The early history of convergence acceleration methods

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Dedicated to the 70th birthday of Claude Brezinski

Abstract

The history of convergence acceleration methods in the 17th century are surveyed. Acceleration methods are classified into three categories.

1 Introduction

The convergence acceleration methods, or the extrapolation methods, were discovered in the 17th century in Europe [4][9] and in Japan [3], independently. The purpose of this paper is to solve two questions. Who are the founders of convergence acceleration methods? What kind of contribution did 17th century Japanese mathematicians Takakazu Seki and Katahiro Takebe make to convergence acceleration?

We classify the acceleration methods into three categories. I. Acceleration methods of sequences of intervals; II. Acceleration methods of real sequences with result verification; III. Usual (or non-verified) acceleration methods of sequences.

In the 17th century the third category contained the following three acceleration methods: 1. The theorem of Huygens, i.e., the first step of the Richardson extrapolation process; 2. The Aitken Δ^2 process; 3. The Richardson extrapolation process.

Almost all convergence acceleration methods in the 17th century were obtained by computing the circumference of a circle, the length of the arc of a sector, the volume of a sphere, and so on. Therefore the following notation [9] will be used throughout in this paper. Let T_n be the perimeter of the n-sided inscribed regular polygon in a circle of diameter 1 and U_n be that of circumscribed regular polygon.

$$T_n = n \sin\left(\frac{\pi}{n}\right), \quad U_n = n \tan\left(\frac{\pi}{n}\right).$$

By the Taylor expansion, we have

$$T_n = n \sin\left(\frac{\pi}{n}\right) = \pi\left(1 - \frac{\pi^2}{6n^2} + \frac{\pi^4}{120n^4} - \frac{\pi^6}{5040n^6} + O\left(\frac{1}{n^8}\right)\right), \tag{1}$$

$$U_n = n \tan\left(\frac{\pi}{n}\right) = \pi\left(1 + \frac{\pi^2}{3n^2} + \frac{2\pi^4}{15n^4} + \frac{17\pi^6}{315n^6} + O\left(\frac{1}{n^8}\right)\right). \tag{2}$$

2 Acceleration of a sequence of intervals

2.1 Definitions

Let $\{I_n\}$ be a sequence of closed intervals such that $(I_0 \supset) I_1 \supset I_2 \supset \cdots$ and the width or the diameter of I_n converges to 0. Then by the completeness there exists unique $s \in \bigcap_{n=1}^{\infty} I_n$. In this case we say that the sequence of intervals $\{I_n\}$ monotonically converges to s.

Let $\mathscr{I}$ be the set of monotonically converging sequences of closed intervals. Let $T : \mathscr{I} \to \mathscr{I}$ be an interval transformation with $T(\{[s_n, t_n]\}) = \{[u_n, v_n]\}$ such that $s_n \leqq u_n < s < v_n \leqq t_n$. Suppose that u_n, v_n are determined only by s_n, t_n, and also suppose that

$$\lim_{n\to\infty} \frac{v_n - u_n}{t_n - s_n} = 0.$$

Then we call T accelerates the sequence of intervals $\{[s_n, t_n]\}$.

2.2 Weighted mean acceleration methods

Let $g : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\lim_{n\to\infty} g(n) = 0$. Let $\{s_n\}, \{t_n\}$ be positive sequences converging to the same limit s. Suppose $\{s_n\}, \{t_n\}$ have the asymptotic formulas

$$s_n = s - cg(n) + o(g(n)), \quad \text{as } n \to \infty, \tag{3}$$
$$t_n = s + dg(n) + o(g(n)), \quad \text{as } n \to \infty, \tag{4}$$

where c, d are known positive constants.

We denote weighted mean acceleration methods by

$$a_n = \frac{ds_n + ct_n}{d + c}, \tag{5}$$
$$h_n = \frac{d + c}{\dfrac{d}{s_n} + \dfrac{c}{t_n}}. \tag{6}$$

By the assumptions (3)(4), we have

$$a_n = s + o(g(n)), \quad h_n = s + o(g(n)),$$

as $n \to \infty$. Acceleration methods (5), (6) are called weighted arithmetic mean and weight harmonic mean, respectively.

2.3 Willebrord Snell

From Archimedes (BC287-BC212) to Ludolph van Ceulen (1540-1610), European mathematicians computed π by using inscribed and circumscribed regular polygons n sides each. Van Ceulen computed 36 digits by using 2^{62}-sided polygons. After Van Ceulen's death, his result was published in 1621 by his student Willebrord Snell (1580-1626) in *Cyclometricus* [19]. Van Ceulen's result was

$$3\frac{14159265358979323846264338327950288}{100000000000000000000000000000000000} < \pi < 3\frac{14159265358979323846264338327950289}{100000000000000000000000000000000000}.$$

It was Snell who surpassed Archimedes and Ceulen. In *Cyclometricus* Snell derived

$$\frac{3\sin\theta}{2 + \cos\theta} < \theta < 2\sin\frac{\theta}{3} + \tan\frac{\theta}{3} \tag{7}$$

without adequate proof, where θ is the central angle of a sector.

Putting $\theta = \pi/n$ in formula (7), we have

$$\frac{3}{\dfrac{2}{T_n} + \dfrac{1}{U_n}} < \pi < \frac{2}{3}T_{3n} + \frac{1}{3}U_{3n}. \tag{8}$$

Snell took the same regular polygons in upper and lower bounds of (8), and he computed π by using

$$\frac{3}{\dfrac{2}{T_n} + \dfrac{1}{U_n}} < \pi < \frac{2}{3}T_n + \frac{1}{3}U_n. \tag{9}$$

The upper bound of (9) is the weighted arithmetic mean and the lower bound is the weighted harmonic mean of T_n and U_n.

By (9) with $n = 1073741824 (= 2^{30})$, Snell gave

$$3\frac{1415926535897932384626433832795028293}{10000000000000000000000000000000000000} < \pi < 3\frac{1415926535897932384626433832795028958}{10000000000000000000000000000000000000}.$$

Snell's result is 1 digit less than his teacher Van Ceulen. This is, in a sense, his declaration to put an end to the endless race for geometric computation of π.

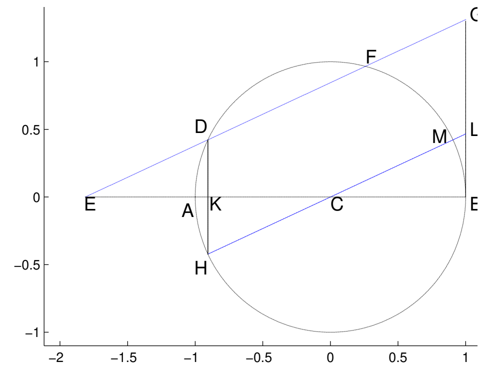

Figure 1: Huygens Theorem XII [7]

By (1)(2), the upper bound of (9) has the asymptotic formula

$$\frac{2}{3}T_n + \frac{1}{3}U_n = \pi\left(1 + \frac{1}{20}\frac{\pi^4}{n^4} + O(\frac{1}{n^6})\right),$$

and the lower bound of (8) has

$$\frac{3}{\dfrac{2}{T_n} + \dfrac{1}{U_n}} = \pi\left(1 - \frac{1}{180}\frac{\pi^4}{n^4} + O(n^{-6})\right).$$

Snell's transformation

$$[T_n, U_n] \mapsto \left[\frac{3}{\dfrac{2}{T_n} + \dfrac{1}{U_n}}, \frac{2}{3}T_n + \frac{1}{3}U_n\right] \tag{10}$$

is the first acceleration method of a sequence of intervals.

2.4 Christiaan Huygens

Christiaan Huygens (1629 - 1695) published *De circuli magnitudine inventa* (1654) [7][8] in which he proved 16 theorems on a circle, an arc, or inscribed or circumscribed polygons in a circle. German translation of this book can be read in [13].

2.4.1 The upper bound of the length of an arc

Huygens proved the upper bound of the length of an arc in Theorem XII in [7].

Theorem XII Choose a point D on the circumference of a circle and fit the point E on the extension of the diameter AB such that ED equals to the radius. The extension of ED intersects F and G with the circumference and the tangent at B, respectively. Then the length of BG is larger than the length of the arc BF. (See Figure 1).

Let C be the center of the circle and $\theta = \angle\text{BCF}$. It follows from Theorem XII and Figure 1 that

$$\theta < 2\sin\frac{\theta}{3} + \tan\frac{\theta}{3},$$

which is the upper bound of Snell's inequality (7).

Huygens wrote at the end of the proof:

> This is one of two theorems on which all of Willebrord Snell's Cyclometricus depend. [7]

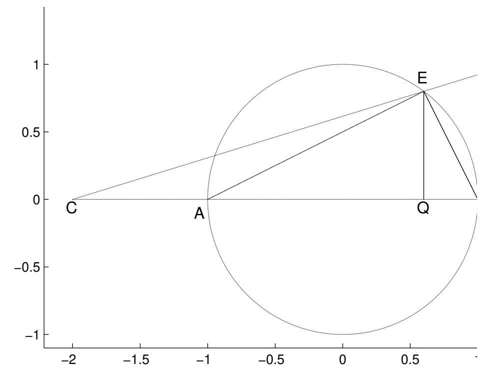

Figure 2: Huygens Theorem XIII [7]

2.4.2 The lower bound of the length of an arc

Huygens proved the lower bound of the length of an arc in Theorem XIII.

Theorem XIII Choose the point C on the the extension of the diameter AB such that AC equals to the radius. Let E be any point on the circumference. The extension of CE intersects L with the tangent at B. Then the length of BL is smaller than the length of the arc BE. (See Figure 2).

Let O be the center of the circle and $\angle \mathrm{BOE} = \theta$. Since EQ/CQ = BL/CB,

$$\mathrm{BL} = \frac{\mathrm{EQ} \cdot \mathrm{CB}}{\mathrm{CQ}} = \frac{3r \sin\theta}{2 + \cos\theta},$$

where r is the radius. It follows from Theorem XIII and Figure 2 that

$$\frac{3\sin\theta}{2 + \cos\theta} < \theta,$$

which is the lower bound of Snell's inequality (7).

3 Verified acceleration of a real sequence

3.1 Definitions

Let $\mathscr{C}$ be the set of real convergent sequences. Let $T : \mathscr{C} \to \mathscr{I}$ be a transformation with $T(\{s_n\}) = \{[u_n, v_n]\}$, where $\{s_n\}$ and $\{[u_n, v_n]\}$ converge to the same limit s. Suppose that

$$\lim_{n\to\infty} \frac{v_n - u_n}{s_{\sigma(n)} - s} = 0,$$

where u_n, v_n are determined by $s_0, s_1, \ldots, s_{\sigma(n)}$, for some integer $\sigma(n)$. Then we call T accelerates the sequence $\{s_n\} \in \mathscr{C}$ with result verification, or T is a verified acceleration method.

3.2 Christian Huygens

The first examples of the verified acceleration of a real sequence are appeared in Theorem XVI and Problem IV in *De circuli magnitudine inventa*.

Theorem XVI Let A be any point of upper semicircle, a be the length of the arc AB, s be the sine AM, s' be the length of the chord AB. (See Figure 3). Then

$$s' + \frac{1}{3}(s' - s) < a < s' + \frac{1}{3}(s' - s)\frac{4s' + s}{2s' + 3s}. \tag{11}$$

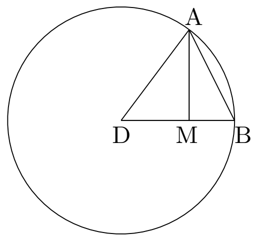

Figure 3: Huygens Theorem XVI

It follows from (11) that

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n) < \pi < T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n}. \tag{12}$$

By (1), the lower and upper bounds of (12) have the asymptotic formulas

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n) = \pi\left(1 - \frac{\pi^4}{480n^4} + O\left(\frac{1}{n^6}\right)\right),$$
$$T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n} = \pi\left(1 + \frac{\pi^6}{22400n^6} + O\left(\frac{1}{n^8}\right)\right).$$

The lower bound of (11) is the Huygens Theorem, i.e. the first step of the Richardson extrapolation process, which will be treated in Section 4.

Huygens' transformation

$$T : \{T_n\} \mapsto \left\{\left[T_{2n} + \frac{1}{3}(T_{2n} - T_n), T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n}\right]\right\}$$

is the first verified acceleration method of a real sequence.

In Problem IV Huygens gave better verified acceleration

$$T : \{T_n\} \mapsto \{[w_n, v_n]\} \tag{13}$$

where

$$u_n = T_{2n} + \frac{1}{3}(T_{2n} - T_n),$$
$$v_n = T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n},$$
$$w_n = T_n + \frac{1}{3}(T_{2n} - T_n)\frac{10(T_{2n} + T_n)}{2T_{2n} + 3T_n + \frac{4}{3}(v_n - u_n)}.$$

See [11]. The width of $[w_n, v_n]$ in (13) satisfies

$$v_n - w_n = O(n^{-6}).$$

Huygens gave a numerical result $[3.1415926533, 3.1415926538]$ by (13) with $T_{30}(= 3.1358538980)$ and $T_{60}(= 3.1401573746)$.

4 The theorem of Huygens

4.1 Definitions

If $\{s_n\}$ can be written in the form

$$s_n = s + \frac{c}{n^2} + \frac{d}{n^4} + O(n^{-6}),$$

where c, d are constants, then

$$t_n^{(1)} = s_{2n} + \frac{s_{2n} - s_n}{3},$$

satisfies

$$t_n^{(1)} = s - \frac{d}{4n^4} + O(n^{-6}).$$

The elimination of the term involving n^{-2} is called the theorem of Huygens. Huygens proved it using Euclidean geometry. The sequence transformation $T : \{s_n\} \mapsto \{t_n^{(1)}\}$ is the first step of the Richardson extrapolation process which we will treat in Section 6.

4.2 Christian Huygens

In *De circuli magnitudine inventa* C. Huygens gave three types of the theorem of Huygens.

Theorem V (On the area of a circle). Let S_n be the area of n-sided inscribed regular polygon in a circle with area S. Then

$$S_{2n} + \frac{1}{3}(S_{2n} - S_n) < S.$$

Theorem VII (On the circumference of a circle). Let T_n be the circumference of n-sided inscribed regular polygon in a circle with circumference C. Then

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n) < C.$$

The last one is the lower bound of Theorem XVI, which has been referred in Section 3.2.

Theorem XVI (On the length of an arc). Let A be any point of upper semicircle, a be the length of the arc AB, s be the sine AM, s' be the length of the chord AB. Then

$$s' + \frac{1}{3}(s' - s) < a < s' + \frac{1}{3}(s' - s)\frac{4s' + s}{2s' + 3s}.$$

4.3 Isaac Newton

Isaac Newton (1642-1727) mentioned the theorem of Huygens at least twice. Newton referred the theorem of Huygens in the letter to Michael Dary on 22 January 1675 [21, p.333][23, p.662]. In this letter Newton applied the theorem of Huygens on a construction of the length of the arc of an ellipse.

Newton also mentioned the theorem of Huygens in the letter *epistola prior* to Oldenburg on 13 June 1676 [22, pp.39-40][23, p.669]. Newton considered " For, being given the chord A of an arc and the chord B of half the arc, to find that arc most nearly, take that arc to be z, and the radius of the circle r:" [22] In Figure 4, let C be the center of a sector CbAB. Let $A = \mathrm{Bb}, B = \mathrm{AB}, z =$ arc bAB and $\angle\mathrm{ACB} = \theta$. Since $z = 2r\theta = 2r \sin^{-1} \frac{A}{2r}$,

$$\frac{A}{2r} = \sin\frac{z}{2r} = \frac{z}{2r} - \frac{z^3}{6 \cdot (2r)^3} + \frac{z^5}{120 \cdot (2r)^5} - \&\text{c}.$$

Thus

$$A = z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} - \&\text{c},$$
$$B = \frac{z}{2} - \frac{z^3}{2 \times 16 \times 6r^2} + \frac{z^5}{2 \times 16 \times 16 \times 120r^4} - \&\text{c}.$$

Eliminating the term z^3/r^2, Newton obtained

$$\frac{8B - A}{3} = z - \frac{z^5}{64 \times 120r^4} + \&\text{c},$$

and wrote

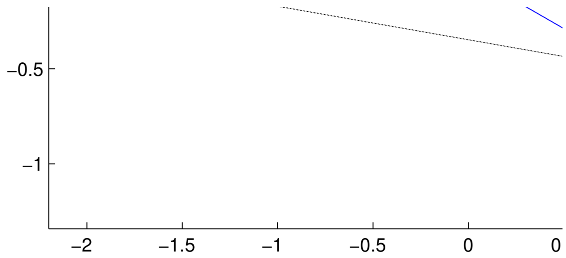

Figure 4: Epistola Prior [22]

> that is, $\frac{1}{3}(8B - A) = z$, with an error of only $z^5/7680r^4$ in excess; which is the theorem of Huygens. [22]

Newton continued, "let C be the center of the circle, d the diameter AK, and x the sagitta AD." "take AH, the fifth part of DH" [22]. Take G on the extension of AK such that KG = HC. The extensions of GB and Gb intersect E and e with the tangent at A, respectively. Then Newton derived

$$\widehat{\mathrm{AB}} - \mathrm{AE} = \frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}} \pm \&\mathrm{c}$$

and wrote

> the error being only $(16x^3/525d^3)\surd(dx) \pm$ etc, much less indeed than in the theorem of Huygens. [22]

In modern notation [4]

$$\mathrm{AE} = \frac{d\sin\theta(14+\cos\theta)}{2(9+6\cos\theta)} = \frac{d}{2}\left(\theta - \frac{1}{2100}\theta^7 + O(\theta^9)\right).$$

Furthermore Newton wrote "if we make 7AK : 3AH :: DH : n and take KG = CH − n", that is,

$$\mathrm{AG} = \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d},$$

"the error will be much smaller still." [22] In modern notation

$$\mathrm{AE} = \frac{d\sin\theta(481+47\cos\theta-3\cos 2\theta)}{2(306+222\cos\theta-3\cos 2\theta)} = \frac{d}{2}\left(\theta - \frac{931}{14112000}\theta^9 + O(\theta^{11})\right).$$

Newton's method is expanding to infinite series of $z/r, (z/r)^3, (z/r)^5, \ldots$ then eliminating terms of lower degree and was not as systematically as the Richardson extrapolation process.

5 Aitken Δ^2 process

5.1 Definitions

The most famous convergence acceleration method is the Aitken Δ^2 process. This method is defined by

$$\begin{aligned}
t_n &= s_n - \frac{(s_{n+1}-s_n)^2}{s_{n+2}-2s_{n+1}+s_n} \left(= s_n - \frac{(\Delta s_n)^2}{\Delta^2 s_n}\right) \\
&= s_{n+1} + \frac{(s_{n+1}-s_n)(s_{n+2}-s_{n+1})}{(s_{n+1}-s_n)-(s_{n+2}-s_{n+1})} \left(= s_{n+1} + \frac{\Delta s_n \Delta s_{n+1}}{\Delta s_n - \Delta s_{n+1}}\right) \\
&= s_{n+2} - \frac{(s_{n+2}-s_{n+1})^2}{s_{n+2}-2s_{n+1}+s_n} \left(= s_{n+2} - \frac{(\nabla s_{n+2})^2}{\nabla^2 s_{n+2}}\right) \\
&= \frac{s_n s_{n+2} - s_{n+1}^2}{s_{n+2}-2s_{n+1}+s_n},
\end{aligned}$$

where Δ and ∇ are the forward and backward difference operators, respectively. All four formulas are mathematical equivalent.

A.C. Aitken (1895 - 1967) used this method in [1] (1926), so it named after Aitken. The Aitken Δ^2 process was discovered by Japanese Mathematician Takakazu Seki (? - 1708) before 1680.

5.2 The circumference ratio in Japan before Seki

Until the mid 17th century, Japanese traditional π was 3.16 or $3.162(\fallingdotseq \sqrt{10})$.

A reprint of Chinese mathematical textbook *Suanxue Qimeng* (Mathematical Enlightenment, 1299) by Zhu Shijie , was published (1658) in Japan. In this book $\pi = 3/1$, $\pi = 157/50(= 3.14)$ and $\pi = 22/7(= 3.142857)$ are written.

The first Japanese mathematician who determined the circumference ratio was Shigekiyo Muramatsu. In 1663 he computed inscribed 2^n-sided regular polygons ($n = 3, \dots, 15$) in a circle of diameter 1. Muramatsu obtained $T_{2^{15}} = 3.14159264877698869248$, and compared with various Chinese values of π, he determined 3.14. See [6, pp.52-56][18, p.78].

In 1673 Yoshimasu Murase determined π as 3.1416 by using

$$T_{2^{17}} = T_{131072} = 131072 \times 0.00002396844980842366(= 3.14159265328970596352).$$

These methods used only inscribed polygons, both Muramatsu and Murase could not determine the number of exact digits.

5.3 Takakazu Seki

Takakazu Seki used the Aitken Δ^2 process in three purposes.

1 Determining the circumference of a circle, in *Katsuyō Sampō* Vol.4 (1712).

2 Computing the length of an arc, in *Katsuyō Sampō*.

3 Determining the volume of a sphere, in *Ritsu-Enritsu-Kai* (1680), *Taisei Sankei* Vol.12 (c.1711), and *Katsuyō Sampō*.

Ritsu-Enritsu-Kai (Solving the volume of the sphere) is a manuscript of Seki's draft dated 1680. *Taisei Sankei* (Accomplished Mathematical Sutra) is manuscripts of books of twenty volumes and is considered to be in collaboration with Takakazu Seki, and his disciples Katahiro Takebe (1664-1739), and his elder brother Kata'akira (or Kataaki) Takebe (1661-1716).

Kata'akira Takebe wrote the story of *Taisei Sankei* in his *Takebe-shi Denki* (Biography of the Takebe family) in 1715. According to *Takebe-shi Denki*, compiling of *Taisei Sankei* began in 1683 under the supervision of Katahiro, and once was assembled and named *Sampō Taisei* (Accomplished Mathematical Methods) in about 1695. Kata'akira by oneself resumed to compile *Taisei Sankei* in 1701 and completed in about 1711. See [6, pp.108-109].

Katsuyō Sampō (Tying the pivots of Mathmatical Methods) was published by Seki's disciple Murahide Araki and was consist of Seki's posthumous drafts.

5.3.1 Formula of the volume of a sphere

In *Ritsu-Enritsu-Kai* Seki gave the formula of the volume of a sphere of diameter D. Seki divided the sphere into m, m is even, segments by parallel equidistant planes. Seki computed

$$v_m = 2 \sum_{i=1}^{m/2} \frac{D}{2m} \left(4 \frac{(i-1)D}{m} \left(D - \frac{(i-1)D}{m} \right) + 4 \frac{iD}{m} \left(D - \frac{iD}{m} \right) \right), \tag{14}$$

with $m = 50, 100, 200$ and $D = 10$, and gave

$$a = v_{50} = 666.4, \quad b = v_{100} = 666.6, \quad c = v_{200} = 666.65.$$

Seki wrote in *Ritsu-Enritsu-Kai* that

> Put the difference a from b, multiply the difference b from c, make the numerator by the product. Put the difference a from b, subtract the difference b from c, make the denominator by the difference. Multiply b with the denominator, and add the numerator, make numerator by the summation, divide the numerator by the denominator, chop under 0.005, and obtain $666\frac{2}{3}$ which is the divided volume. [14]

This means

$$\frac{((b-a)-(c-b))b+(b-a)(c-b)}{(b-a)-(c-b)}\left(=b+\frac{(b-a)(c-b)}{(b-a)-(c-b)}\right)=666\frac{2}{3}(=\frac{2}{3}D^3), \tag{15}$$

which is the Aitken Δ^2 process. Since the ratio of the areas of the square and the circle at cut plane is $1:\pi/4$, Seki determined that the volume of the sphere is $\frac{1}{6}\pi D^3$.

The volumes a, b, c are represented as

$$a=\frac{2D^3}{3}-\frac{2D^3}{3m^2},\quad b=\frac{2D^3}{3}-\frac{D^3}{6m^2},\quad c=\frac{2D^3}{3}-\frac{D^3}{24m^2}.$$

where $m=50, D=10$. Therefore

$$\frac{2D^3}{3}-\frac{D^3}{6m^2}+\frac{\frac{D^3}{2m^2}\frac{D^3}{8m^2}}{\frac{D^3}{2m^2}-\frac{D^3}{8m^2}}=\frac{2D^3}{3}-\frac{D^3}{6m^2}+\frac{D^3}{6m^2}=\frac{2D^3}{3}.$$

This fact was proved [5] by Matsusaburo Fujiwara (1881-1946). He also pointed out that if we divide $2m, 2km, 2k^2m$ instead of $50, 100, 200$, the formula (15) gives the constant independent of m and k.

From the numerical analysis view point, the formula (14) is the m-panels trapezoidal rule for the numerical integration

$$\int_0^D 4y(D-y)dy=\frac{2D^3}{3}.$$

Therefore we can say that Seki accelerated the trapezoidal rule by the Aitken Δ^2 process.

Seki wrote only the formula of (15) in *Ritsu-Enritsu-Kai* and *Katsuyō Sampō*, but Katahiro Takebe wrote in *Taisei Sankei* Vol. 12 as follows:

> Let $b-a$ be the front difference, $c-b$ the behind difference. Add the number by applying *zōyaku-jutsu* (the art of summation of geometric series) to both differences to b, we get the divided volume. [16][17]

Takebe called $b-a, c-b$ the front difference and the behind difference, respectively, and called to obtain

$$\frac{(b-a)(c-b)}{(b-a)-(c-b)}$$

zōyaku-jutsu. Therefore Takebe considered that a, b, c approximately equal to the consecutive partial sums of a geometric series, say

$$a=k(1+r+\cdots+r^{n-1}), b=k(1+r+\cdots+r^n), c=k(1+r+\cdots+r^{n+1}).$$

Since $b-a=kr^n$ and $c-b=kr^{n+1}$, we have

$$b+\frac{(b-a)(c-b)}{(b-a)-(c-b)}=k(1+r+\cdots+r^n)+\frac{kr^{n+1}}{1-r}=\frac{k}{1-r}. \tag{16}$$

(See Horiuchi [6, pp.248-249]). When $n=1$ Yoshisuke Matsunaga (? - 1747) who was a second-generation pupil of Seki proved (16) in *Kigenkai* or *Sampō Shūsei* [10].

5.3.2 Computing the circumference of a circle

In *Katsuyō Sampō* [15] Seki determined the circumference of a circle by using (see [6, pp.252-253][18, p.111])

$$T_{2^{16}} + \frac{(T_{2^{16}} - T_{2^{15}})(T_{2^{17}} - T_{2^{16}})}{(T_{2^{16}} - T_{2^{15}}) - (T_{2^{17}} - T_{2^{16}})} = \text{(very little less than)}3.14159265359, \tag{17}$$

which is the Aitken Δ^2 process.

By (1),

$$V_n = T_{2n} + \frac{(T_{2n} - T_n)(T_{4n} - T_{2n})}{(T_{2n} - T_n) - (T_{4n} - T_{2n})} = \pi \left(1 + \frac{\pi^4}{1920n^4} + \frac{11\pi^6}{516096n^6} + O(n^{-8})\right). \tag{18}$$

Therefore the Δ^2 process accelerates $\{T_{2^n}\}$. Moreover the inequality

$$T_{2^n} < \pi < V_{2^{n-2}} \tag{19}$$

holds. Furthermore $\{V_{2^n}\}$ monotonically decreases and $\{T_{2^n}\}$ monotonically increases, and both converge to π. Seki did not mention these facts.

By using the definite circumference of a circle, Ski derived the rational approximate 355/113(= 3.14159292) of π. See [6, p.248]. We consider that Seki gave only 12 digits of π for this purpose.

Seki computed the length of an arc similarly to the circumference of a circle.

6 Richardson extrapolation process

6.1 Definitions

Suppose a sequence $\{s_n\} \in \mathscr{S}$ has the asymptotic representation

$$s_n = s + c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + \ldots + c_m\lambda_m^n + o(\lambda_m^n), \tag{20}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m, (1 > |\lambda_1| > |\lambda_2| > \ldots > |\lambda_m|)$ are known constants and $c_1, c_2, \ldots, c_m$ are constants. The double sequence $\{T_n^{(k)}\}$ is defined by

$$\begin{aligned} T_n^{(0)} &= s_n, \quad n = 1, 2, \ldots, \\ T_n^{(k)} &= T_{n+1}^{(k-1)} + \frac{\lambda_k}{1 - \lambda_k}(T_{n+1}^{(k-1)} - T_n^{(k-1)}), \quad k = 1, 2, \ldots, m; n = 1, 2, \ldots. \end{aligned} \tag{21}$$

Then

$$T_n^{(k)} = s + c_{k+1} \prod_{i=1}^{k} \frac{\lambda_{k+1} - \lambda_i}{1 - \lambda_i} \lambda_{k+1}^n + o(\lambda_{k+1}^n), \quad k = 1, 2, \ldots, m-1. \tag{22}$$

The procedure to get $\{T_n^{(k)}\}$ by the sequential elimination of the terms involving $\lambda_1^n, \lambda_2^n, \ldots$ is called the Richardson extrapolation process.

When a sequence $\{s_n\} \in \mathscr{S}$ has the asymptotic representation

$$s_n = s + \frac{c_1}{n^2} + \frac{c_2}{n^4} + \frac{c_3}{n^6} + \ldots + \frac{c_m}{n^{2m}} + o(n^{-2m}), \tag{23}$$

where $c_1, c_2, \ldots, c_m$ are constants, the subsequence $\{s_{2^n}\}$ satisfies (20) with $\lambda_k = 2^{-2k}(k = 1, \ldots, m)$. In this case the Richardson extrapolation process are written by

$$\begin{aligned} T_n^{(0)} &= s_{2^n}, n = 1, 2, \ldots, \\ T_n^{(k)} &= T_{n+1}^{(k-1)} + \frac{1}{2^{2k} - 1}(T_{n+1}^{(k-1)} - T_n^{(k-1)}), \quad k = 1, 2, \ldots; n = 1, 2, \ldots. \end{aligned} \tag{24}$$

Since $\lambda_k = 2^{-2k}$, it follows from (22) that

$$T_n^{(k)} = s + (-1)^k c_{k+1} 2^{(k+1)(-k-2n)} + o(2^{(k+1)(-k-2n)}), \quad k = 1, 2, \ldots, m-1. \tag{25}$$

In 1927 Lewis Fry Richardson [12] (1881-1953) treated (21) and (24). By this reason the sequence transformation $\{s_n\} \mapsto \{T_n^{(k)}\}$ is called the Richardson extrapolation process.

The Richardson extrapolation process was discovered by Seki's disciple Katahiro Takebe before 1710, probably before 1695.

6.2 Katahiro Takebe

In *Taisei Sankei* Vol. 12 [16][17], Takebe computed $T_n^{(0)} = (T_{2^n})^2 = 2^{2n}100\sin^2(\pi/2^n)$ $(n = 1, \ldots, 9)$, i.e.

$$\begin{aligned}
T_1^{(0)} &= (T_2)^2 = 400, \\
T_2^{(0)} &= (T_4)^2 = 800, \\
&\ldots \\
T_9^{(0)} &= (T_{512})^2 = \text{(little over than)}986.94805369467323179511059076580.
\end{aligned}$$

Takebe watched

$$\frac{T_{n+1}^{(0)} - T_n^{(0)}}{T_{n+2}^{(0)} - T_{n+1}^{(0)}}$$

tends to 4. Replacing $(T_{n+1}^{(0)} - T_n^{(0)})/(T_{n+2}^{(0)} - T_{n+1}^{(0)})$ by 4 in Master Seki's method, i.e. the Δ^2 process

$$T_{n+1}^{(0)} + \frac{(T_{n+1}^{(0)} - T_n^{(0)})(T_{n+2}^{(0)} - T_{n+1}^{(0)})}{(T_{n+1}^{(0)} - T_n^{(0)}) - (T_{n+2}^{(0)} - T_{n+1}^{(0)})} = T_{n+1}^{(0)} + \frac{T_{n+1}^{(0)} - T_n^{(0)}}{\frac{T_{n+1}^{(0)} - T_n^{(0)}}{T_{n+2}^{(0)} - T_{n+1}^{(0)}} - 1},$$

Takebe computed

$$T_n^{(1)} = T_{n+1}^{(0)} + \frac{T_{n+1}^{(0)} - T_n^{(0)}}{4 - 1}, \quad n = 1, \ldots, 8,$$

such as

$$\begin{aligned}
T_1^{(1)} &= \text{(little over than)}933.33333333333333333333333333333, \\
T_2^{(1)} &= \text{(very little over than)}983.01106693739722922306405501931, \\
&\ldots \\
T_8^{(1)} &= \text{(little less than)}986.96043986022713417377439639209.
\end{aligned}$$

Moreover Takebe watched

$$\frac{T_{n+1}^{(1)} - T_n^{(1)}}{T_{n+2}^{(1)} - T_{n+1}^{(1)}}$$

tends to 16, he computed

$$T_n^{(2)} = T_{n+1}^{(1)} + \frac{T_{n+1}^{(1)} - T_n^{(1)}}{16 - 1}, \quad n = 1, \ldots, 7.$$

Similarly Takebe watched

$$\frac{T_{n+1}^{(k-1)} - T_n^{(k-1)}}{T_{n+2}^{(k-1)} - T_{n+1}^{(k-1)}}$$

tends to 4^k, he computed

$$T_n^{(k)} = T_{n+1}^{(k-1)} + \frac{T_{n+1}^{(k-1)} - T_n^{(k-1)}}{4^k - 1}, \quad k = 1, \ldots, 8; n = 1, \ldots, 9 - k, \tag{26}$$

and got

$$T_1^{(8)} = \text{(little less than)}986.96044010893586188344909998747,$$

and he determined the squared definite circumference

(very little less than)986.96044010893586188344491,

and then he extracted the square root

(little over than)31.415926535897932384626 43,

which was his definite circumference. Takebe's method is just the Richardson extrapolation process.

The squared length of inscribing polygon $(T_{2^n})^2 = 100 \times 2^{2n} \sin^2(\pi/2^n)$ has the asymptotic expansion

$$(T_{2^n})^2 = 100\pi^2 \left(1 + \sum_{j=1}^{\infty} (-1)^j \frac{\pi^{2j}}{(2j+2)!} 2^{-2jn+2j+1}\right).$$

By (25), we have

$$T_1^{(8)} - 100\pi^2 \sim -\frac{\pi^{20}}{20!} \times 100 \times 2^{-71} \risingdotseq -1.53 \times 10^{-28},$$
$$\sqrt{T_1^{(8)}} - 10\pi \sim -\frac{\pi^{19}}{20!} \times 10 \times 2^{-72} \risingdotseq -2.43 \times 10^{-30}.$$

Takebe's method can give exact 30 decimal digits.

More than two decades after, Takebe gave exact 41 decimal digits in *Tetsujutsu Sankei* (1722) [20] of which English translation [?] will be published.

Takebe noted in *Tetsujutsu Sankei*

> Since all kinds of numbers by the art of summation of geometric series were published in *Enritsu* (i.e. *Taisei Sankei* Vol.12 Sec.1), we omitted them here. [20]

According to *Takebe-shi Denki* and the above note, Katahiro Takebe discovered the Richardson extrapolation process probably before 1695.

7 Conclusion

In Europe the circumference ratio was computed using inscribed and circumscribed polygons in a circle from Archimedes to Van Ceulen. In Japan the ratio was computed using inscribed polygons only. W. Snell (1621) and T. Seki (before 1680) made breakthroughs to these methods. Snell used weighted mean methods and Seki used the Δ^2 process. Snell's methods were proved by C. Huygens (1654) using Euclidean geometry. Seki's method was explained and partially proved by Katahiro Takebe and Y. Matsunaga.

The verified acceleration method was given and proved by C. Huygens.

The theorem of Huygens, the first step of the Richardson extrapolation process, was also given and proved by Huygens. I. Newton (1675) treated the theorem of Huygens by infinite series. Newton's method was expanding to infinite series then eliminating terms of lower degree and was not as systematically as the Richardson extrapolation process. Takebe (before 1695) discovered the Richardson extrapolation process by improving Seki's Δ^2 process.

The Δ^2 process was discovered by Seki. This fact was found by Claude Brezinski [2].

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