 \leq u_1 \leq u_2 \leq u_3 \leq u_4$ because $b_1 = 1$, $u_1 = u_2 = u_3 = u_4 = 1$. But this contradicts the fact that at least one inequality must hold strictly.

Next assume that $\mathbf{b} \leq \mathbf{u}_2$. That is, $(b_1, b_2, b_3) \leq (u_1, u_2, u_3)$. Then $1 = b_1 \leq u_2 \leq u_3 \leq u_4$. So $\mathbf{u}_2 = (1, 1, 1)$ and it follows that $b_2 = 1$ by lemma 3.1.3. This contradicts to $b_2 = 0$.

Finally, assume that $\mathbf{b} \leq \mathbf{u}_3$. That is $(b_1, b_2, b_3) \leq (u_3, u_4, u_5)$, then we have $1 = b_1 \leq u_3 \leq u_4$. Therefore, $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \mathbf{u}_3$ (i.e. $(1, 0, b_3) = (1, 1, u_5)$) since $\mathbf{b} \not\leq \mathbf{u}_1$ and $\mathbf{b} \not\leq \mathbf{u}_2$. This is a contradiction.

Hence, $b_1 \leq b_2$ holds.

(ii) Similar to case (i).

(iii) Suppose that $\mathbf{u}_1 \leq \mathbf{u}_3$ and $\mathbf{u}_1 \neq \mathbf{u}_3$. Then $u_1 \leq u_3, u_2 \leq u_4, u_3 \leq u_5$ and at least one inequality holds strictly. Assume that $b_1 > b_3$, that is, $b_1 = 1, b_3 = 0$. With the same reason as in the beginning of part (i), there exists some $i \in \{1, 2, 3\}$ such that $\mathbf{b} \leq \mathbf{u}_i$.

First assume that $\mathbf{b} \leq \mathbf{u}_1$ (i.e. $(b_1, b_2, b_3) \leq (u_1, u_2, u_3)$), then $1 = b_1 \leq u_1 \leq u_3 \leq u_5$ and the local arrangement is identified as

$$\begin{array}{ccccc} 1 & u_2 & 1 & u_4 & 1 \\ & 1 & b_2 & 0. & \end{array}$$

If $u_4 = 1$, then $\mathbf{u}_3 = (1, 1, 1)$. By lemma 3.1.3, it follows that $b_3 = 1$. However this contradicts to $b_3 = 0$. Therefore we have $u_4 = 0$. u_2 is also equal to 0 since $u_2 \leq u_4$. Thus the local arrangement is identified as

$$\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ & 1 & b_2 & 0. & \end{array}$$

Then we get $\mathbf{b} \notin \overline{\mathcal{L}}$ by proposition 3.1.4. But this contradicts to $\mathbf{b} \in \overline{\mathcal{L}}$.

Next suppose that $\mathbf{b} \leq \mathbf{u}_3$. That is $(b_1, b_2, b_3) \leq (u_3, u_4, u_5)$, then $1 = b_1 \leq u_3 \leq u_5$. Now the local arrangement can be identified as

$$\begin{array}{ccccc} u_1 & u_2 & 1 & u_4 & 1 \\ & 1 & b_2 & 0. & \end{array}$$

We also have $u_4 = 0$ by the same reason in the proof of $\mathbf{b} \not\leq \mathbf{u}_1$. Consider $u_2 \leq u_4$, we identify the local arrangement as

$$\begin{array}{ccccc} u_1 & 0 & 1 & 0 & 1 \\ & 1 & b_2 & 0, & \end{array}$$

then $\mathbf{b} = (1, b_2, 0) \not\leq (0,1,0) = \mathbf{u}_2$. On the other hand, since $\mathbf{b} \not\leq \mathbf{u}_1$, $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \mathbf{u}_3$ (i.e. $(1, b_2, 0) = (1, 0, 1)$). This is a contradiction.

Finally, Assume that $\mathbf{b} \leq \mathbf{u}_2$, that is, $(b_1, b_2, b_3) \leq (u_2, u_3, u_4)$. Then we have $1 = b_1 \leq u_2 \leq u_4$ and so $\mathbf{u}_2 = (1, u_3, 1)$. Since both $\mathbf{b} \not\leq \mathbf{u}_1$ and $\mathbf{b} \not\leq \mathbf{u}_3$ hold, $\mathbf{b} = \bigwedge_{\mathbf{b} \leq \mathbf{u} \in \mathcal{L}} \mathbf{u} = \mathbf{u}_2$ (i.e. $(1, b_2, 0) = (1, u_3, 1)$). This is a contradiction.

Thus, $b_1 \leq b_3$ must hold. ∎

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Figure captions

Figure 1. Construction of $\overline{\mathcal{L}}$, $\widetilde{\mathcal{L}}$, $\overline{f}$ and $\widetilde{f}$ from $\mathcal{L}$ and f. All ordered sets are expressed as Hasse diagrams. Circles filled in black and white on elements of Ω^3 denote 1 and 0, respectively. Banished nodes are indefinite elements. (a) $RAN(\mathbf{u}^*, \mathbf{b}) = 43$. (b) $RAN(\mathbf{u}^*, \mathbf{b}) = 107$.

Figure 2. The time developments of CD-ELCA. The system size is equal to 100. Here we show the initial 200 time steps. The left side pictures are shown by the state of cells. Black and white cells correspond to 1 and 0 as their values, respectively. The right side pictures are shown by whether the time development of the cell is definite or not. In here, black denotes definite time development and white denotes indefinite one. (a) With the periodic boundary condition. (b) With the random boundary condition.

Figure 3. The time developments of WCD-ELCA. The experimental conditions are the same as one of CD-ELCA. (a) With the periodic boundary condition. (b) With the random boundary condition.

Figure 4. Log-log plot of the frequency distribution of sizes for time-space value 1 clusters. The frequency of each cluster size is calculated from 200 runs of 5000 time steps from different random initial conditions, where system size is equal to 200.

Figure 5. Log-log plots of the frequency distributions of ranks of local arrangements in CD-ELCA and in WCD-ELCA with the random boundary condition. The frequency of each local arrangement is calculated from one run of 50500 time steps from random initial condition and initial 500 time steps are thrown out. The system size is equal to 200. (a) The CD-ELCA case. It seems that the distribution can be divided into two parts which scales as different exponential laws. The border seems to be around rank 70. Type (iii) local arrangements appear only at ranks lower than the border. (b) The WCD-ELCA case. The distribution can scale as power law with exponent -1.0 over wide range from 1 to around 200 in rank. All types of local arrangement appear in the range. The part of graph from 30 to 100 in rank is enlarged in the small picture.

Figure 6. The time developments of the (2,5,2)-type LCA from random initial conditions. The system size is 50×50. The boundary conditions are periodic. Time steps from 70 to 91 per three steps are shown. The pictures are shown by state of cells, black and white cells correspond to 1 and 0 as their values, respectively. (a) Driven by the closure. (b) Driven by the weak closure.

Figure 7. The time developments of the (1,5,2)-type LCA from random initial conditions. The system size equals to 101. The boundary conditions of left side pictures are periodic and those of right side ones are random. Initial 200 time steps are shown. The pictures are shown by state of cells, black and white cells correspond to 1 and 0 as their values, respectively. (a) Driven by the closure. (b) Driven by the weak closure.

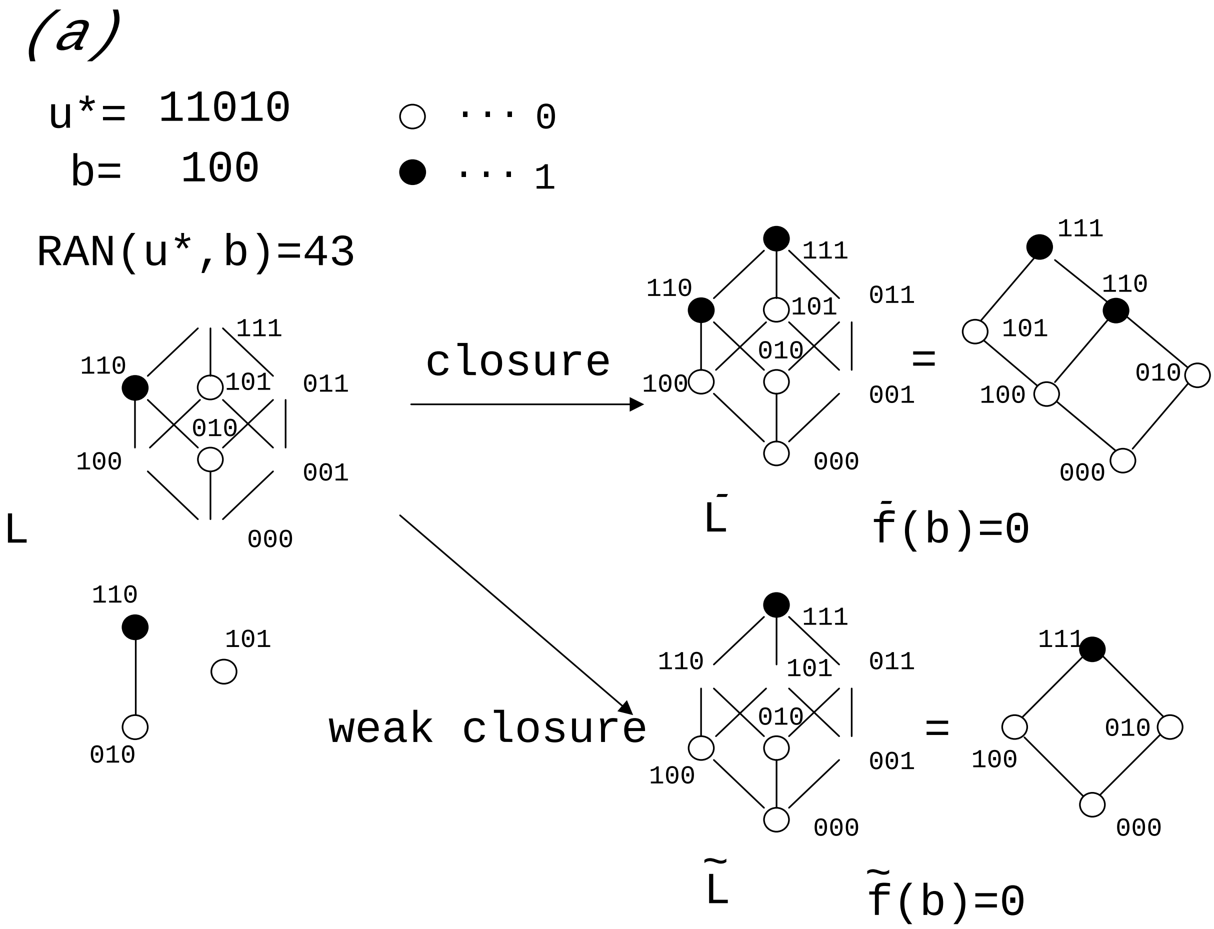

(a)
u*= 11010
b= 100
··· 0
··· 1
RAN(u*,b)=43
111
110
101
011
010
100
001
000
L
closure
weak closure
L̄
f̄(b)=0
L̃
f̃(b)=0

Fig. 1a

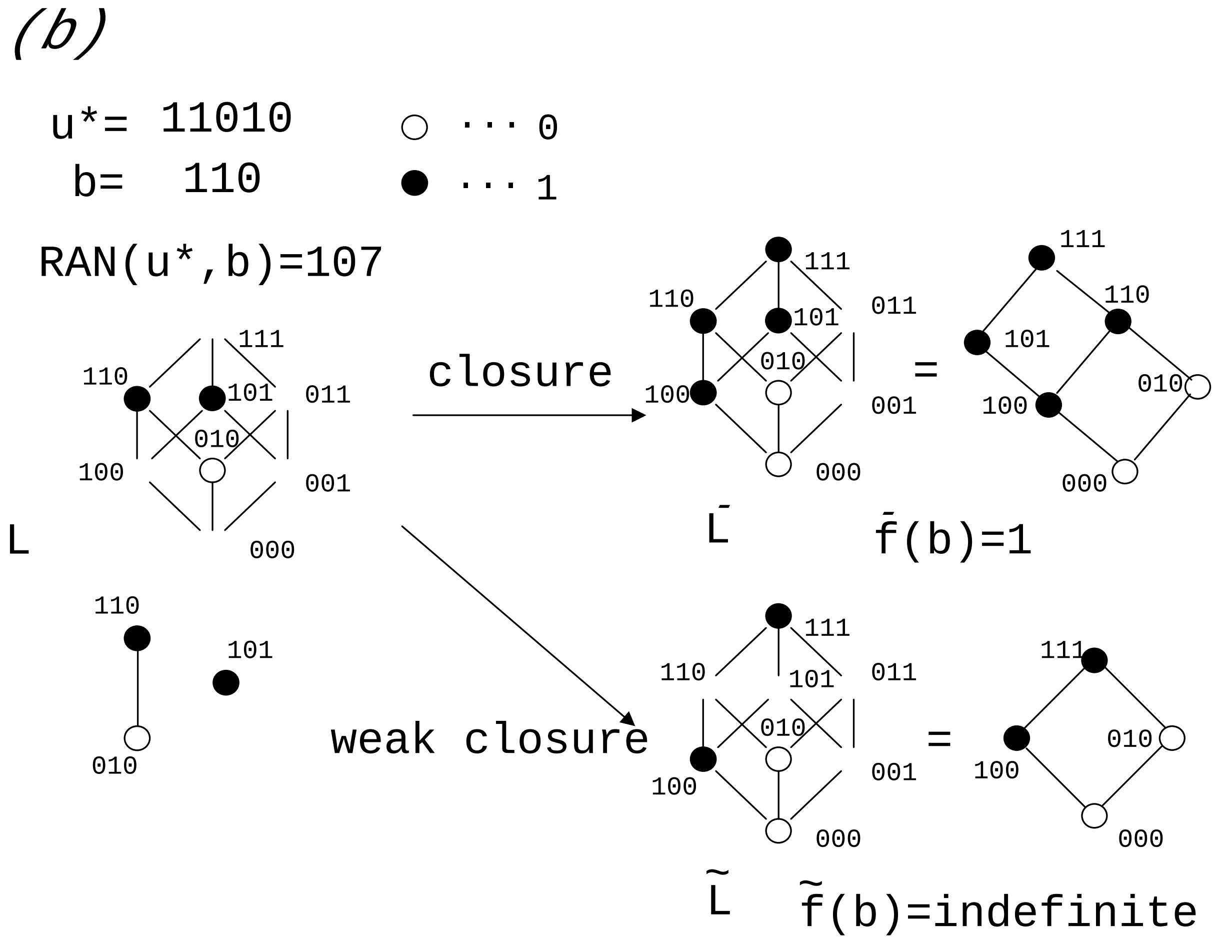

(b)
u*= 11010
b= 110
··· 0
··· 1
RAN(u*,b)=107
L
111
110
101
011
010
100
001
000
closure
weak closure
=
f̄(b)=1
f̃(b)=indefinite

Fig. 1b

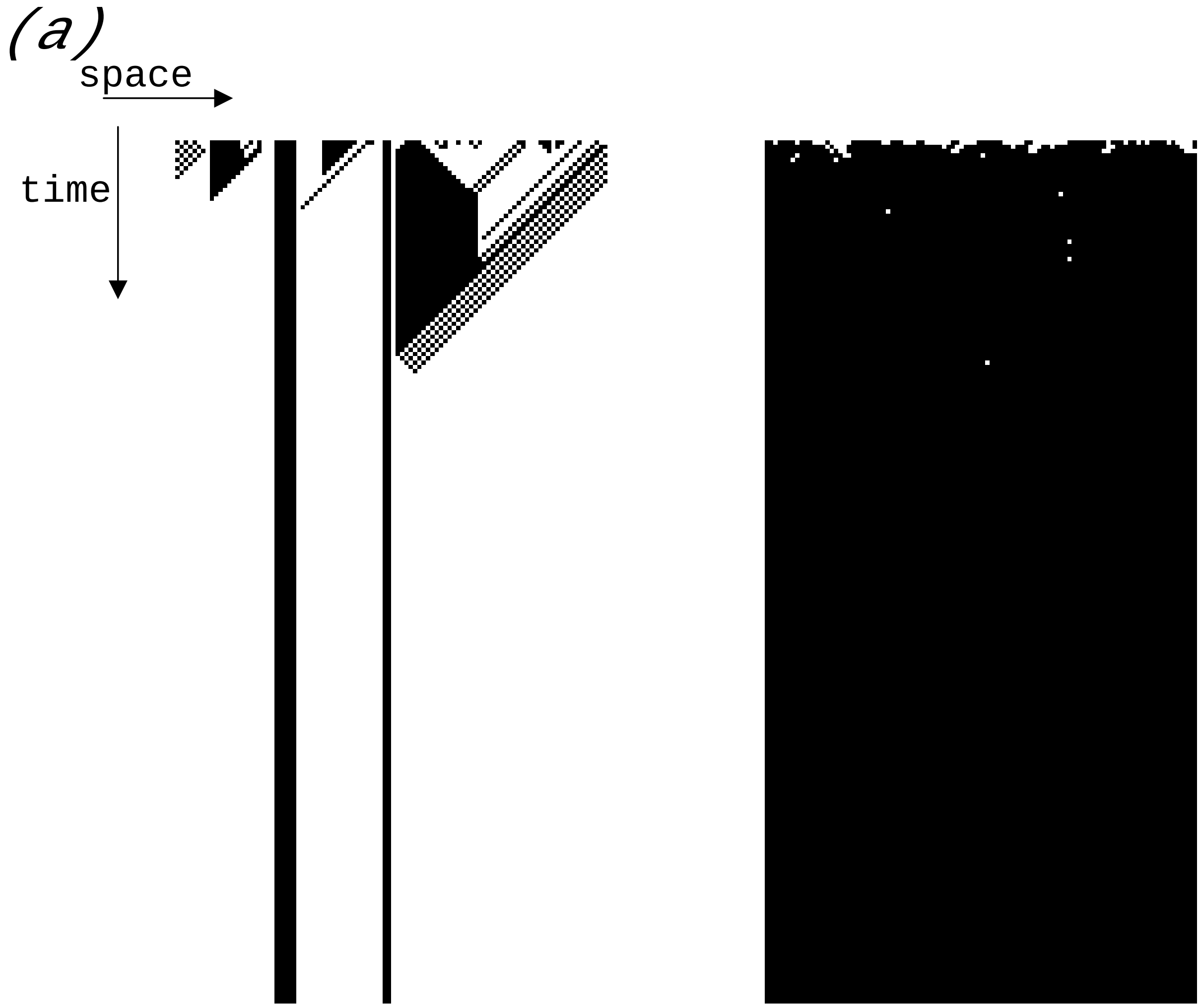


Fig. 2a

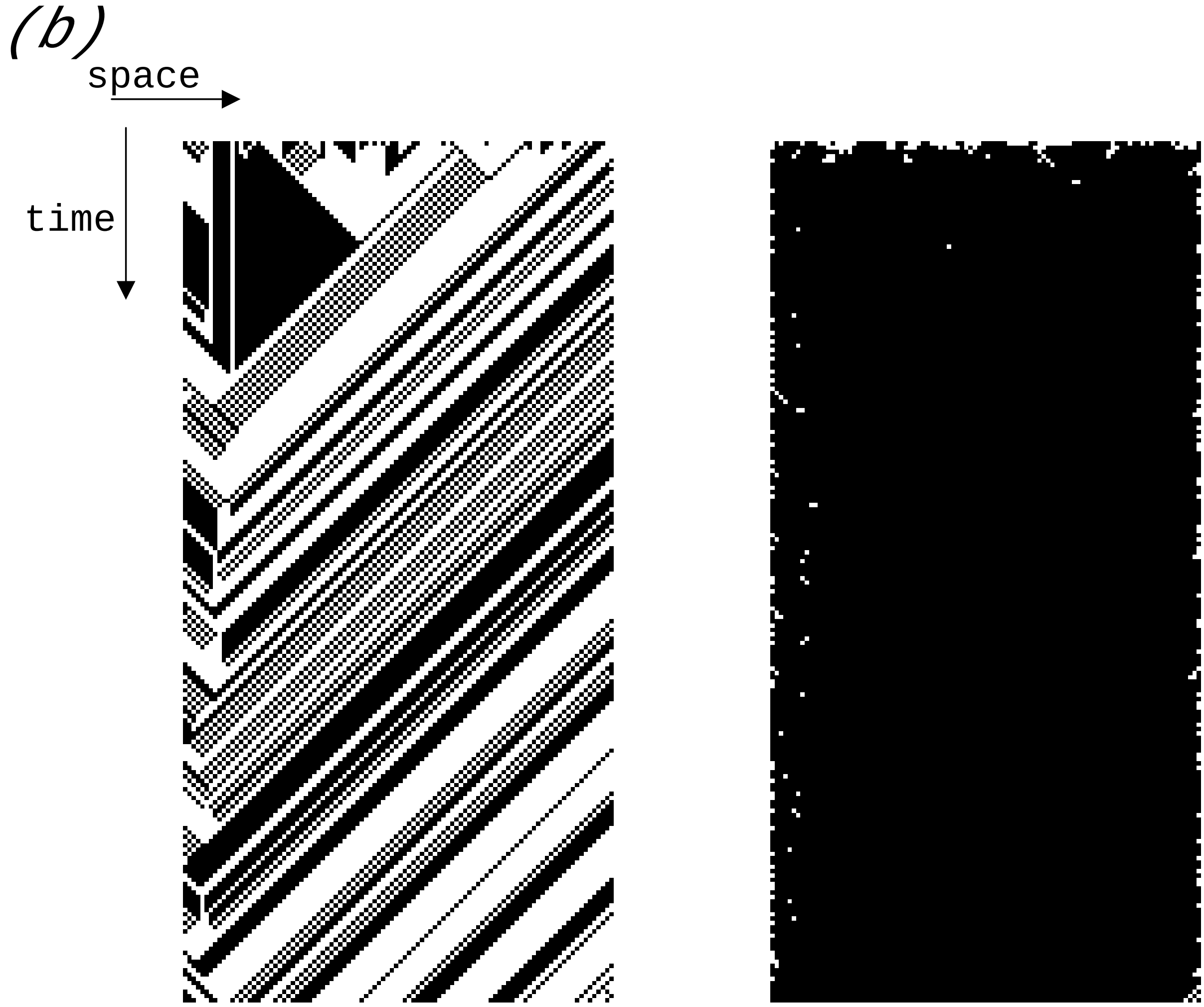


Fig. 2b

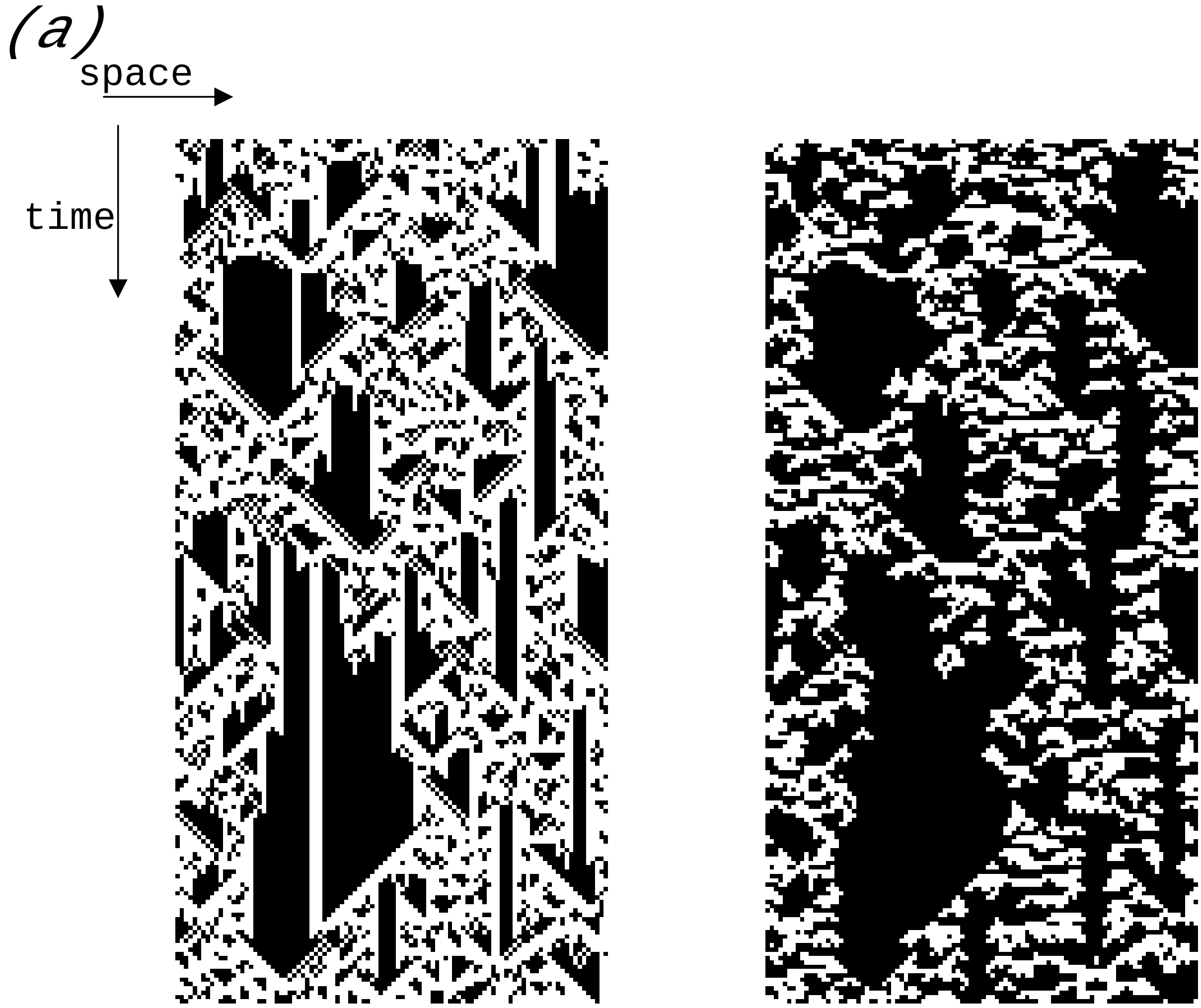


Fig. 3a

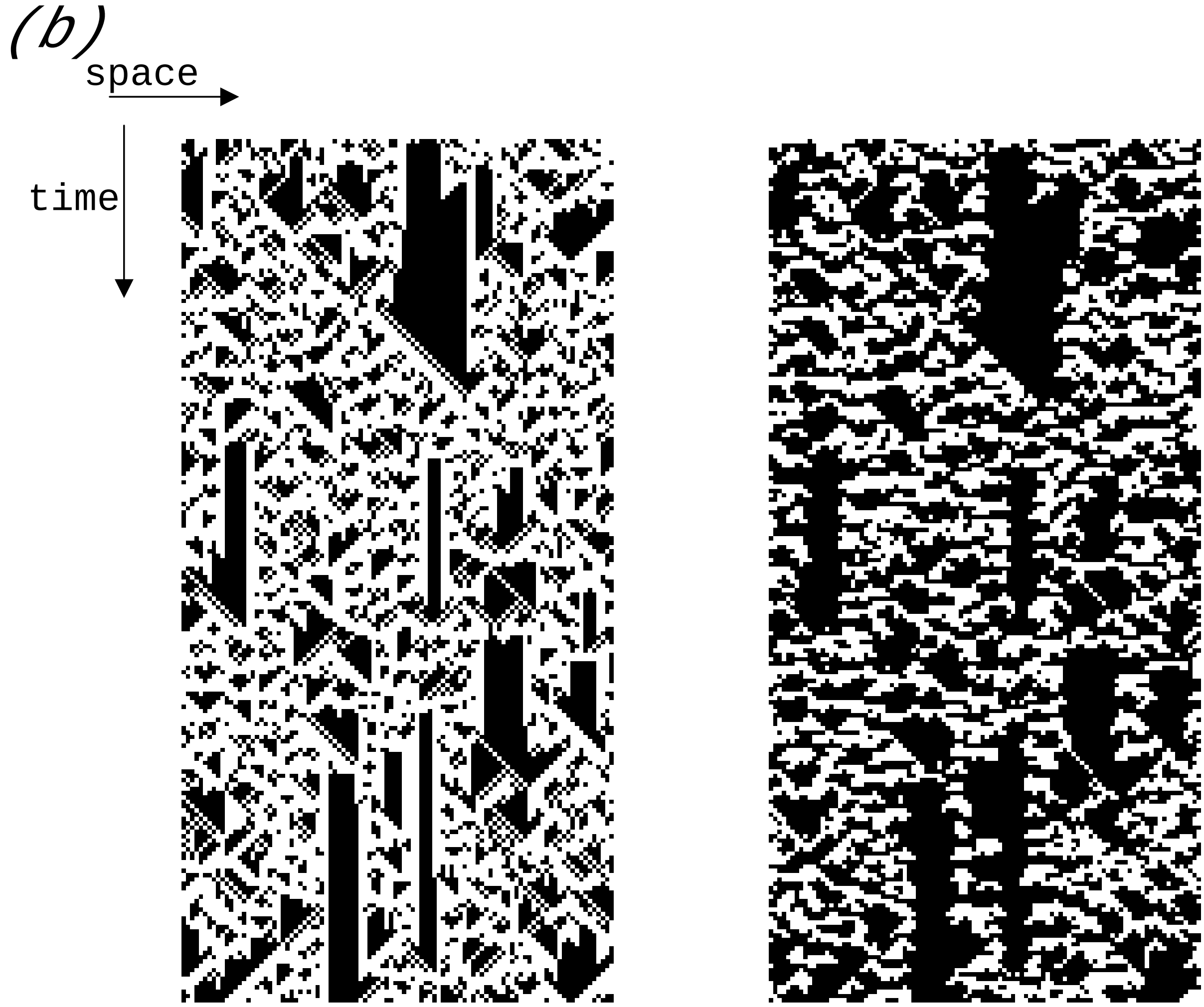


Fig. 3b

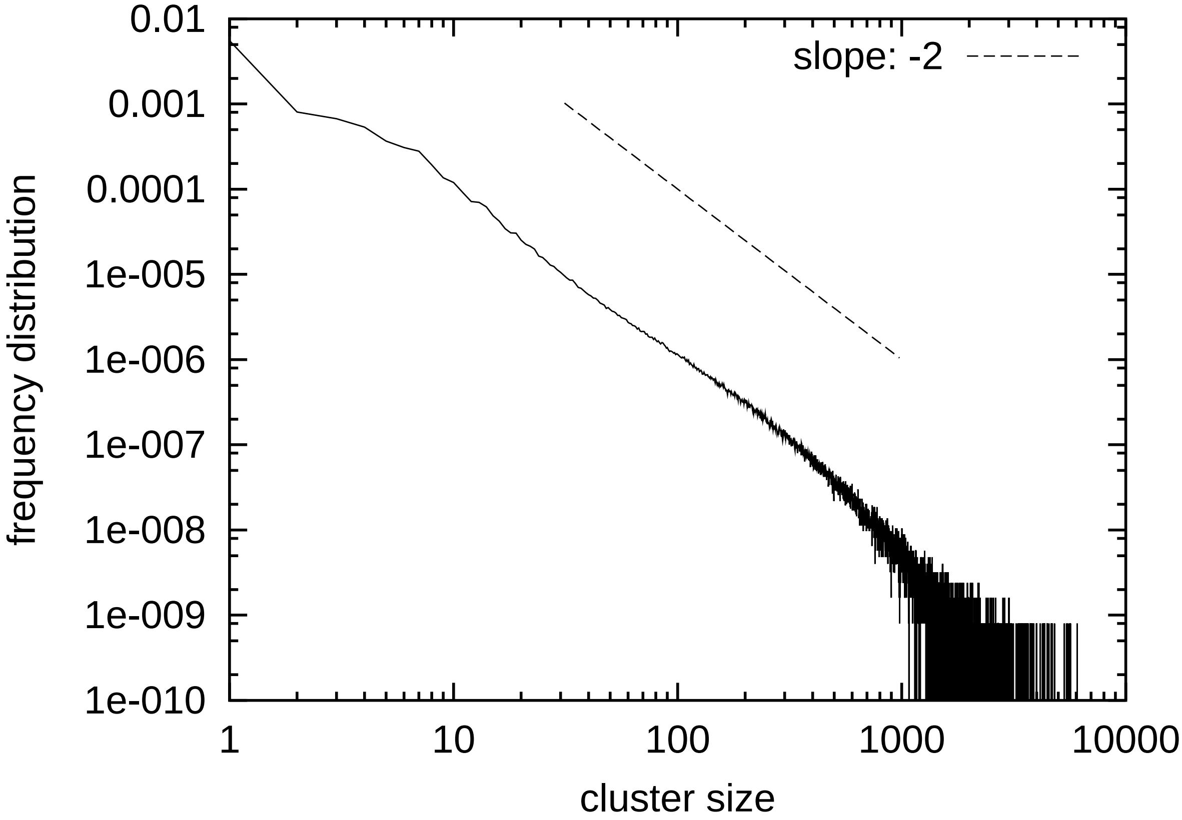

0.01
0.001
0.0001
1e-005
1e-006
1e-007
1e-008
1e-009
1e-010
frequency distribution
slope: -2
1
10
100
1000
10000
cluster size

Fig. 4

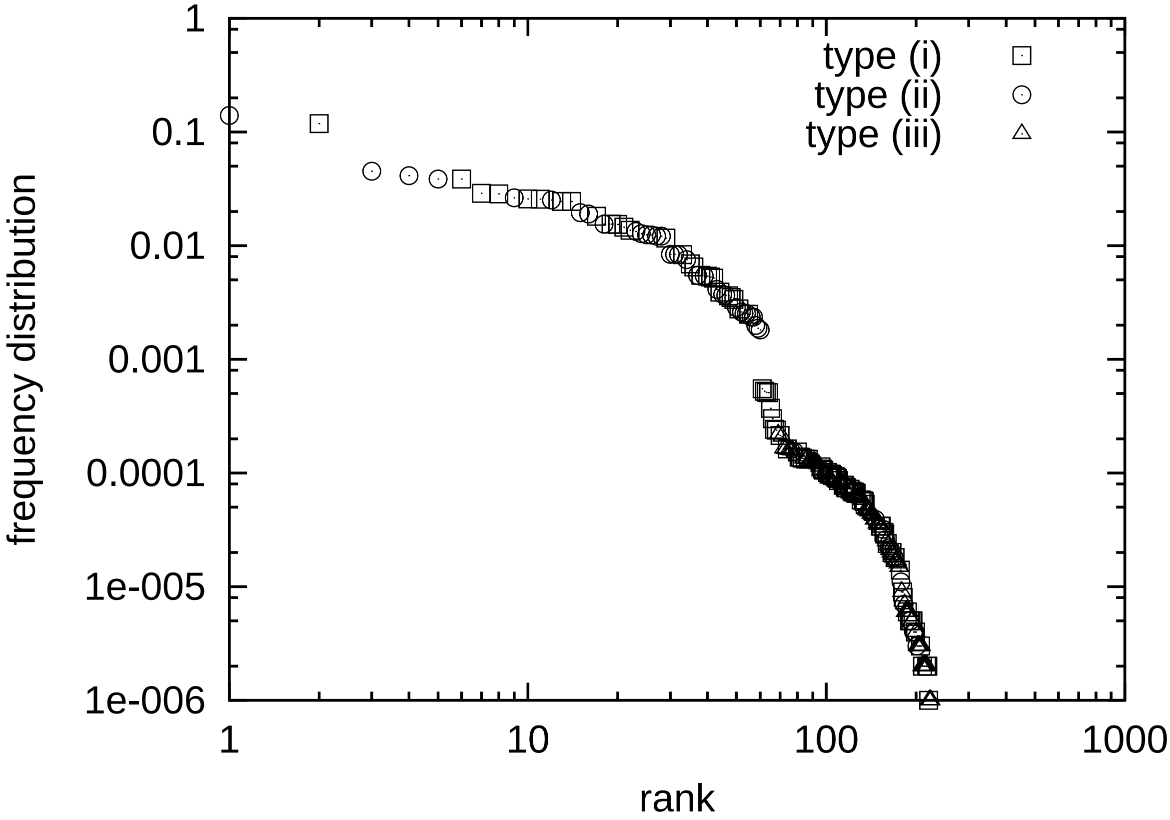
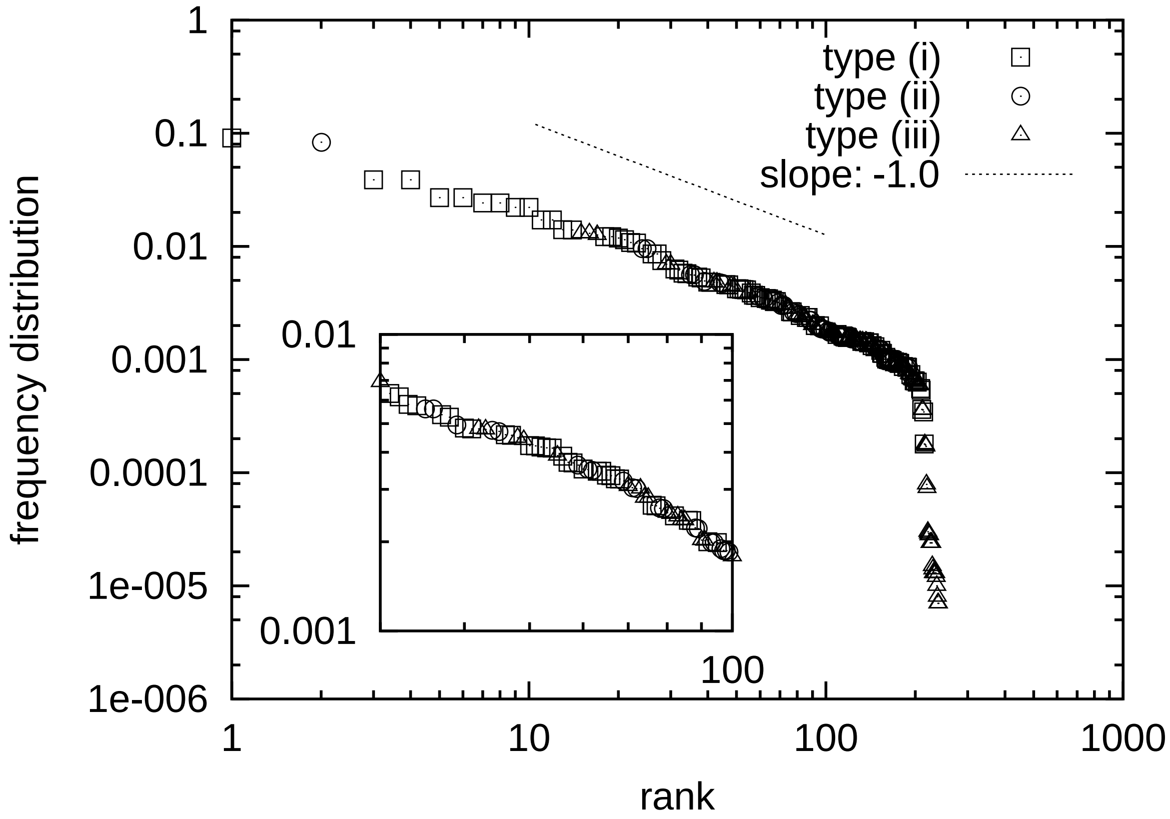

1
0.1
0.01
0.001
0.0001
1e-005
1e-006
frequency distribution
1
10
100
1000
rank
type (i)
type (ii)
type (iii)

Fig. 5a



frequency distribution
rank
type (i)
type (ii)
type (iii)
slope: -1.0
1
0.1
0.01
0.001
0.0001
1e-005
1e-006
1
10
100
1000
0.01
0.001
100

Fig. 5b

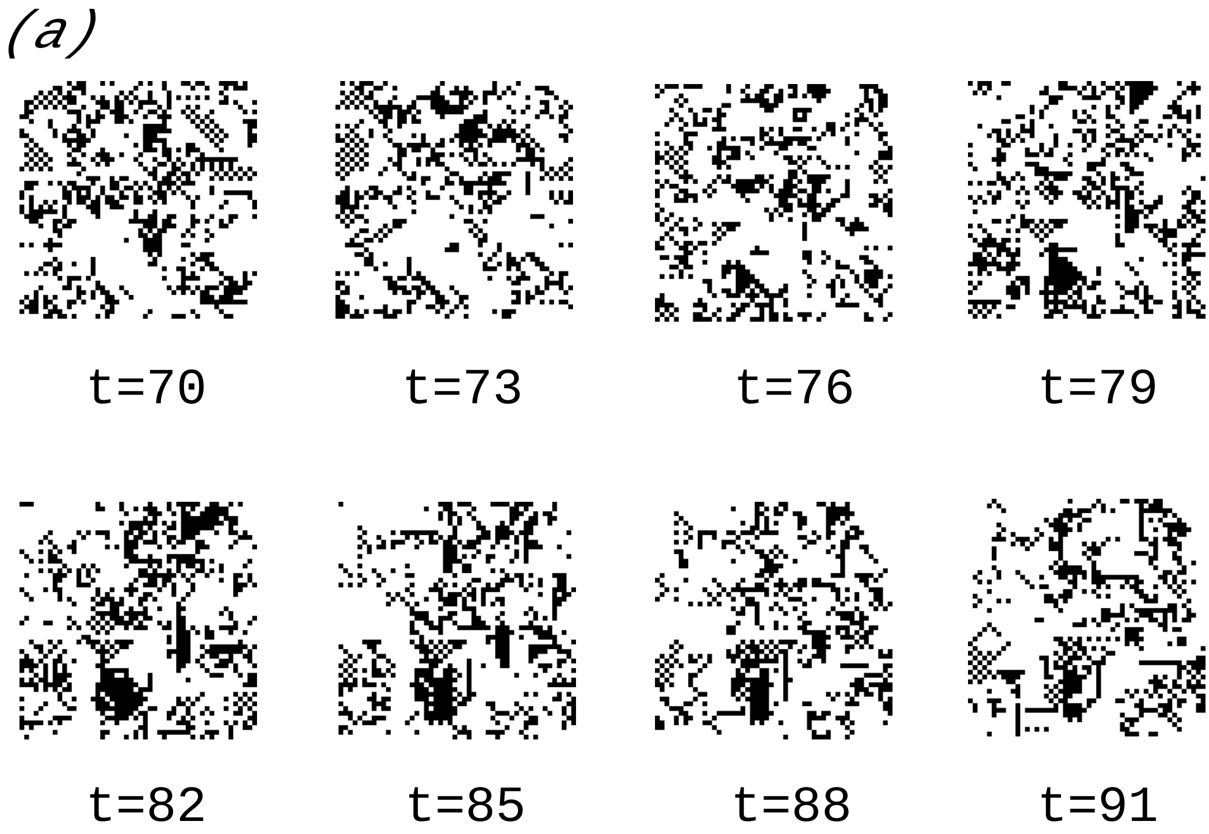


Fig. 6a

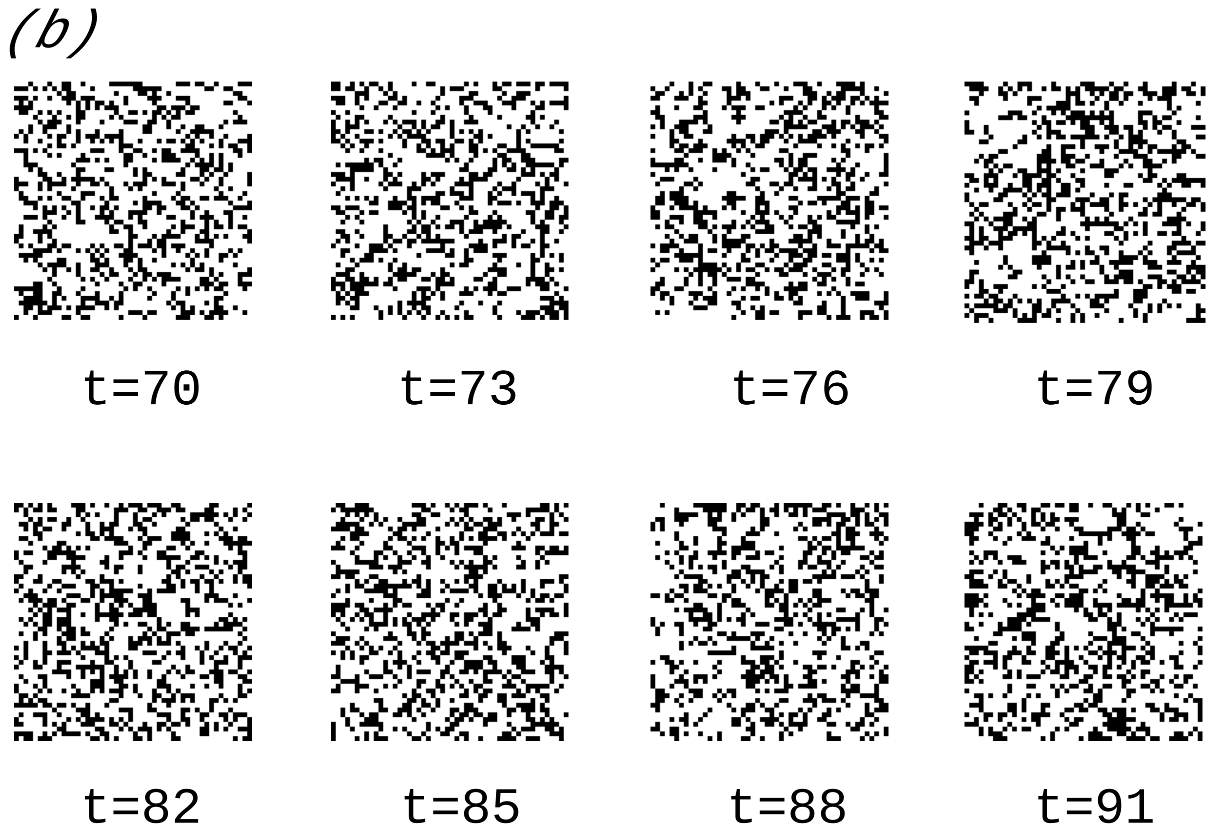


Fig. 6b

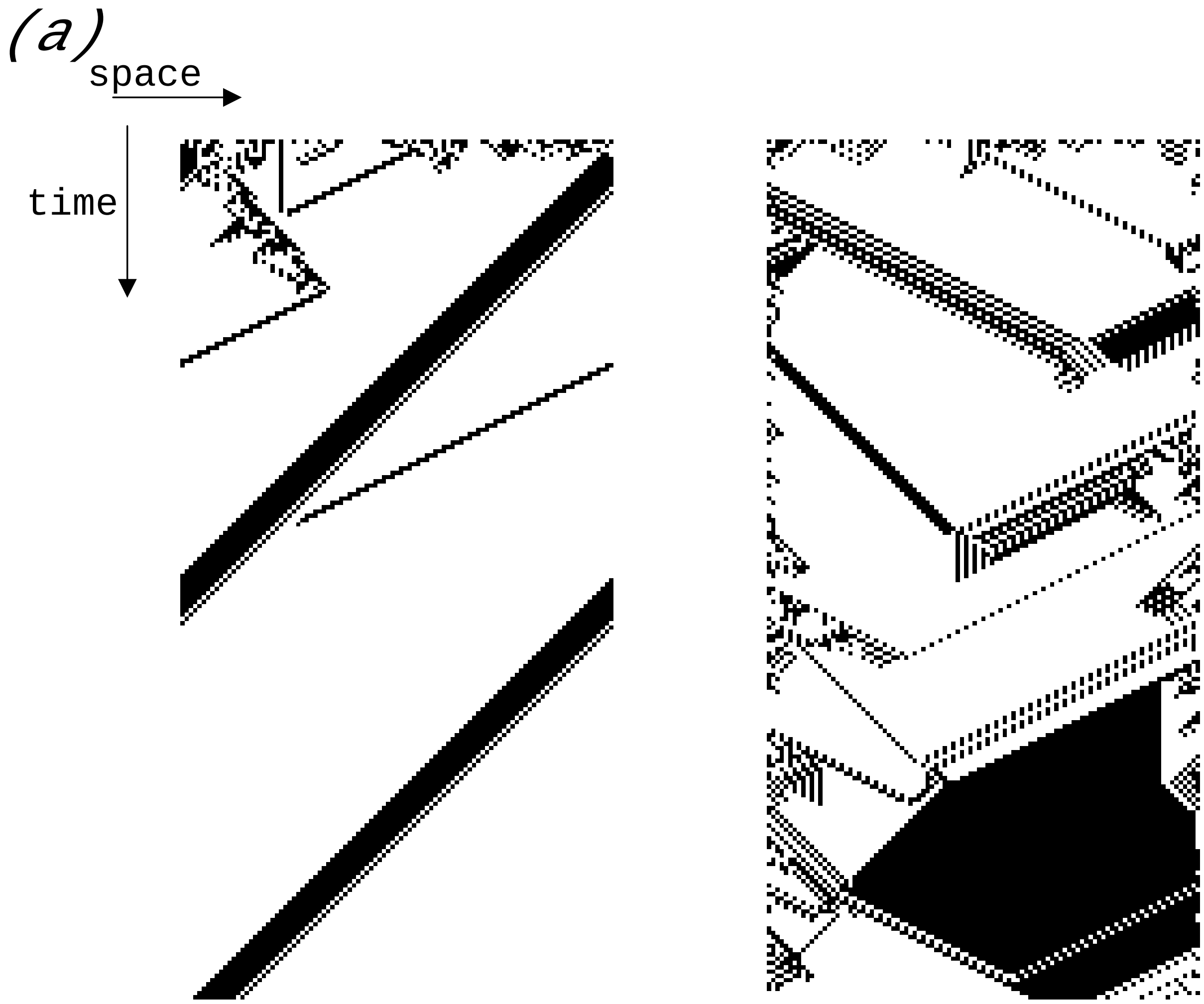


Fig. 7a

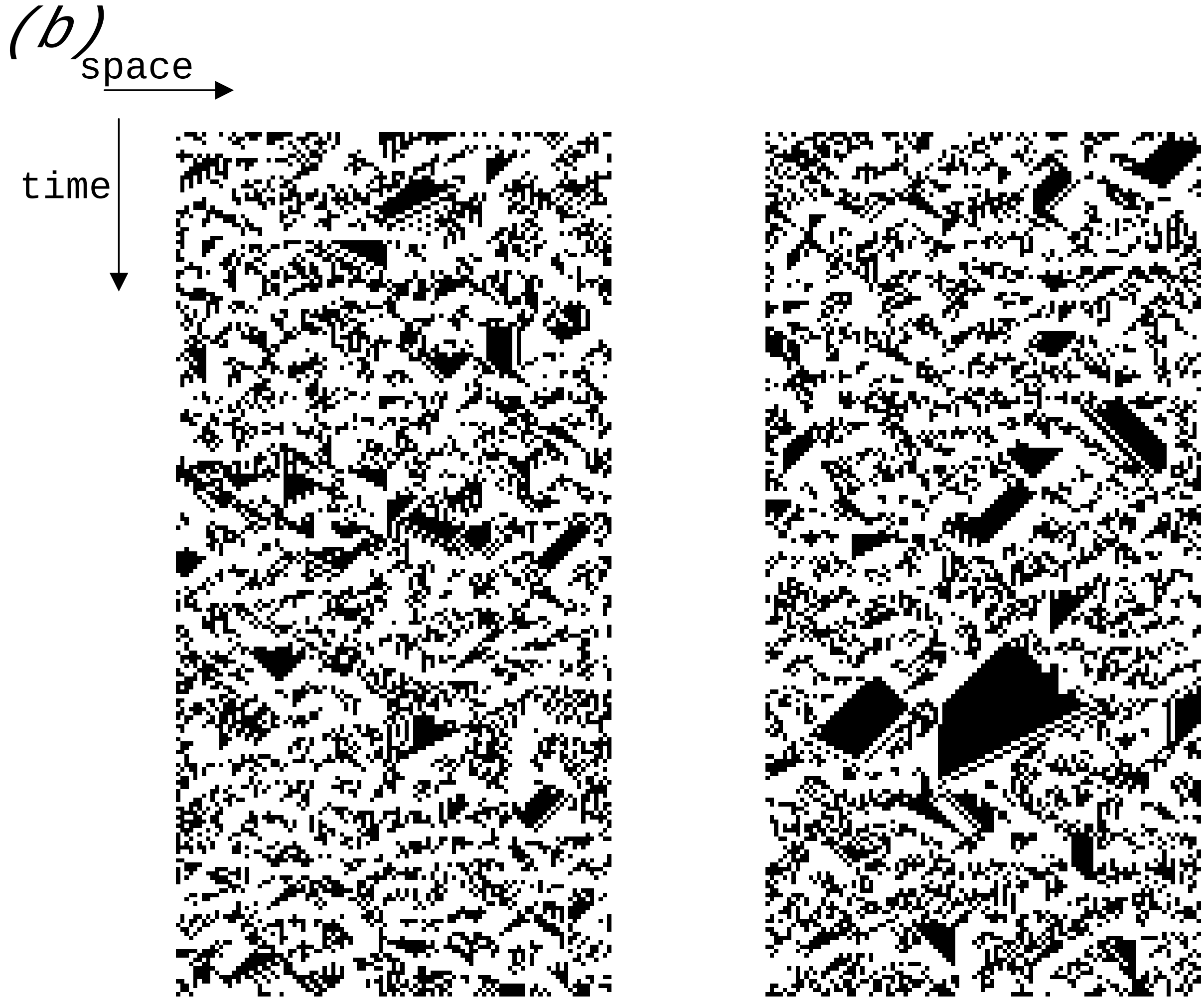


Fig. 7b