Newton's *De Analysi* and the priority dispute*

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Abstract

In 1669 Newton wrote *De Analysi* to claim the priority of the method of infinite series. Newton did not want to publish his method of fluxions, so he introduced the moment instead of the ratio of fluxions, which is the differential quotient in modern calculus, and expressed the fluent, which is the antiderivative in modern calculus, in terms of the pair of the indefinite region drawn by the motion of the ordinate and its signed area.

In the priority dispute, Newton suspected that Leibniz had read *De Analysi* in 1676, so he used *De Analysi* as evidence that Newton was the first inventor of the method of fluxions and that Leibniz had plagiarized differences from the moments of *De Analysi*. However Newton did not use fluxions when he wrote *De Analysi* and the moment in *De Analysi* was differential quotient, not differential, in modern calculus. In 1715, in his anonymous *An Account of the Book*, Newton explained how he had represented the fluxions in *De Analysi* and rewrote the note he had given in *De Analysi* regarding the unit of moment so that moment meant differential.

Keywords: Isaac Newton, *De Analysi*, priority dispute, moment, *Commercium Epistolicum*

1. Introduction

In 1669 Isaac Newton wrote the *De Analysi per Æquationes numero Terminorum Infinitas*, abbreviated as *De Analysi*, to claim the priority of the method of infinite series against Nicolaus Mercator. When Newton wrote *De Analysi*, because he did not want to publish his method of fluxions, he did not use velocities, which he later called fluxions. Therefore he introduced the moment instead of the ratio of fluxions, which is the differential quotient in modern calculus, and he expressed the quantity to be sought, which he later called fluent, in terms of the pair of the indefinite region drawn by the motion of the ordinate and its signed area.

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In the priority dispute with Leibniz on calculus, Newton tried to prove that Leibniz had plagiarized from *De Analysi* and the two letters, *Epistola Prior* and *Epistola Posterior.* Westfall[1] and Guicciardini[2] have described the accusations of plagiarism from the two letters in details. In this paper we will focus on the accusations of plagiarism from *De Analysi* [3].

2. Rule I and II in *De Analysi*

The procedure of the method of infinite series is to expand the differential quotient of a quantity to be sought into an infinite series of the form

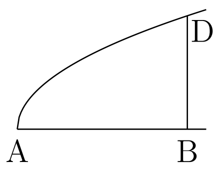
$$\sum_i a_i x^{\frac{m_i}{n_i}}, \tag{1}$$

and then to obtain the quantity by termwise integration. In order to perform this method it is sufficient that

1. To find the antiderivative of $ax^{\frac{m}{n}}$.
2. The possibility of termwise integration.
3. To expand the differential quotient of the quantity into an infinite series of the form (1).

Newton gave Rule I, II, and III in *De Analysi* which correspond to the above 1,2, and 3, respectively. Rule I and Rule II are as follows.

> To the base AB of some curve AD let the ordinate BD be perpendicular and let AB be called x and BD y. Let again $a, b, c, \ldots$ be given quantities and m, n integers. Then



[1] Richard Westfall, *Never at Rest*, Cambridge University Press, 1980, pp.712-720.

[2] Niccolò Guicciardini, *Isaac Newton On Mathematical Certainty and Method*, The MIT press, 2009, pp.373-381.

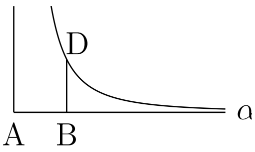
[3] The main purpose of *De Analysi* was to show how to express y as an infinite series of x given an implicit function $f(x, y) = 0$, but this is not closely related to the priority dispute and is therefore omitted. For the series expansion of implicit functions, see the author's paper: Naoki Osada, "Literal resolution of affected equations by Isaac Newton," RIMS Kôkyûroku Bessatsu B73 (2019): 1-20.

The Quadrature of simple Curves
RULE I. If $ax^{\frac{m}{n}} = y$, then will $\frac{na}{m+n}x^{\frac{m+n}{n}}$ equal the area ABD. The matter will be evident by example.

And of those compounded of simple ones
RULE II. If the value of y is compounded of several terms of that kind the area also will be compounded of the areas which arise separately from each of those terms.[4]

Since Newton said "The matter will be evident by example", we shall look at the fourth example.

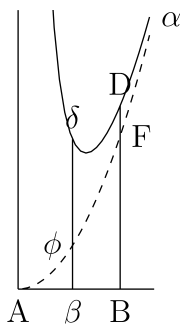
Example 4 If $\frac{1}{x^2}(= x^{-2}) = y$, that is, if $a = n = 1$ and $m = -2$, then $\left(\frac{1}{-1}x^{\frac{-1}{1}} =\right) - x^{-1}\left(= \frac{-1}{x}\right) = \alpha$BD infinitely extended in the direction of α: the computation sets its sign negative because it lies on the further side of the line BD.[5]



Example 4 shows that when $\frac{m}{n} < -1$, Newton defined the indefinite region described by the line segment BD $= ax^{\frac{m}{n}}$ by αBD, and its area is $\dfrac{na}{n+m}x^{\frac{m+n}{n}}$. Newton made the area negative because it matched the sign of the antiderivative.

Next we see the third example of Rule II.

Third examples. If $x^2 + x^{-2} = y$, then $\frac{1}{3}x^3 - x^{-1}$ =the surface described. But here you should note that the parts of the said surface thus found lie on opposite sides of the line BD: precisely, on setting BF $= x^2$ and FD $= x^{-2}$, then $\frac{1}{3}x^3 =$ the surface ABF described by BF and $-x^{-1} =$ DFα describes by DF. And this always happens when the indices $\left(\dfrac{m+n}{n}\right)$ of the ratios of the base x in the values of the surface sought are affected with different signs.[6]



[4] D.T. Whiteside, *The Mathematical Papers of Isaac Newton, Vol.II,* Cambridge University Press, 1968, pp.206-209. This book will be abbreviated as MP II in this paper.
[5] Ibid., pp.208-209.
[6] Ibid., pp.210-211.

Example 4 in Rule I and the above example show that Rule I should be viewed as giving the antiderivative rather than the area of the curve. That is, the antiderivative of the simple curve $y = ax^{\frac{m}{n}}$ corresponds to the signed area of the indefinite region drawn by the line segment BD $= ax^{\frac{m}{n}}$.

$$\frac{na}{m+n}x^{\frac{m+n}{n}} = \begin{cases} \left[\int_0^x ax^{\frac{m}{n}}dx =\right] \text{ABD} & \text{if } \frac{m}{n} > 0, \\ \left[\int_\infty^x ax^{\frac{m}{n}}dx =\right] \alpha\text{BD} & \text{if } \frac{m}{n} < -1. \end{cases}$$

See our recent paper[7] for details.

The equivalent of Rule I was given in the October 1666 tract as follows:

> If two Bodys A&B, by their velocitys $p\&q$ describe y$^{\text{e}}$ lines $x\&y$.
> [...]
> As if $ax^{\frac{m}{n}} = \frac{q}{p}$. Then is $\frac{na}{n+m}x^{\frac{n+m}{n}} = y$.[8]

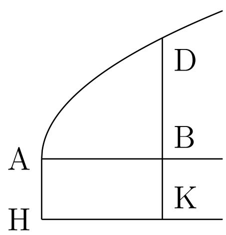
The ratio[9] $\frac{q}{p}$ of velocities in the October 1666 tract was represented as the line segment BD in Rule I in *De Analysi*.

3. Moments in *De Analysi*

3.1 Introduction of moments

Newton introduced the new term and concept of moment as follows:

> Let ABD be any curve and AHKB a rectangle whose side AH or BK is unity. And consider that the straight line DBK describes the areas ABD and AK as it moves uniformly away from AH; that BK(1) is the moment by which AK(x) gradually increases and BD(y) that by which ABD does so; and that, when given



> continuously the moment of BD, you can by the foregoing rules investigate the area ABD described by it [...]
> The matter will be clarified by example.[10]

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[7] Naoki Osada, "Area and moment in *De Analysi* by Isaac Newton," *RIMS Kôkyûroku Bessatsu*, B85 (2021): 15-34.

[8] D.T. Whiteside, *The Mathematical Papers of Isaac Newton, Vol.I,* Cambridge University Press, 1967, p.403.

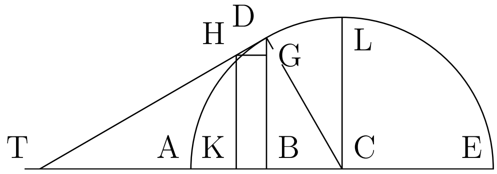
[9] In modern notation, $p = \frac{dx}{dt}$ and $q = \frac{dy}{dt}$, thus $\frac{q}{p} = \frac{dy}{dx}$.

[10] MP II, pp.232-233

It is not clear from the above explanation alone.[11]

3.2 The length of an arc

Newton gave an example of finding the length of an arc using moments.



Let ADLE be a circle whose arc length AD is to be discovered. On drawing the tangent DHT, completing the indefinitely small rectangle HGBK and setting AE = 2AC = 1, there will then be BK or GH (the moment of the base AB) to DH (the moment of the arc AD)[= BK : DH]

$= \mathrm{BT} : \mathrm{DT} = \mathrm{BD}(\sqrt{x-x^2}) : \mathrm{DC}(\frac{1}{2}) = 1(\mathrm{BK}) : \dfrac{1}{2\sqrt{x-x^2}}(\mathrm{DH})$

so that $\dfrac{1}{2\sqrt{x-x^2}}$ or $\dfrac{\sqrt{x-x^2}}{2(x-x^2)}$ is the moment of the arc AD. When reduced this becomes $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ *&c.* Therefore by Rule II the length of the arc AD is

$$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}} \ \&c.$$ [12]

Since HGBK is an indefinitely small rectangle, the moment BK of the base AB and the moment DH of the arc AD, denoted by $\widehat{\mathrm{AD}}$, are indefinitely small line segments. However Newton put BK = 1 and $\mathrm{DH} = \dfrac{1}{2\sqrt{x-x^2}}$. The fact that BK is indefinitely small and BK = 1[13] are contradictory.

3.3 Newton's true intention of the concept of moment

There are two main interpretations of moment in *De Analysi*: one is the fluxional velocity by Whiteside[14], other is the infinitesimal increments of

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[11] Malet and Panza† wrote "Nothing else is said about the nature and properties of those moments, although it seems to be implicit that (the line DBK being in motion) they are something else that mere Cavalierian indivisibles – in any case, whatever Newton had in mind, here, he left to the reader the task of clarifying the matter." (p.377)
†Antoni Malet and Marco Panza, "Newton on indivisibles" in Vincent Jullien, ed. *Seventeenth-Century Indivisibles Revisited*, 365-390, Birkhäuser, 2015.
`https://halshs.archives-ouvertes.fr/halshs-01172653/document`

[12] MP II, pp.232-233

[13] The author wrote "if BK = 1 is corrected to BK = o, the moment in *De Analysi* becomes the differential and coincides with that in *De Methodis*." (footnote 7, p.31.) If HGBK were indefinitely small rectangle, BK = o, but Newton did not treat the moments in *De Analysi* in that way. See Section 3.3.

[14] MP II, p.233 (101).

variable quantity by Boyer[15], Whitrow[16] and Guicciardini[17], or differential[18] by Dunham[19] and others.

In this section, we consider Newton's true intention of the concept of moment from the description in *De Analysi*. After the example of finding the length of an arc, Newton made the following note about the units of moments.

> But it must be noted that unit[20] which is set for the moment is a surface when the question concerns solids, a line when it relates to surfaces and a point when (as in this example) it has to do with lines. Nor am I afraid to talk of a unit in points or infinitely small lines inasmuch as geometers now consider proportions in these while using indivisible methods.[21]

Let us consider the area of ABD in Section 3.1. In modern notation, since the unit of ABD is a surface, the unit of $\frac{d}{dt}\mathrm{ABD}$ is a velocity of changing area, the unit of the differential $d\mathrm{ABD}$ is an infinitesimal increment of surface and the unit of the differential quotient $\frac{d}{dx}\mathrm{ABD}$ is a line, it is appropriate to determine that Newton represented differential quotients by moments.

We will verify by modern calculus that Newton represented the differential quotient by moments, using the example of the length of an arc (Section 3.2). It follows from $\mathrm{AB} = x$ that $\mathrm{BK} = \frac{d}{dx}\mathrm{AB} = 1$. Since $\angle\mathrm{DCA} = \sin^{-1}\dfrac{\sqrt{x - x^2}}{\frac{1}{2}}$, we have $\overset{\frown}{\mathrm{AD}} = \dfrac{1}{2}\sin^{-1}\dfrac{\sqrt{x-x^2}}{\frac{1}{2}}$ and

$$\mathrm{DH} = \frac{d}{dx}\overset{\frown}{\mathrm{AD}} = \frac{1}{2}\frac{\frac{1-2x}{\sqrt{x-x^2}}}{\sqrt{1-4(x-x^2)}} = \frac{1}{2\sqrt{x-x^2}}, \quad (0 < x < \frac{1}{2}),$$

which is consistent with Newton's calculation.

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[15] Carl B. Boyer, The History of the Calculus and its Conceptual Development, Dover, 2018, (Hafner Publishing, first published 1949), p.191.

[16] G.J. Whitrow, "Newton's Role in the History of Mathematics," *Notes and Records of the Royal Society of London*, Vol.43, No.1, (1989), pp.71-92.

[17] Footnote 2, p.155.

[18] Let $y = f(x)$ be a function. The differential dy is defined by $dy = f'(x)dx$, where $f'(x)$ is the derived function of f, and dx is the infinitesimal increment of the variable x.

[19] William Dunham, *The Calculus Gallery: Masterpieces from Newton to Lebesgue*, Princeton University Press, 2009.
`http://assets.press.princeton.edu/chapters/s7905.pdf`

[20] The Latin original is *unitas*, which Whiteside translates as "unity", while Malet and Panza (footnote 11, p.378.) translated as "unit".

[21] MP II, pp.234-235.

In 1671 Newton defined the moment in the *De Methodis Serierum et Fluxionum*,[22] abbreviated as *De Methodis*, as follows:

> The moments of the fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed m and an infinitely small quantity o (that is, by mo), then the moments of the others, v, y, z, will be expressed by lo, no, ro, seeing that lo, mo, no, and ro are to one another as l, m, n, and r.[23]

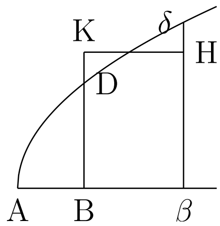
The symbol o is considered to represent dt in modern notation. Thus $mo = \frac{dx}{dt}dt = dx$ and $lo = \frac{dv}{dt}dt = dv$ are the differentials of x and v, respectively.

In the sense of *De Methodis*, BK $= o$ and DH $= \frac{o}{2\sqrt{x-x^2}}$. The representation "indefinitely small" was not appropriate in *De Analysi*, but became an important phrase in *De Methodis*.[24]

3.4 The proof of Rule I in *De Analysi*

In *De Analysi* Newton proved Rule I for the case $\frac{m}{n} > 0$. In the preparation of the proof he used a tool which is equivalent to differential.

> **Preparation for demonstrating the first rule.**
> 1. The quadrature of simple curves in Rule I. Let then any curve ADδ have base AB $= x$, perpendicular ordinate BD $= y$ and area ABD $= z$, as before. Likewise take B$\beta = o$, BK $= v$ and the rectangle BβHK(ov) equal to the space B$\beta\delta$D. It is, therefore, A$\beta = x + o$ and A$\delta\beta = z + ov$.[25]



In modern notation, since $o = dx$ and $v = \frac{dz}{dx}$, $ov = \frac{dz}{dx}dx = dz$ is the differential of z, and v is the moment (differential quotient) of the area $z =$ ABD. The ordinate $v =$ BK represents the moment of z in terms of

[22] D.T. Whiteside, *The Mathematical Papers of Isaac Newton, Vol.III*, Cambridge University Press, 1969, pp.32-329.

[23] Ibid.,, pp.78-81.

[24] Kokiti Hara‡ wrote "The argument of the next work [*De Methodis*], which calls the differential, not velocity, the moment, has already started here." (*Early-modern Mathematics*, p.307.)

‡Kokiti Hara, *Early-modern Mathematics* (in Japanese), in *A History of Mathematics*, Chikuma Shobō, 1975.

[25] MP II, pp.242-245.

De Analysi, and the fluxion of z in terms of *De Methodis*. If we assume the speed of the base AB is 1,

$$\frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}} = \frac{dz}{dt}.$$

Thus moments in *De Analysi* and the fluxions in *De Methodis* have equal values but different units. The physical unit of the former is length, that of the latter is a velocity of changing area.

4. Accusations of Plagiarism in the priority dispute

Until the discovery of the Collins' letters and papers by William Jones in 1708, Newton and mathematicians around him believed that Leibniz had plagiarized from the two letters (*Epistola Prior* and *Epistola Posterior*) in 1676. In 1699, John Wallis published the two letters with Newton's permission in his *Opera Mathematica*.

In 1704, Newton published the *Tractatus de Quadratura Curvarum* (abbreviated as *De Quadratura*) as an appendix to his *Opticks*. *De Quadratura* was the first published treatise on the method of fluxions, nearly 40 years after Newton's discovery of it. In the preface (advertisement) to the *Opticks*, Newton wrote:

> In a Letter written to Mr. Leibniz in the Year 1676 and published by Dr. Wallis, I mentioned a Method by which I had found some general Theorems about squaring Curvilinear Figures, or comparing them with the Conic Sections, or other the simplest Figures with which they may be compared.[26]

Newton claimed the priority of the method of fluxions using *Epistola Posterior* as evidence, but did not directly mention Leibniz's plagiarism[27]. And in the introduction to *De Quadratura*, Newton explicitly dated the discovery of the method of fluxions to 1665-1666 as follows:

> I was led to seek a method of determining quantities out of the speeds of motion or increment by which they are generated; and, naming these speeds of motion or increment 'fluxions' and the quantities so born 'fluents', I fell in the years 1665 and 1666 upon the method of fluxions which I have here employed in the quadrature of curves.[28]

[26] I. Newton, *Opticks*, Printed for Sam. Smith, and Benj. Walford, London, 1704 `https://library.si.edu/digital-library/book/optickstreatise00newta`

[27] Leibniz might have taken Newton to imply Leibniz's plagiarism in the above preface.

[28] D.T. Whiteside, *The Mathematical Papers of Isaac Newton, Vol.VIII*, Cambridge University Press, 1981, p.123.

In 1705, Leibniz published an anonymous review of the *De Quadratura* in the *Acta Eruditorum*.

> when some magnitude grows continuously, as (for instance) a line grows by the flowing of a point which describes it, the instantaneous increments are called 'differences', [...] whose elements have been delivered in these *Acts* by its inventor Mr Gottfried Wilhelm Leibniz, [...] In place of Leibnizian differences, accordingly, Mr Newton employs, and has ever employed, 'fluxions',[29]

Leibniz claimed the priority of the differential calculus. He also implied that Newton's fluxions were plagiarism from Leibniz's differences.

In 1708, John Keill, in his paper published in the *Philosophical Transactions*, inserted Leibniz's plagiarism.

> All of these [propositions] follow from the now highly celebrated Arithmetic of Fluxions which Mr. Newton, beyond all doubt, First Invented, as anyone who reads his Letters published by Wallis can easily determine; the same Arithmetic under a different name and using a different notation was later published in the Acta eruditorum, however, by Mr. Leibniz.[30]

In the same year 1708, William Jones obtained the letters and papers left by John Collins, which were later called the Collins papers. The Collins papers consisted of Collins' handwritten copy of *De Analysi* and correspondence with mathematicians. In 1711, Jones published *De Analysi*, *De Quadratura* and others with Newton's permission.

In 1711, Leibniz learned that he had been accused of plagiarism by Keill, so he wrote a letter of protest to Hans Sloan, the secretary of the Royal Society, asking Keill to retract his accusation. In March 1712, the Royal Society appointed a committee and adopted the report[31] attributing Newton as the first inventor, and published a book in evidence, *Commercium Epistolicum D. Johannis Collins, et aliorum de Analysi promota* (Printed 1712, distributed 1713; abbreviated as *Commercium Epistolicum*). The report and the *Commercium Epistolicum* were substantially prepared by Newton himself.

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[29] Ibid., p.26.

[30] Footnote 1, pp.715-716.

[31] Westfall stated "the published report did not name the committee, whose membership remained concealed in the society's records until the nineteenth century."(ibid., p.725)

5. *De Analysi* in the priority dispute

The following statement appears in the report included at the end of the *Commercium Epistolicum*.

> I. *That Mr.* Leibniz *was in* London *in the beginning of the Year 1673, and went thence in or about* March *to* Paris, *where he kept a Correspondence with Mr.* Collins *by means of Mr.* Oldenburg, *till about* September *1676, and then return'd by* London *and* Amsterdam *to* Hannover: *And that Mr.* Collins *was very free in communicating to able Mathematicians what he had receiv'd from Mr.* Newton *and Mr.* Gregory.[32]

Since Newton learned the above facts from the Collins papers, he would have had a strong suspicion, though not conclusive evidence[33] that Leibniz had read or transcribed *De Analysi* from Collins during his stopover in London in 1676. It is believed that Newton was trying to prove that Leibniz plagiarized from *De Analysi*, *Epistola Prior* and *Epistola Posterior.*

While the two letters had already been published by Wallis in his *Opera Mathematica*, *De Analysi* was published by Jones in 1711 and reproduced in the *Commercium Epistolicum.* Furthermore, Newton decided to use *De Analysi* and the two letters as evidence for his claim of priority of the method of fluxions. The report appears as follows:

> III. *That by Mr. Newton's Letter of the 13th of* June *1676 it appears, that he had the Method of* Fluxions *above five Years before the writing of that Letter. And by his Analysis* per Æquationes numero Terminorum Infinitas, *communicated by Dr.* Barrow *to Mr.* Collins *in* July *1669, we find that he had invented the Method before that time.*[34]

With *De Analysi*, Newton attempted to prove that Newton was the first inventor of the method of fluxions and that Leibniz plagiarized from moments to make differential calculus.

However, when Newton wrote *De Analysi*, he did not use fluxions, which caused two problems: First, if one reads *De Analysi* in contrast to the October

[32] Royal Society, *Commercium Epistolicum Collinii & Aliorum, De Analysi promota*, 1713. p.120. `https://archive.org/details/bub_gb_r1BCoilZ3PwC`

[33] Westfall writes that "Leibniz never breathed a word about *De Analysi* [...] Only when Leibniz, shortly before his death, inadvertently revealed the extent of Collins's liberality in the fall of 1676, did Newton begin to realize that he might have seen the tract as well." (footnote 1, p.720.)

[34] Footnote 32, p.121.

1666 tract and *De Methodis*, one can understand that this paper is a paper on the method of fluxions, but if one only reads *De Analysi* by itself, one would not have thought that this paper dealt with the method of fluxions. Second, the moments in *De Analysi* were differential quotients, not differentials, in modern calculus.

We discuss the first problem in the rest of this section and the second problem in the next section. In 1715,[35] Newton published anonymously the *An Account of the Book* (abbreviated as *An Account*) in the *Philosophical Transactions*. In it, he explained how he represented the method of fluxions when writing *De Analysi*.

> When Mr. *Newton* had in this Compendium explained these three Rules, and illustrated them with various Examples, he laid down the Idea of deducing the Area from the Ordinate, by considering the Area as a Quantity, growing or increasing by continual Flux, in proportion to the Length of the Ordinate, supposing the Abscissa to increase uniformly in proportion to Time.[36]
>
> Mr. *Newton* doth not place his Method in Forms of Symbols, nor confine himself to any particular Sort of Symbols for Fluents and Fluxions. Where he puts the Areas of Curves for Fluents, he frequently puts the Ordinates for Fluxions, and denotes the Fluxions by the Symbols of the Ordinates, as in his *Analysis*.[37]

In *De Analysi*, Newton used symbols that differed from those in his other works on the method of fluxions. He explained that he expressed the fluent in terms of the area of the curve and the fluxion in terms of the ordinate. Thus, the area $\frac{na}{m+n}x^{\frac{m+n}{n}} = z$ of Rule I is the fluent, and the ordinate $ax^{\frac{m}{n}} = y$ is the fluxion.

6. Differentials in the priority dispute

As mentioned in Section 4, Leibniz wrote in his review of the *De Quadratura* that Newton plagiarized Leibniz's differences (instantaneous increment) as a fluxion. On the other hand, in the *Commercium Epistolicum* perhaps Newton himself added the following footnote to moments in *De Analysi*.

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[35] Derek Gjertsen, *The Newton Handbook*, Routledge & Kegan Paul, 1986, p.505.

[36] Royal Society, "An Account of the Book entitles Commercium Epistolicum Collinii & Aliorum, De Analysi promota", *Philosophical Transactions*, Vol. 29 (1714 - 1716), p.178, https://www.jstor.org/stable/103050

[37] Ibid., p.204.

> Here is described the method of operating by fluents and their moments. These moments were afterwards called differences by Mr Leibniz: and so came the name of differential method.[38]

Furthermore, Newton essentially rewrote in the *An Account* the note on the units of moments (see Section 3.3) as follows:

> And from the Moments of Time he gave the Name of Moments to the momentaneous Increases, or infinitely small Parts of the Abscissa and Area, generated in Moments of Time. The Moment of a Line he called a Point, in the Sense of *Cavallerius*, tho' it be not a geometrical Point, but a Line infinitely short, and the Moment of an Area or Superficies he called a Line, in the Sense of *Cavallerius*, tho' it be not a geometrical Line, but a Superficies infinitely narrow.[39]

If the note on the unit of moment given in *De Analysi* is revised as above, the moment becomes an momentaneous increases, i.e., a differential in modern calculus. The moment of AK(x) is BK($1 \times dx$), because BK is not a line [segment] but an infinitely narrow surface, and the moment of ABD(z) is BD($y \times dx$) because BD is not a line [segment] but an infinitely narrow surface. It is thought that Newton changed moments in *De Analysi* from differential quotients to differentials in order to claim priority of differentials and Leibniz's plagiarism from moments.

7. Why did Newton choose *De Analysi* as evidence?

What Newton described in *De Analysi* was the method of infinite series and its applications. Why did Newton choose this paper, which did not explicitly use the method of fluxions, as the most important evidence for his claim of priority of calculus?

The method of infinite series is a method of finding a quantity by expanding the differential quotient (moment) of it to infinite series, and then by integrating it term by term (Rule II). Newton applied the method of infinite series to the length of an arc, the area of a cycloid and the area of a quadratrix. In the arc length problem discussed in Section 3.2, the moment of the arc AD is DH $= \frac{1}{2\sqrt{x-x^2}}$. Expanding DH to an infinite series gives

$$\left(\frac{1}{2\sqrt{x-x^2}} =\right) \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}} + \cdots$$

[38] MP II, p.232 (99).

[39] *An Account*, p.178.

and then integrating term by term yields

$$x^{\frac{1}{2}} + \tfrac{1}{6}x^{\frac{3}{2}} + \tfrac{3}{40}x^{\frac{5}{2}} + \tfrac{5}{112}x^{\frac{7}{2}} + \tfrac{35}{1152}x^{\frac{9}{2}} + \tfrac{63}{2816}x^{\frac{11}{2}} + \cdots \left(= \sin^{-1}\sqrt{x}\right),$$

which is the arc length. In modern notation, Newton derived

$$\int_0^x \frac{1}{2\sqrt{x-x^2}}dx = \sin^{-1}\sqrt{x}.$$

Based on these applications, Newton concluded *De Analysi* as follows.

> It should, finally, merit consideration as a relevant part of analysis since by its aid the areas and lengths of curves and so on (provided that may be done) may be exactly and geometrically determined. But this is not the place to dwell on these matters.[40]

Newton regarded the method of infinite series to be a powerful tool for quadrature. Furthermore, as Guicciardini wrote[41], Newton considered the method of integration more important than the method of differentiation, in modern terms.

8. Conclusion

There are three possible reasons why Newton choose *De Analysi* as the most important evidence in the priority dispute.

First, the Collins papers made it extremely likely that Leibniz had read or transcribed *De Analysi* in 1676, so Newton probably thought that if he could prove Leibniz's plagiarism from *De Analysi* it would not be necessary to publish the October 1666 tract or *De Methodis*.

Second, as we described in sections 5 and 7, Newton regarded *De Analysi* as a treatise on the method of fluxions.

Third, Newton showed *De Analysi* to Collins through Barrow, and Collins circulated it to many mathematicians, so Newton would have regarded the paper as having been published[42]. In the early 18th century, there was no rule established today that the first paper submitted to an academic journal has the right of priority.

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[40] MP II, pp.242-243.

[41] Guicciardini wrote "When one analyzes the mathematical examples adduced in *Commercium Epistolicum*, it emerges that Newton and his acolytes who were slavishly editing it were referring to the inverse method of fluxions applied to problems of quadrature." (footnote 2, pp.375-376.)

[42] For example, "And by his Analysis per Æquationes numero Terminorum Infinitas, communicated by Dr. Barrow to Mr. Collins in July 1669, we find that he had invented the Method before that time." (the report in *Commercium Epistolicum*, p.121)

As Newton suspected, Leibniz is confirmed to have viewed *De Analysi* and made excerpts[43] from it in London in mid-October 1676. Leibniz transcribed the parts on infinite series, replacing them with his own symbols, but did not transcribe the introduction of moments, the application of moments to the length of an arc and the area of a cycloid, or the proof of Rule I. The parts that Leibniz did not transcribe are important parts of the method of fluxions, but since Newton concealed the fluxion, i.e., avoiding the terms and symbols of the method of fluxions, Leibniz probably did not consider it a paper on the method of fluxions.

[43] We can read the excerpt in MP II, pp.248-259.