Advanced Studies in Pure Mathematics 79, 2018
Mathematics of Takebe Katahiro and History of Mathematics in East Asia
pp. 83–99

Determinants by Seki Takakazu from the group-theoretic viewpoint

Naoki Osada

Abstract.

In the *Kai Fukudai no Hō* (*Methods of Solving Concealed Problems*) re-revised in 1683, Seki Takakazu gave two eliminating procedures for a system of n polynomial equations of degree $n-1$. Both procedures are derived from a formula in which the determinant of the coefficient matrix of the system of equations vanishes. Seki called the first procedure the successive multiplication of equations by coefficients of other equations (*chikushiki kōjō*). Because the first procedure was so complicated, he invented another procedure consisting of shuffles (*kōshiki*) and oblique multiplications (*shajō*), an extension of the rule of Sarrus.

Although there are some errors when $n \geq 5$, we can prove that Seki's conception of the second procedure is essentially correct. We can summarize that Seki's second procedure is based on a coset decomposition of the symmetric group S_n with respect to the dihedral subgroup D_n. We also clarify Seki's terms *forward* (*jun*) and *backward* (*gyaku*) which have not yet been explained by historians of mathematics.

§1. Introduction

The *Kai Fukudai no Hō* [解伏題之法] (*Methods of Solving Concealed Problems*) re-revised by Seki Takakazu [関孝和] (1640s–1708) in 1683 was passed down among students of his school in the form of manuscripts such as [15, 16]. A *concealed problem* (*fukudai* [伏題]) means the problem which is solved using a system of equations with several unknowns.

Received January 3, 2015.
Revised March 26, 2017.
2010 *Mathematics Subject Classification.* Primary 01A27; Secondary11C20.
Key words and phrases. history of Japanese mathematics, permutation group theory.

Let x be an unknown to be found and y an auxiliary unknown. Seki considers a pair of polynomial equations for y of degree n:

$$\text{(1)} \qquad \begin{cases} a_0(x) + a_1(x)y + \cdots + a_n(x)y^n = 0, \\ b_0(x) + b_1(x)y + \cdots + b_n(x)y^n = 0. \end{cases}$$

Eliminating y^n in (1), Seki transforms the pair (1) into a system of [simultaneous] n polynomial equations of degree $n-1$:

$$\text{(2)} \qquad \begin{cases} x_{11} + x_{12}y + \cdots + x_{1n}y^{n-1} = 0, \\ x_{21} + x_{22}y + \cdots + x_{2n}y^{n-1} = 0, \\ \cdots \\ x_{n1} + x_{n2}y + \cdots + x_{nn}y^{n-1} = 0, \end{cases}$$

where x_{ij} are polynomials in terms of the required unknown x. Seki refers to (2) as the [*transformed*] *system of* n *equations* (*kanshiki* [換式]).

For the systen of n equations (2) to have a solution y, it is necessary for the determinant of the coefficients to vanish:

$$\text{(3)} \qquad \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} = 0.$$

In the *Kai Fukudai no Hō*, an algorithm for transforming the pair (1) into the system of n equations (2) was explicitly given. See e.g. M. Fujiwara [1, p. 203] or H. Katō [7, pp. 138–139]. For deriving (3) from (2), however, Seki gave two different procedures: The first is referred to as the *successive multiplication of each equation by coefficients of other equations* (*chikushiki kōjō* [逐式交乗]) and tables of multiplications are given only for $n = 2,\ 3$ and 4. For this procedure, see N. Osada [13].

This first procedure was so complicated that Seki introduced a second procedure called *shuffles* (*kōshiki* [交式]) and oblique multiplications (*shajō* [斜乗]), which generalize the rule of Sarrus.

Seki's choice of *shuffles* for the system of n equations (See Fig. 1) gives incorrect multiplications when $n = 5$, but a corrected procedure for any n greater than 3 is given by Matsunaga Yoshisuke [松永良弼] (1694–1744), a second-generation pupil of Seki, in the *Kai Fukudai Kōshiki Shajō no Genkai* [解伏題交式斜乗之諺解] (*Commentary on the Shuffles and Oblique Multiplications in the Kai Fukudai no Hō*) [10] re-revised in 1715. However, the algorithm used by Matsunaga is different from that of Seki who used the expression

Table 1. List of abbreviations

abbreviation	Seki's term in English	Seki's term in Japanese
b.	*backward*	*gyaku* [逆]
f.	*forward*	*jun* [順]
3-*system*	*system of* 3 *equations*	*kan-san-shiki* [換三式]
4-systm	*system of* 4 *equations*	*kan-shi-shiki* [換四式]
5-*system*	*system of* 5 *equations*	*kan-go-shiki* [換五式]

"*forward* and *backward* [rearrangements] proceed and are followed by attaching 1 in turn"[1]. Matsunaga did not mention the terms *forward* (*jun* [順]) and *backward* (*gyaku* [逆]).

Toita Yasusuke [戸板保佑] (1708–1784), a second-generation pupil of Matsunaga, regarded the terms *forward* and *backward* as signs of cofactors in his *Seikoku Inhō Den* [生尅因法伝] (*Commentary on Creative and Annihilative Terms Using Multiplicative Methods*) [18] in 1759. Most modern and present day historians of mathematics, such as T. Hayashi [3], M. Fujiwara [1], H. Katō [7], A. Hirayama [4] and H. Komatsu [9] have treated the terms *forward* and *backward* as signs of *oblique multiplications* or cofactors. On the other hand, Y. Mikami [11, p. 14] notes: "Though there are added in these figures the ideograms *jun* (regular order) and *gyaku* (reverse order), we are not yet enabled to decipher them correctly". K. Satō [14] is the only historian who correlated the terms *forward* and *backward* with permutations. For these, see N. Osada [12].

In this paper we resolve the procedure of *shuffles* and *oblique multiplications* by means of the group theory. In particular, we clarify Seki's terms *forward* and *backward*.

In the sequel, we shall use the abbreviations as listed in Table 1.

§2. *Shuffles*

2.1. Original text for *shuffles*

The original text[2] of referring to *shuffles* (*kōshiki*) in the *Kai Fukudai no Hō* is as follows[3].

> The [*shuffles* of the] 4-*system* is derived from [those of] the 3-*system*. The [*shuffles* of the] 5-*system* is derived from

[1]順逆共遁添一

[2]從換三式起換四式, 從換四式起換五式, 逐如此.（換二式換三式者, 不及交式也）順逆共遁添一, 得次. 乃式数奇者, 皆順. 偶者, 順逆相交也.

[3]We referred to Goto and Komatsu [2] for English translation of the original text.

5-*system*

5	4	3	2	1
4	5	2	3	1
3	2	5	4	1
2	3	4	5	1
3	5	4	2	1
5	3	2	4	1
4	2	3	5	1
2	4	5	3	1
4	3	5	2	1
3	4	2	5	1
5	2	4	3	1
2	5	3	4	1

4-*system*

b.	f.	b.	f.
4	3	2	1
2	4	3	1
3	2	4	1

3-*system*

f.	f.	f.
3	2	1

Fig. 1. The permutations corresponding to Seki's choice of *shuffles* in the original form.

> [those of] the 4-*system*, and so on. (The 2-*system* and the 3-*system* require no *shuffles*.)
>
> *forward* and *backward* [rearrangements] proceed and are followed by attaching 1 in turn so as to produce the next [*shuffles*]. That is, if the number of equations is odd, all [rearrangements] are *forward*; if even, alternately *forward* and *backward*.

Next, Seki gives lists of the permutations corresponding to *shuffles* for the 3-, 4- and 5-*systems*. These lists are shown in Fig. 1.

2.2. Interpretation of a *shuffle*

As can be seen in Fig. 1, the original text, based on the Japanese writing system, has the permutations set out in rows from right to left [5, p. 193] [6, p. 194]. But from here on, we will rearrange them into columns running from top to bottom as shown in Fig. 2.

Applying three *shuffles* for 4-*systems* in Fig. 2 to the 4-*system* (4), we obtain three 4-*systems* (4), (5) and (6):

$$\text{(4)} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array} \qquad \begin{cases} x_{11}+x_{12}y+x_{13}y^2+x_{14}y^3=0\\ x_{21}+x_{22}y+x_{23}y^2+x_{24}y^3=0\\ x_{31}+x_{32}y+x_{33}y^2+x_{34}y^3=0\\ x_{41}+x_{42}y+x_{43}y^2+x_{44}y^3=0, \end{cases}$$

3-*system*

f.	1
f.	2
f.	3

4-*system*

f.	1	1	1
b.	2	3	4
f.	3	4	2
b.	4	2	3

5-*system*

[f.]	1	1	1	1	1	1	1	1	1	1	1	1
[f.]	2	3	4	5	2	4	5	3	2	5	3	4
[f.]	3	2	5	4	4	2	3	5	5	2	4	3
[f.]	4	5	2	3	5	3	2	4	3	4	2	5
[f.]	5	4	3	2	3	5	4	2	4	3	5	2

Fig. 2. Seki's permutations rotated $90°$ in counterclockwise to fit the representation of *shuffles* adopted in this paper.

$$\text{(5)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 2\\\hline\end{array} \qquad \begin{cases} x_{11}+x_{12}y+x_{13}y^2+x_{14}y^3=0\\ x_{31}+x_{32}y+x_{33}y^2+x_{34}y^3=0\\ x_{41}+x_{42}y+x_{43}y^2+x_{44}y^3=0\\ x_{21}+x_{22}y+x_{23}y^2+x_{24}y^3=0, \end{cases}$$

$$\text{(6)} \qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline 2\\\hline 3\\\hline\end{array} \qquad \begin{cases} x_{11}+x_{12}y+x_{13}y^2+x_{14}y^3=0\\ x_{41}+x_{42}y+x_{43}y^2+x_{44}y^3=0\\ x_{21}+x_{22}y+x_{23}y^2+x_{24}y^3=0\\ x_{31}+x_{32}y+x_{33}y^2+x_{34}y^3=0. \end{cases}$$

Fig. 3 and Fig. 4 show the algorithm of giving rise to *shuffles* for the 4- and 5-*systems*, respectively. Here, a line connecting two letters means a *forward* or *backward* rearrangement, and arrows $\searrow$ and $\rightarrow$ mean *attaching 1 in turn*.

§3. *Oblique multiplications*

3.1. Translation of the original text of *oblique multiplications*

Subsequently, Seki states his procedure of *oblique multiplications*. Seki's original text[4] is as follows.

> At each [simultaneous equations obtained by] a *shuffle*, we perform *oblique multiplications* from left and from right and

[4]交式各布之, 從左右斜乘, 而得生尅也. (若當空級者, 除之) 換式數奇者, 以左斜乘爲生, 以右斜乘爲尅. 偶者左斜乘右斜乘共生尅相交也.

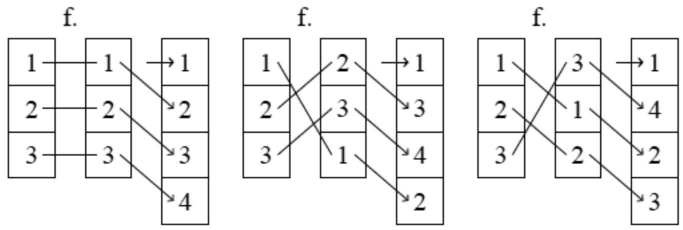

Fig. 3. The algorithm of giving rise to *shuffles* for the 4-*system*.

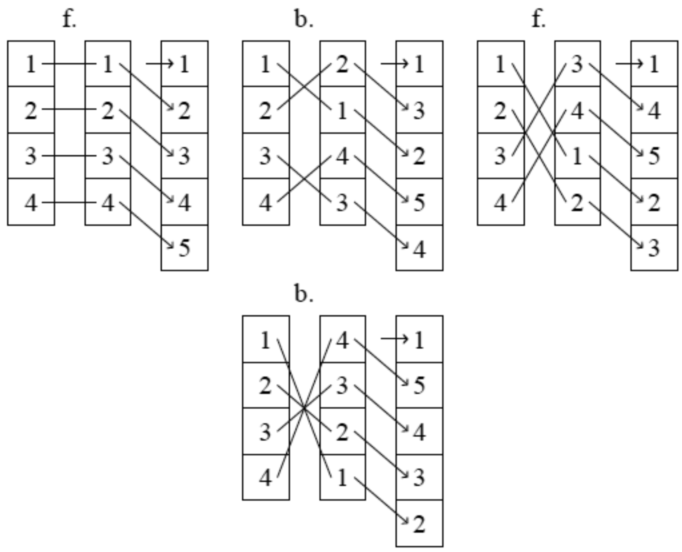

Fig. 4. The algorithm of giving rise to the first four *shuffles* for 5-*system*.

thus obtain *creative* and *annihilative* terms. (If a multiplication hits an empty term, then delete it.)

If the number of the [*transformed*] *equations* is odd, the *oblique multiplications* from the left are *creative*, and *oblique multiplications* from the right are *annihilative*. If the number is even, then *oblique multiplications* from both left and right are alternately *creative* and *annihilative*.

Seki explains further using so-called *diagrams of oblique multiplications*.

3.2. Interpretation of *oblique multiplications*

Let us now apply *shuffles* and *oblique multiplications* to the 3-, 4- and 5-*systems*.

For a *transformed system* (2) *of* n *equations* of degree $n-1$ we define the coefficient matrix by

$$X^{(n)} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \cdots & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}.$$

Let $\mathscr{S}(X^{(n)})$ be the set of all unsigned *oblique multiplications* of $X^{(n)}$, that is the set of all unsigned multiplications of n elements parallel to the diagonal or anti-diagonal, and $\mathscr{S}_{\pm}(X^{(n)})$ be the set of all *oblique multiplications* of $X^{(n)}$.

For the coefficient matrix $X^{(3)}$ of a 3-*system*, according to the *diagrams of oblique multiplications*, we have

$$\begin{aligned}\mathscr{S}_{\pm}(X^{(3)}) = \{ &+ x_{11}x_{22}x_{33}, +x_{21}x_{32}x_{13}, +x_{31}x_{12}x_{23}, \\ &- x_{11}x_{32}x_{23}, -x_{21}x_{12}x_{33}, -x_{31}x_{22}x_{13}\}.\end{aligned}$$

The summation of all 6 terms of $\mathscr{S}_{\pm}(X^{(3)})$ is the determinant of $X^{(3)}$. This is commonly called the rule of Sarrus.

Let $X_1^{(4)}, X_2^{(4)}$ and $X_3^{(4)}$ be matrices of coefficients of the 4-*systems* (4), (5) and (6), respectively. Then we have

$$\begin{aligned}&\mathscr{S}_{\pm}(X_1^{(4)}) \\ &= \{ + x_{11}x_{22}x_{33}x_{44}, -x_{21}x_{32}x_{43}x_{14}, +x_{31}x_{42}x_{13}x_{24}, -x_{41}x_{12}x_{23}x_{34}, \\ &\quad - x_{11}x_{42}x_{33}x_{24}, +x_{21}x_{12}x_{43}x_{34}, -x_{31}x_{22}x_{13}x_{44}, +x_{41}x_{32}x_{23}x_{14}\}, \\ &\mathscr{S}_{\pm}(X_2^{(4)}) \\ &= \{ + x_{11}x_{32}x_{43}x_{24}, -x_{31}x_{42}x_{23}x_{14}, +x_{41}x_{22}x_{13}x_{34}, -x_{21}x_{12}x_{33}x_{44}, \\ &\quad - x_{11}x_{22}x_{43}x_{34}, +x_{31}x_{12}x_{23}x_{44}, -x_{41}x_{32}x_{13}x_{24}, +x_{21}x_{42}x_{33}x_{14}\}, \\ &\mathscr{S}_{\pm}(X_3^{(4)}) \\ &= \{ + x_{11}x_{42}x_{23}x_{34}, -x_{41}x_{22}x_{33}x_{14}, +x_{21}x_{32}x_{13}x_{44}, -x_{31}x_{12}x_{43}x_{24}, \\ &\quad - x_{11}x_{32}x_{23}x_{44}, +x_{41}x_{12}x_{33}x_{24}, -x_{21}x_{42}x_{13}x_{34}, +x_{31}x_{22}x_{41}x_{14}\}.\end{aligned}$$

The summation of all 24 terms of $\mathscr{S}_{\pm}(X_1^{(4)}) \cup \mathscr{S}_{\pm}(X_2^{(4)}) \cup \mathscr{S}_{\pm}(X_3^{(4)})$ is the determinant of $X_1^{(4)}$ (See T. Hayashi [3, p. 588]).

Let $X_1^{(5)}$ be the coefficient matrix of a 5-*system*, i.e. the systen of n equations (2) with $n = 5$. Let $X_4^{(5)}$ be the coefficient matrix applying the permutation $\begin{array}{|c|}\hline 1\\ \hline 5\\ \hline 4\\ \hline 3\\ \hline 2\\ \hline\end{array}$. Then we have

$$\begin{aligned}
&\mathscr{S}_{\pm}(X_1^{(5)})\\
={}&\{+x_{11}x_{22}x_{33}x_{44}x_{55}, +x_{21}x_{32}x_{43}x_{54}x_{15}, +x_{31}x_{42}x_{53}x_{14}x_{25},\\
&+x_{41}x_{52}x_{13}x_{24}x_{35}, +x_{51}x_{12}x_{23}x_{34}x_{45}, -x_{11}x_{52}x_{43}x_{34}x_{25},\\
&-x_{21}x_{12}x_{53}x_{44}x_{35}, -x_{31}x_{22}x_{13}x_{54}x_{45}, -x_{41}x_{32}x_{23}x_{14}x_{55},\\
&-x_{51}x_{42}x_{33}x_{24}x_{15}\},\\
&\mathscr{S}_{\pm}(X_4^{(5)})\\
={}&\{+x_{11}x_{52}x_{43}x_{34}x_{25}, +x_{51}x_{42}x_{33}x_{24}x_{15}, +x_{41}x_{32}x_{23}x_{14}x_{55},\\
&+x_{31}x_{22}x_{13}x_{54}x_{45}, +x_{21}x_{12}x_{53}x_{44}x_{35}, -x_{11}x_{22}x_{33}x_{44}x_{55},\\
&-x_{51}x_{12}x_{23}x_{34}x_{45}, -x_{41}x_{52}x_{13}x_{24}x_{35}, -x_{31}x_{42}x_{53}x_{14}x_{25},\\
&-x_{21}x_{32}x_{43}x_{54}x_{15}\}.
\end{aligned}$$

Since $\mathscr{S}(X_1^{(5)}) = \mathscr{S}(X_4^{(5)})$, Seki's choice of *shuffles* for the 5-*system* is incorrect (See H. Katō [7, p. 45]).

§4. Notation and definitions in group theory

For a permutation μ the sign of μ is defined by

$$\operatorname{sgn}(\mu) = \begin{cases} 1 & \text{if } \mu \text{ is an even permutation,}\\ -1 & \text{if } \mu \text{ is an odd permutation.}\end{cases}$$

The group S_n of all permutations on the set $\{1, 2, \ldots, n\}$ is called the symmetric group of degree n. The product is composition of permutations from right to left. The subgroup A_n of all even permutations of S_n is called the alternating group of degree n.

Let n be an integer greater than 2. We define σ, $\tau \in S_n$ as

$$\sigma = \sigma_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n\\ 2 & 3 & 4 & \cdots & 1\end{pmatrix}, \tag{7}$$

$$\tau = \tau_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n\\ 1 & n & n-1 & \cdots & 2\end{pmatrix}, \tag{8}$$

respectively. Since $\sigma = (123\cdots n) = (1\ n)(1\ n-1)\cdots(1\ 2)$ and

$$\tau = \begin{cases} (2\ n)(3\ n-1)\cdots(m\ m+1) & \text{if } n = 2m-1, \\ (2\ n)(3\ n-1)\cdots(m\ m+2) & \text{if } n = 2m, \end{cases}$$

we have

(9) $\quad \sigma \in A_n \quad$ if and only if $\quad n \equiv 1 \pmod 2$,

(10) $\quad \tau \in A_n \quad$ if and only if $\quad n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$,

and $\sigma^n = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}$. Let D_n be the subgroup of S_n generated by σ and τ. The subgroup D_n is called the *dihedral subgroup* and we have the following equality:

(11) $$D_n = \langle \sigma, \tau \rangle = \{1, \sigma, \ldots, \sigma^{n-1}, \tau, \tau\sigma, \ldots, \tau\sigma^{n-1}\}.$$

Let G be a finite group and H a subgroup of G. For $a \in G$, the set of the form $aH = \{ah | h \in H\}$ is called a left coset of H. Two elements $a, b \in G$ are contained in the same coset of H if and only if $a^{-1}b \in H$. The group G is partitioned into a disjoint union of left cosets of H. A subset T of G is said to be a set of left representatives (or transversal) of H if T contains exactly one element of each left coset of H. When $T = \{a_1, \ldots, a_m\}$ is a set of representatives of H, we write

$$G = \sum_{i=1}^{m} a_i H.$$

This is called the left coset decomposition of G with respect to H.

§5. *Shuffles* and *oblique multiplications* from a group-theoretic viewpoint

Let n be an integer greater than 2. Let σ and τ be the permutations defined by (7) and (8), respectively. Recall the dihedral subgroup D_n of S_n is generated by σ and τ.

The determinant of an $n \times n$-matrix $(x_{i,j})$ is defined by

(12) $$\det(x_{i,j}) = \sum_{\eta \in S_n} \mathrm{sgn}(\eta) x_{\eta(1),1} \cdots x_{\eta(n),n}.$$

We define the action of a permutation $\rho \in S_n$ on a matrix $X = (x_{i,j})$ by

$$\rho(X) = \begin{pmatrix} x_{\rho(1),1} & x_{\rho(1),2} & x_{\rho(1),3} & \cdots & x_{\rho(1),n} \\ x_{\rho(2),1} & x_{\rho(2),2} & x_{\rho(2),3} & \cdots & x_{\rho(2),n} \\ x_{\rho(3),1} & x_{\rho(3),2} & x_{\rho(3),3} & \cdots & x_{\rho(3),n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{\rho(n),1} & x_{\rho(n),2} & x_{\rho(n),3} & \cdots & x_{\rho(n),n} \end{pmatrix}.$$

Lemma 1. For a permutation $\rho \in S_n$ and a matrix $X = (x_{i,j})$ of degree n, the set of all unsigned oblique multiplications of $\rho(X)$ is given as follows:

$$\mathscr{S}(\rho(X)) = \left\{ \prod_{j=1}^{n} x_{\rho(\tau^t\sigma^s(j)),j} \mid t = 0, 1; s = 0, \ldots, n-1 \right\}.$$

Proof. The left unsigned oblique multiplications (parallel to the principal diagonal) of the matrix $\rho(X)$ are

$$\begin{cases} \quad x_{\rho(1),1}x_{\rho(2),2}x_{\rho(3),3}\cdots x_{\rho(n),n}, \\ \quad x_{\rho(2),1}x_{\rho(3),2}x_{\rho(4),3}\cdots x_{\rho(1),n} \\ = x_{\rho(\sigma(1)),1}x_{\rho(\sigma(2)),2}x_{\rho(\sigma(3)),3}\cdots x_{\rho(\sigma(n)),n}, \\ \quad x_{\rho(3),1}x_{\rho(4),2}x_{\rho(5),3}\cdots x_{\rho(2),n} \\ = x_{\rho(\sigma^2(1)),1}x_{\rho(\sigma^2(2)),2}x_{\rho(\sigma^2(3)),3}\cdots x_{\rho(\sigma^2(n)),n}, \\ \cdots \\ \quad x_{\rho(n),1}x_{\rho(1),2}x_{\rho(2),3}\cdots x_{\rho(n-1),n} \\ = x_{\rho(\sigma^{n-1}(1)),1}x_{\rho(\sigma^{n-1}(2)),2}x_{\rho(\sigma^{n-1}(3)),3}\cdots x_{\rho(\sigma^{n-1}(n)),n}. \end{cases}$$

The right unsigned oblique multiplications (parallel to the anti-diagonal) of the matrix $\rho(X)$ are

$$\begin{cases} \quad x_{\rho(1),1}x_{\rho(n),2}x_{\rho(n-1),3}\cdots x_{\rho(2),n} \\ = x_{\rho(\tau(1)),1}x_{\rho(\tau(2)),2}x_{\rho(\tau(3)),3}\cdots x_{\rho(\tau(n)),n}, \\ \quad x_{\rho(2),1}x_{\rho(1),2}x_{\rho(n),3}\cdots x_{\rho(3),n} \\ = x_{\rho(\tau\sigma^{n-1}(1)),1}x_{\rho(\tau\sigma^{n-1}(2)),2}x_{\rho(\tau\sigma^{n-1}(3)),3}\cdots x_{\rho(\tau\sigma^{n-1}(n)),n}, \\ \quad x_{\rho(3),1}x_{\rho(2),2}x_{\rho(1),3}\cdots x_{\rho(4),n} \\ = x_{\rho(\tau\sigma^{n-2}(1)),1}x_{\rho(\tau\sigma^{n-2}(2)),2}x_{\rho(\tau\sigma^{n-2}(3)),3}\cdots x_{\rho(\tau\sigma^{n-2}(n)),n}, \\ \cdots \\ \quad x_{\rho(n),1}x_{\rho(n-1),2}x_{\rho(n-2),3}\cdots x_{\rho(1),n} \\ = x_{\rho(\tau\sigma(1)),1}x_{\rho(\tau\sigma(2)),2}x_{\rho(\tau\sigma(3)),3}\cdots x_{\rho(\tau\sigma(n)),n}. \end{cases}$$

Q.E.D.

Proposition 1. If

$$S_n = \sum_{k=1}^{(n-1)!/2} \rho_k D_n$$

is a left coset decomposition of S_n with respect to D_n, then the determinant of some matrix $(x_{i,j})$ is given as follows:

$$\det(x_{i,j}) = \begin{cases} \sum_{k=1}^{(n-1)!/2} \sum_{t=0}^{1} \sum_{s=0}^{n-1} (-1)^{s+t} \mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k(\tau^t\sigma^s(j)),j}, \\ \qquad \text{if } n \equiv 0 \pmod 4, \\ \sum_{k=1}^{(n-1)!/2} \sum_{t=0}^{1} \sum_{s=0}^{n-1} \mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k(\tau^t\sigma^s(j)),j}, \\ \qquad \text{if } n \equiv 1 \pmod 4, \\ \sum_{k=1}^{(n-1)!/2} \sum_{t=0}^{1} \sum_{s=0}^{n-1} (-1)^{s} \mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k(\tau^t\sigma^s(j)),j}, \\ \qquad \text{if } n \equiv 2 \pmod 4, \\ \sum_{k=1}^{(n-1)!/2} \sum_{t=0}^{1} \sum_{s=0}^{n-1} (-1)^{t} \mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k(\tau^t\sigma^s(j)),j}, \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. This proposition follows from the above assumption of a left coset decomposition, the definition of determinant (12), and Lemma 1. The signs are determined by (9) and (10). Q.E.D.

Proposition 1 implies Propositions 2 and 3.

Proposition 2. (Criterion of a good choice of *shuffles*). A set of permutations $\{\rho_1, \dots, \rho_{(n-1)!/2}\} \subset S_n$ corresponds to a good choice of *shuffles* if and only if $\{\rho_1, \dots, \rho_{(n-1)!/2}\}$ is a set of representatives of S_n with respect to D_n.

Proposition 3. (Criterion of *oblique multiplications*). Let $\{\rho_1, \dots, \rho_{(n-1)!/2}\}$ be a set of representatives of S_n with respect to D_n. The signs of *oblique multiplications* are correct if and only if they satisfy

$$\begin{array}{ll} (-1)^{s+t}\mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k\tau^t\sigma^s(j),j}, & \text{if } n \equiv 0 \pmod 4, \\ \mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k\tau^t\sigma^s(j),j}, & \text{if } n \equiv 1 \pmod 4, \\ (-1)^{s}\mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k\tau^t\sigma^s(j),j}, & \text{if } n \equiv 2 \pmod 4, \\ (-1)^{t}\mathrm{sgn}(\rho_k) \prod_{j=1}^{n} x_{\rho_k\tau^t\sigma^s(j),j}, & \text{if } n \equiv 3 \pmod 4, \end{array}$$

for $k = 1, \dots, (n-1)!/2$.

§6. A group-theoretical meaning of *shuffles*

We denote the permutations corresponding to the *shuffles* for the system of n equations by

$$\rho_1^{(n)}, \ \dots, \ \rho_{(n-1)!/2}^{(n)}.$$

We define the stabilizer of 1 by

$$(S_n)_1 = \{\rho \in S_n \mid \rho(1) = 1\}.$$

Lemma 2. Let n be an integer greater than 1. The mapping

$$\iota : S_{n-1} \to (S_n)_1$$

defined by

$$\begin{aligned} &\iota \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ i_1 & i_2 & \cdots & i_{n-2} & i_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & i_1+1 & \cdots & i_{n-2}+1 & i_{n-1}+1 \end{pmatrix} \end{aligned} \tag{13}$$

is an isomorphism.

Remark 1. The embedding ι corresponds to *attaching* 1 *in turn* (see footnote 1) in Seki's original text.

6.1. From the 3-*system* to the 4-*system*

Let n be an integer greater than 2. Let 1_n be the identity permutation of S_n. Let σ_n be the cyclic permutation defined in (7), i.e.

$$\sigma_n = \begin{pmatrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & 1 \end{pmatrix} \in S_n.$$

The permutations corresponding to Seki's choice of *shuffles* for 4-*systems* are

$$\begin{aligned} \rho_1^{(4)} &= 1_4 = \iota(1_3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \\ \rho_2^{(4)} &= \rho = \iota(\sigma_3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \\ \rho_3^{(4)} &= \rho^2 = \iota(\sigma_3^2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}. \end{aligned}$$

Since

$$(\rho_1^{(4)})^{-1}\rho_2^{(4)} = (\rho_2^{(4)})^{-1}\rho_3^{(4)} = (\rho_3^{(4)})^{-1}\rho_1^{(4)} = \rho \notin D_4,$$

the set $\{\rho_1^{(4)}, \rho_2^{(4)}, \rho_3^{(4)}\}$ is a set of representatives of S_4 with respect to D_4.

Seki understood the series of three arrangements $1_3, \sigma_3$ and σ_3^2 is *forward*.

6.2. From the 4-*system* to the 5-*system*

The permutations corresponding to Seki's choice of *shuffles* for 5-*systems* are

$$\begin{aligned} &\rho_1^{(5)} = 1_5,\ \rho_2^{(5)} = (23)(45),\ \rho_3^{(5)} = (24)(35),\ \rho_4^{(5)} = (25)(34), \\ &\rho_5^{(5)} = (345),\ \rho_6^{(5)} = (243),\ \rho_7^{(5)} = (254),\ \rho_8^{(5)} = (235), \\ &\rho_9^{(5)} = (354),\ \rho_{10}^{(5)} = (253),\ \rho_{11}^{(5)} = (234),\ \rho_{12}^{(5)} = (245). \end{aligned}$$

We define permutations ρ, λ and μ by

$$\rho = \iota(\sigma_3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \quad \lambda = \sigma_4^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$
$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in S_4,$$

respectively. Then the permutations $\rho_k^{(5)}, k = 1, \ldots, 12$ can be represented as

$$\rho_1^{(5)} = \iota(1_4),\ \rho_2^{(5)} = \iota(\mu),\ \rho_3^{(5)} = \iota(\lambda),\ \rho_4^{(5)} = \iota(\lambda\mu),$$
$$\rho_5^{(5)} = \iota(\rho),\ \rho_6^{(5)} = \iota(\rho\mu),\ \rho_7^{(5)} = \iota(\rho\lambda),\ \rho_8^{(5)} = \iota(\rho\lambda\mu),$$
$$\rho_9^{(5)} = \iota(\rho^2),\ \rho_{10}^{(5)} = \iota(\rho^2\mu),\ \rho_{11}^{(5)} = \iota(\rho^2\lambda),\ \rho_{12}^{(5)} = \iota(\rho^2\lambda\mu).$$

Since the permutations corresponding to Seki's choice of *shuffles* for 5-*systems* are

$$\rho_1^{(5)}\tau_5 = \rho_4^{(5)}, \quad \rho_2^{(5)}\tau_5 = \rho_3^{(5)}, \quad \rho_5^{(5)}\tau_5 = \rho_8^{(5)},$$
$$\rho_6^{(5)}\tau_5 = \rho_7^{(5)}, \quad \rho_9^{(5)}\tau_5 = \rho_{12}^{(5)}, \quad \rho_{10}^{(5)}\tau_5 = \rho_{11}^{(5)},$$

we have

$$\rho_{4k-3}^{(5)}D_5 = \rho_{4k}^{(5)}D_5, \quad \rho_{4k-2}^{(5)}D_5 = \rho_{4k-1}^{(5)}D_5 \quad (k = 1, 2, 3).$$

Thus, the set $\{\rho_k^{(5)}\}_{k=1,\ldots,12}$ is not a set of representatives. That is, Seki's choice of *shuffles* for 5-*systems* is not correct.

Seki understood the two permutations 1_4 and σ_4^2 to be *forward*, and the two permutations λ and $\lambda\mu$ *backward*. In Table 2 we show permutations corresponding to the shuffles for 5-*systems* which are derived form those for 4-*systems* (See Fig. 4).

Table 2. Permutations corresponding to the shuffles for 5-*systems* derived from those for 4-*systems*

	$\rho_1^{(4)} = 1_4$	$\rho_2^{(4)} = \rho$	$\rho_3^{(4)} = \rho^2$
f. (1)	$\iota(1_4) = \rho_1^{(5)}$	$\iota(\rho) = \rho_5^{(5)}$	$\iota(\rho^2) = \rho_9^{(5)}$
b. (μ)	$\iota(\mu) = \rho_2^{(5)}$	$\iota(\rho\mu) = \rho_6^{(5)}$	$\iota(\rho^2\mu) = \rho_{10}^{(5)}$
f. (λ)	$\iota(\lambda) = \rho_3^{(5)}$	$\iota(\rho\lambda) = \rho_7^{(5)}$	$\iota(\rho^2\lambda) = \rho_{11}^{(5)}$
b. ($\lambda\mu$)	$\iota(\lambda\mu) = \rho_4^{(5)}$	$\iota(\rho\lambda\mu) = \rho_8^{(5)}$	$\iota(\rho^2\lambda\mu) = \rho_{12}^{(5)}$

6.3. The terms *forward* and *backward*

In the *Sandatsu no Hō* [算脱之法] (*Method of Counting and Removing*) revised by Seki Takakazu in 1683, he called the *forward counting* (*junsan* [順算]) counting and removing black and white stones around a circle in the clockwise direction, and the *backward counting* (*gyakusan* [逆算]) counting and removing in the counterclockwise direction. With this in mind, we can formulate Seki's usage of the terms *forward* and *backward* as follows:

Let n be an integer greater than 2. Let ρ be a permutation of degree n. We lay out letters $\rho(1)$, $\rho(2)$, $\ldots$, $\rho(n)$ on a circle at equal distances in a clockwise order. The permutation ρ is called *forward* (resp., backward) if $\rho(1)$, $\rho(2)$, $\cdots$, $\rho(n)$ are in the same direction of 1, 2, $\cdots$, n (resp., n, $n-1$, $\cdots$, 1). By the definitions, the permutation σ defined in (7) is *forward* and τ defined in (8) is *backward*.

Let μ and λ be the same as in Section 6.2. Then 1_4, μ, λ and $\lambda\mu$ are represented as

$$\begin{array}{ccc} & 1 & \\ 4 & \circlearrowright & 2 \\ & 3 & \\ & 1_4 & \end{array} \qquad \begin{array}{ccc} & 2 & \\ 3 & \circlearrowleft & 1 \\ & 4 & \\ & \mu & \end{array} \qquad \begin{array}{ccc} & 3 & \\ 2 & \circlearrowright & 4 \\ & 1 & \\ & \lambda & \end{array} \qquad \begin{array}{ccc} & 4 & \\ 1 & \circlearrowleft & 3 \\ & 2 & \\ & \lambda\mu & \end{array}$$

respectively. Thus permutations 1_4 and λ are *forward*, while μ and $\lambda\mu$ are *backward*.

6.4. Seki's choice of *shuffles* for the general case

Lemma 3. The group D_n is a subgroup of A_n if and only if $n \equiv 1 \pmod 4$.

Proof. By (9) and (10), D_n is a subgroup of A_n if and only if $n \equiv 1 \pmod 4$. Q.E.D.

Lemma 4. Suppose $n \not\equiv 1 \pmod 4$. There exist $\rho_1, \ldots, \rho_{(n-1)!/2} \in A_n$ such that

$$S_n = \sum_{k=1}^{(n-1)!/2} \rho_k D_n.$$

Proof. By Lemma 3 and from the assumption in Lemma 4, there exists $\eta \in D_n \setminus A_n$. Let $S_n = \displaystyle\sum_{k=1}^{(n-1)!/2} \rho'_k D_n$ be any left coset decomposition. For $k = 1, \ldots, (n-1)!/2$, if we put

$$\rho_k = \begin{cases} \rho'_k & \text{if } \rho'_k \in A_n \\ \rho'_k \eta & \text{if } \rho'_k \notin A_n \end{cases}$$

then the set $\{\rho_1, \ldots, \rho_{(n-1)!/2}\}$ is qualified as a set of representatives and included in A_n. Q.E.D.

Lemma 5. Let n be an integer greater than 2. For any permutation $\rho \in S_n$, there exists $q \in \{1, 2, \ldots, n\}$ such that $\rho\sigma_n^{q-1} \in (S_n)_1$.

Proof. Let

$$\rho = \begin{pmatrix} 1 & \cdots & q & \cdots \\ p & \cdots & 1 & \cdots \end{pmatrix}.$$

If $p = 1$, then $q = 1$ and $\rho \in (S_n)_1$. If $p \neq 1$, then $q \neq 1$. Since

$$\sigma_n^{q-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ q & q+1 & \cdots & q-1 \end{pmatrix},$$

$\rho\sigma_n^{q-1} \in (S_n)_1$. Q.E.D.

Presumably, Seki must have performed some rapid judgment that a set of representatives is to be found only in even permutations for which the letter 1 is fixed, i.e. in $(S_n)_1 \cap A_n$. In the case of $n \equiv 1 \pmod 4$, by Lemma 3, no subset of A_n can qualify as a set of representatives of S_n with respect to D_n.

Proposition 4. If $n \not\equiv 1 \pmod 4$, there is a set of representatives of S_n with respect to D_n in $(S_n)_1 \cap A_n$.

Proof. This follows from Lemma 4 and Lemma 5. Q.E.D.

§7. Conclusion

Seki's procedure of *shuffles* (*kōshiki*) and *oblique multiplications* (*shajō*) is hereby completely explained. In particular, the meaning of the terms *forward* and *backward* has been made clear, i.e. they are not signs but the processes of rearrangement. This procedure is based on a coset decomposition of the symmetric group of degree n with respect to the dihedral subgroup D_n. Seki was able to offer an explicit solution for the choice of *shuffles* and the sign of *oblique multiplication* for $n = 4$, but failed for $n = 5$. Matsunaga's shuffles [10] are correct for any integer n greater than 3, but his oblique multiplications are not correct (See Osada [12]).

Acknowledgment

I would like to thank my colleague, Associate Professor Hiroshi Yamauchi for reading the draft of [12] and making useful comments.

References

[1] Academy of Japan ed. [日本学士院編] (Fujiwara, Matsusaburo [藤原松三郎]): *History of Japanese Mathematics before the Meiji Era* [明治前日本数学史], Vol. 2, Iwanami-shoten [岩波書店], Tokyo (1956). Fifth printing (2008).

[2] Goto Takefumi [後藤武史] and Komatsu Hikosaburo [小松彦三郎]: Seki's theory of elimination as compared with the others', pp. 553–574 in [8].

[3] Hayashi Tsuruichi [林鶴一]: The "fukudai" and determinants in Japanese mathematics, in *Dr. Tsuruichi Hayashi Wasan Kenkyū Shūroku*, Vol. 1, Tokyo Kaiseikan [東京開成館], Tokyo (1937). (*Proc. Tokyo Mathematico-Physical Soc.*, Ser. 2, **5**, pp. 254–271 (1910).)

[4] Hirayama Akira [平山諦]: *Seki Takakazu – His Accomplishment and Biography –* [関孝和 —その業績と伝記—], enlarged and revised edition, Kōseisha [恒星社], Tokyo (1974).

[5] Horiuchi, Annick: *Les Mathématiques japonaises à l'époque d'Edo 1600–1868*, Librairie Philosophique J. Vrin (1994).

[6] Horiuchi, Annick: *Japanese Mathematics in the Edo Period* (*1600-1868*), Translated from [5] by S. Wimmer-Zagier, Birkhäuser (2010).

[7] Katō Heizaemon [加藤平左エ門]: *The Works of Mathematical Genius Seki Takakazu* [算聖関孝和の業績], Maki-shoten [槇書店], Tokyo (1972).

[8] Knobloch, E., H. Komatsu and D. Liu (eds.): *Seki, Founder of Modern Mathematics in Japan*, Springer Japan (2013).

[9] Komatsu Hikosaburo [小松彦三郎]: All kinds of *kōshiki* and *shajō* after Seki Takakazu [関孝和以後の交式と斜乗さまざま], Institute for Mathematics and Computer Science, Tsuda College, **33**, pp. 103–121 (2012).

[10] Matsunaga, Yoshisuke [松永良弼]: *Kaifukudai Kōshiki shajō no Genkai* [解伏題交式斜乗之諺解] (manuscript) (re-revised 1715), Tohoku University Library, Okamoto Collection MS [岡本写] 0180.

[11] Mikami Yoshio [三上義夫]: On the Japanese theory of determinants, *Isis.* **2**, pp. 9–36 (1914).

[12] Osada Naoki [長田直樹]: On *Kai-fuku-dai no hō* (1) — Seki's Resolution of the determinants using group theory — [『解伏題之法』について (I) —交式斜乗の群論による解説 —] *RIMS Kôkyûroku*, to appear.

[13] Osada Naoki: On *Kai-fuku-dai no hō* (II) — Seki's Resolution of the elimination method — [『解伏題之法』について (II)— 逐式交乗の解説 —], *RIMS Kôkyûroku Bessatsu*, submitted.

[14] Satō, Ken'ichi [佐藤賢一]: The reexamination of the determinant by Seki Takakazu [関孝和の行列式の再検討], *History of Science and Philosophy of Science*, **11**, pp. 3–33 (1993); reprinted in *RIMS Kōkyūroku* **1392**, pp. 214–224 (2004).

[15] Seki, Takakazu [関孝和]: *Kaifukudai* [解伏題] (manuscript) (re-revised 1683), *Seki-san Zenden* [関算前伝] No. 2 in [17].

[16] Seki, Takakazu: *Kaifukudai-no-hō* [解伏題之法] (manuscript) (re-revised 1683), Japan Academy [日本学士院] 118.

[17] Society for History of East Asian Mathematics [東アジア数学史研究会編]: *Collected Works of Japanese Mathematics of Seki School* [関流和算書大成], Vol. 1, (*Seki-san-shiden-sho* [関算四伝書]), Bensei-shuppan [勉誠出版], Tokyo (2008).

[18] Toita Yasusuke [戸板保佑], *Seikoku inhō den* [生尅因法伝] (manuscript) (1759). *Sekisan Zenden* [関算前伝] No. 3 in [17].

Tokyo Woman's Christian University
2-6-1 Zempukuji, Suginami, Tokyo, 167-8585, JAPAN
E-mail address: osada@lab.twcu.ac.jp

page:100 (blank)