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Literal resolution of affected equations by Isaac Newton

By

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Abstract

In 1669 and 1671, Isaac Newton resolved an algebraic equation $f(x, y) = 0$ by expressing y as an infinite series of x. In this paper, we formulate Newton's resolution as a contemporary algorithm along the line of his original text and prove that the series converges asymptotically to the implicit function or one of the branches under certain conditions.

§ 1. Introduction

Isaac Newton gave the literal resolution of an affected equation in *De Analysi* (1669) and *De Methodis* (1671). An affected equation is an algebraic equation that is not binomial, such as $y^3 - 2y - 5 = 0$ or $y^3 + a^2y - 2a^3 + axy - x^3 = 0$. Newton referred to the former equation as numerical and the latter as literal. The literal resolution of an affected equation $f(x, y) = 0$ is to express y as an infinite series

$$y = \sum_{i=k}^{\infty} c_i x^{\pm i/r}, \quad c_i \in \mathbb{R}, k \in \mathbb{Z}, \text{ and } r \in \mathbb{N}, \tag{1.1}$$

where the double sign is set to $+$ when x is close to 0, and it is set to $-$ when x is sufficiently large. The series (1.1) is called a Puiseux series if the double sign is $+$. In *De Analysi*, Newton elucidated the two cases, namely, one when x is close to 0 and there exists c such that $f(0, c) = 0$ and $\frac{\partial}{\partial y} f(0, c) \neq 0$, and another when x is sufficiently large. In the first case, $k = 0, c_0 = c$ and $r = 1$ in (1.1). In *De Methodis*, Newton improved the above algorithm so that it can be applied easily even if $(0, c)$ is a singular point, i.e.,

$$f(0, c) = 0, \frac{\partial}{\partial x} f(0, c) = 0, \frac{\partial}{\partial y} f(0, c) = 0,$$

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using the Newton diagram. For the Newton diagram method, see [3, pp.191-196] or [4, pp.158-164]. Neither *De Analysi* nor *De Methodis* was published at that time, but Newton's algorithm using the Newton diagram became publicly known because Newton wrote it in the letter *epistola posterior* [9] sent to Henry Oldenburg for Gottfried Wilhelm Leibniz in 1676, and John Wallis reproduced Newton's algorithm in his *A Treatise of Algebra* [11] in 1685.

In 1850, Victor Puiseux [7, p.401] proved that for a complex algebraic equation $f(u,z)=0$, u can be represented as convergent infinite series

$$u_j = b + \gamma_j (z-a)^{\frac{1}{p}} + a_j (z-a)^{\frac{2}{p}} + b_j (z-a)^{\frac{3}{p}} + \cdots, \quad (j = 1, \ldots, p).$$

See [8, p.194] for more details. Today the following theorem is known:

Theorem 1.1. (Puiseux theorem)
Let $\mathbb{C}\{z\}$ be the ring of all convergent power series. Let $f(z,y) \in \mathbb{C}\{z\}[y]$ be a monic irreducible polynomial of the form

$$f(z,y) = y^n + a_{n-1}(z) y^{n-1} + \cdots + a_1(z) y + a_0(z), \quad a_j(z) \in \mathbb{C}\{z\}.$$

Then, there is $g(z) \in \mathbb{C}\{z\}$ such that

$$f(z,y) = \prod_{j=0}^{n-1} (y - g(\zeta^j z^{1/n})),$$

where ζ is a primitive n-th root of unity.

Proof. See [2, pp.15-26] or [5, pp.235-237]. □

In regard to the Puiseux theorem Abyhyankar wrote

> Newton's theorem was revived by Puiseux in 1850. [...] Puiseux's proof, being based upon Cauchy's integral theorems, applies only to convergent power series with complex coefficients. On the other hand, Newton's proof, being algorithmic, applies equally well to power series, whether they converge or not. Moreover, and that is the main point, Newton's algorithmic proof leads to numerous other existence theorems while Puiseux's existential proof does not do so. [1, p.417]

In this paper, we formulate Newton's algorithm to express an infinite series as a contemporary algorithm along the line of his original text. Additionally we prove under certain conditions that the infinite series (1.1) is the asymptotic expansion of the implicit function or one of the branches.

Newton's papers are cited from *Mathematical Papers of Isaac Newton* [12] edited by Whiteside and abbreviated as MP.

§ 2. The asymptotic expansion of the implicit function as $x \to 0$

Newton elucidated the literal resolution of affected equations in *De Analysi* as follows.

> Suppose now that the algebraic equation $y^3 + a^2y - 2a^3 + axy - x^3 = 0$ has to be resolved. First I seek out the value of y when x is zero, that is, I elicit the root of this equation $y^3 + a^2y - 2a^3 = 0$, and find it to be $+a$. And so I write $+a$ in the quotient. Again, supposing $y = a + p$, for y I substitute that value and the terms $p^3 + 3ap^2 + 4a^2p$... which thence result I set in the margin. Out of these I take $4a^2p + a^2x$, in which p and x separately are of least dimension and suppose them nearly equal to zero, that is, $p = -\frac{x}{4}$ nearly or $p = -\frac{x}{4} + q$. [...]

		$a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \cdots$
$+a + p = y.)$	$+y^3$	$+a^3 + 3a^2p + 3ap^2 + p^3$
	$+a^2y$	$+a^3 + a^2p$
	$+axy$	$+a^2x + axp$
	$-2a^3$	$-2a^3$
	$-x^3$	$-x^3$
$-\frac{1}{4}x + q = p.)$	$+p^3$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{3}{4}xq^2 + q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2 - \frac{3}{2}axq + 3aq^2$
	$+4a^2p$	$-a^2x + 4a^2q$
	$+axp$	$-\frac{1}{4}ax^2 + axq$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a} + r = q.)$	$+3aq^2$	$+\frac{3x^4}{4096a} + \frac{3}{32}x^2r + 3ar^2$
	$+4a^2q$	$+\frac{1}{16}ax^2 + 4a^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3 - \frac{1}{2}axr$
	$+\frac{3}{16}x^2q$	$+\frac{3x^4}{1024a} + \frac{3}{16}x^2r$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$
$+4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{131}{128}x^3 - \frac{15x^4}{4096a}\ ($		$+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ [...]

MP II, pp.222-225

We now formulate the literal resolution of an affected equation $f(x,y)=0$ under the condition that there exists a root c of $f(0,y)=0$ with $\frac{\partial}{\partial y}f(0,c)\neq 0$.

Algorithm 2.1. **(The asymptotic expansion as $x\to 0$.)** Let $f(x,y)$ be a polynomial of the form

$$f(x,y)=\sum_{i=0}^{l}a_{i,0}x^i+\sum_{j=1}^{n}\left(\sum_{i=0}^{m}a_{i,j}x^i\right)y^j.$$

Suppose $f(0,y)=0$ has a root c with $\frac{\partial}{\partial y}f(0,c)\neq 0$.

(i) Put $f_0(x,y)=f(x,y)$ and $d_0(x)=c$.

(ii) Repeat below for $\nu=1,2,\dots,N$:
calculate

$$\begin{aligned}f_\nu(x,y)&=f_{\nu-1}(x,d_{\nu-1}(x)+y)\\&=\sum_{i=i_\nu}^{l_\nu}a_{i,0}^{(\nu)}x^i+\sum_{j=1}^{n}\left(\sum_{i=0}^{m_\nu}a_{i,j}^{(\nu)}x^i\right)y^j,\quad a_{i_\nu,0}^{(\nu)}\neq 0\end{aligned}$$

and

$$d_\nu(x)=-\frac{a_{i_\nu,0}^{(\nu)}}{a_{0,1}^{(\nu)}}x^{i_\nu}.$$

Then, $y_N(x)=d_0(x)+\cdots+d_{N-1}(x)+d_N(x)$ satisfies

$$f(x,y_N(x))=o(x^{i_N})\quad \text{as } x\to 0.$$

It is further assumed that there exists a positive integer $\mu\in\mathbb{N}$ such that

$$f(x,y)=\sum_{i=0}^{l/\mu}a_{i\mu,0}x^{i\mu}+\sum_{j=1}^{n}\left(\sum_{i=0}^{m/\mu}a_{i\mu,j}x^{i\mu}\right)y^j. \tag{2.1}$$

(iii) Let $\mathcal{N}(x)$ be the first N terms of $\displaystyle\sum_{i=i_N/\mu}^{l_N/\mu}a_{i\mu,0}^{(N)}x^{i\mu}$, and let $\mathcal{D}(x)$ be the first N terms of $\displaystyle\sum_{i=0}^{m_N/\mu}a_{i\mu,1}^{(N)}x^{i\mu}$. Expand $-\mathcal{N}(x)/\mathcal{D}(x)$ to N terms by division, and put this expansion as $\tilde{d}_N(x)$.

Then, $\tilde{y}_N(x)=d_0(x)+\cdots+d_{N-1}(x)+\tilde{d}_N(x)$ satisfies

$$f(x,\tilde{y}_N(x))=O(x^{i_N+N\mu})\quad \text{as } x\to 0.$$

Example 2.2. We now apply Algorithm 2.1 to Newton's example $f(x,y) = y^3 + a^2y - 2a^3 + axy - x^3 = 0$ with $N = 3$. We take $c = a$ as the root of the equation $f(0,y) = y^3 + a^2y - 2a^3 = 0$. Then, $\frac{\partial}{\partial y} f(0,a) = 4a^2 \neq 0$.

$\nu = 0$ Put $f_0(x,y) = y^3 + a^2y - 2a^3 + axy - x^3$ and $d_0(x) = a$.

$\nu = 1$ Since $f_1(x,y) = f_0(x, a+y) = a^2x - x^3 + (4a^2 + ax)y + 3ay^2 + y^3$, we have $i_1 = 1, a_{1,0}^{(1)} = a^2, a_{0,1}^{(1)} = 4a^2$ and thus, $d_1(x) = -\frac{1}{4}x$.

$\nu = 2$ Since

$$f_2(x,y) = -\frac{1}{16}ax^2 - \frac{65}{64}x^3 + \left(4a^2 - \frac{1}{2}ax + \frac{3}{16}x^2\right) y + \left(3a - \frac{3}{4}x\right) y^2 + y^3,$$

we have $i_2 = 2, a_{2,0}^{(2)} = -\frac{1}{16}a, a_{0,1}^{(2)} = 4a^2$ and thus, $d_2(x) = \frac{1}{64a}x^2$.

$\nu = 3$ Since

$$f_3(x,y) = -\frac{131}{128}x^3 + \frac{15}{4096a}x^4 - \frac{3}{16384a^2}x^5 + \frac{1}{262144a^3}x^6 + \left(4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2 - \frac{3}{128a}x^3 + \frac{3}{4096a^2}x^4\right) y + \left(3a - \frac{3}{4}x + \frac{3}{64a}x^2\right) y^2 + y^3,$$

we have $\mathcal{N}(x) = -\frac{131}{128}x^3 + \frac{15}{4096a}x^4 - \frac{3}{16384a^2}x^5$ and $\mathcal{D}(x) = 4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2$. Expanding $\mathcal{N}(x)/\mathcal{D}(x)$, we obtain

$$\frac{\mathcal{N}(x)}{\mathcal{D}(x)} = -\frac{131}{512a^2}x^3 - \frac{509}{16384a^3}x^4 + \frac{1843}{131072a^4}x^5$$

and

$$\tilde{d}_3(x) = \frac{131}{512a^2}x^3 + \frac{509}{16384a^3}x^4 - \frac{1843}{131072a^4}x^5.$$

Thus, $f_3(x, \tilde{d}_3(x)) = \mathcal{N}(x) + \mathcal{D}(x)\tilde{d}_3(x) = O(x^6) \quad \text{as } x \to 0.$

Therefore,

$$\begin{aligned} \tilde{y}_3(x) =& d_0(x) + d_1(x) + d_2(x) + \tilde{d}_3(x) \\ =& a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} - \frac{1843x^5}{131072a^4}, \\ f(x, \tilde{y}_3(x)) =& f_3(x, \tilde{d}_3(x)) = O(x^6) \quad \text{as } x \to 0. \end{aligned}$$

The next theorem shows that the sequence of functions $\{f(x, y_\nu(x))\}$ generated by Algorithm 2.1 (i)(ii) asymptotically converges to 0, where $y_\nu(x) = d_0(x) + \cdots + d_\nu(x)$.

Theorem 2.3. *Let $f(x,y)$ be a polynomial of the form*

$$f(x,y) = \sum_{i=0}^{l} a_{i,0}x^i + \sum_{j=1}^{n}\left(\sum_{i=0}^{m} a_{i,j}x^i\right) y^j.$$

Suppose $f(0,y)=0$ has a root c with $\frac{\partial}{\partial y}f(0,c) \neq 0$. Let the polynomials $f_\nu(x,y)$ and the monomials $d_\nu(x)$ be the same as Algorithm 2.1 (i)(ii). Then, the following (1), (2) and (3) hold for $\nu = 1,2,\ldots,N$:

(1) $a_{0,1}^{(\nu)} = a_{0,1}^{(1)} = \dfrac{\partial}{\partial y}f(0,c) \neq 0,$

(2) $1 \leq i_1 < \cdots < i_{\nu-1} < i_\nu,$

(3) $y_\nu(x) = d_0(x) + d_1(x) + \cdots + d_\nu(x)$ *satisfies*

$$f(x, y_\nu(x)) = o(x^{i_\nu}) \quad \text{as } x \to 0. \tag{2.2}$$

Proof. We prove by mathematical induction on ν. By Taylor's theorem,

$$f_1(x,y) = f(x,c+y) = f(x,c) + \frac{\partial}{\partial y}f(x,c)y + \sum_{j=2}^{n}\frac{1}{j!}\frac{\partial^j}{\partial y^j}f(x,c)y^j.$$

When $x=0$, we have

$$f_1(0,y) = f(0,c) + \frac{\partial}{\partial y}f(0,c)y + \sum_{j=2}^{n}\frac{1}{j!}\frac{\partial^j}{\partial y^j}f(0,c)y^j.$$

Therefore, $a_{0,0}^{(1)} = f(0,c) = 0$ and $a_{0,1}^{(1)} = \frac{\partial}{\partial y}f(0,c) \neq 0$, and thus, $i_1 \geq 1$ and

$$d_1(x) = -\frac{a_{i_1,0}^{(1)}}{a_{0,1}^{(1)}}x^{i_1}.$$

Then,

$$\begin{aligned}
f(x,y_1(x)) =& f(x, d_0(x)+d_1(x)) = f_1(x, d_1(x)) \\
=& \sum_{i=i_1}^{l_1} a_{i,0}^{(1)}x^i + \sum_{j=1}^{n}\left(\sum_{i=0}^{m_1} a_{i,j}^{(1)}x^i\right) d_1(x)^j \\
=& \left(a_{i_1,0}^{(1)}x^{i_1} + a_{0,1}^{(1)}d_1(x)\right) + \sum_{i=i_1+1}^{l_1} a_{i,0}^{(1)}x^i \\
& + \left(\sum_{i=1}^{m_1} a_{i,1}^{(1)}x^i\right) d_1(x) + \sum_{j=2}^{n}\left(\sum_{i=0}^{m_1} a_{i,j}^{(1)}x^i\right) d_1(x)^j \\
=& o(x^{i_1}) \quad \text{as } x \to 0.
\end{aligned}$$

It holds for $\nu = 1$.

Next we assume that it holds for $\nu(\nu > 1)$. Let

$$f_\nu(x,y) = \sum_{i=i_\nu}^{l_\nu} a_{i,0}^{(\nu)} x^i + \sum_{j=1}^{n} \left(\sum_{i=0}^{m_\nu} a_{i,j}^{(\nu)} x^i \right) y^j, \quad a_{i_\nu,0}^{(\nu)} \neq 0.$$

By the induction hypothesis, $a_{0,1}^{(\nu)} = a_{0,1}^{(1)} \neq 0$ and

$$d_\nu(x) = -\frac{a_{i_\nu,0}^{(\nu)}}{a_{0,1}^{(1)}} x^{i_\nu}.$$

Since

$$\begin{aligned} f_{\nu+1}(x,y) =& f_\nu(x, d_\nu(x) + y) \\ =& \sum_{i=i_\nu}^{l_\nu} a_{i,0}^{(\nu)} x^i + \sum_{j=1}^{n} \left(\sum_{i=0}^{m_\nu} a_{i,j}^{(\nu)} x^i \right) (d_\nu(x) + y)^j \\ =& \left(a_{i_\nu,0}^{(\nu)} x^{i_\nu} + a_{0,1}^{(1)} d_\nu(x) \right) + \sum_{i=i_\nu+1}^{l_\nu} a_{i,0}^{(\nu)} x^i + \left(\sum_{i=1}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) d_\nu(x) \\ &+ \left(\sum_{i=0}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) y + \sum_{j=2}^{n} \left(\sum_{i=0}^{m_\nu} a_{i,j}^{(\nu)} x^i \right) (d_\nu(x) + y)^j \end{aligned}$$

and $a_{i_\nu,0}^{(\nu)} x^{i_\nu} + a_{0,1}^{(1)} d_\nu(x) = 0$, it holds that $i_{\nu+1} > i_\nu$ and $a_{0,1}^{(\nu+1)} = a_{0,1}^{(\nu)} = a_{0,1}^{(1)} \neq 0$. By the definition of $i_{\nu+1}$,

$$\sum_{i=1}^{i_{\nu+1}-i_\nu-1} \left(a_{i+i_\nu,0}^{(\nu)} x^{i+i_\nu} + a_{i,1}^{(\nu)} x^i d_\nu(x) \right) = 0.$$

Thus,

$$\begin{aligned} f_{\nu+1}(x,y) =& \sum_{i=i_{\nu+1}}^{l_\nu} a_{i,0}^{(\nu)} x^i + \left(\sum_{i=i_{\nu+1}-i_\nu}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) d_\nu(x) \\ &+ \left(\sum_{i=0}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) y + \sum_{j=2}^{n} \left(\sum_{i=0}^{m_\nu} a_{i,j}^{(\nu)} x^i \right) (d_\nu(x) + y)^j \\ =& a_{i_{\nu+1},0}^{(\nu+1)} x^{i_{\nu+1}} + \sum_{i=i_{\nu+1}+1}^{l_\nu} a_{i,0}^{(\nu)} x^i + \left(\sum_{i=i_{\nu+1}-i_\nu+1}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) d_\nu(x) \\ &+ \left(\sum_{i=0}^{m_\nu} a_{i,1}^{(\nu)} x^i \right) y + \sum_{j=2}^{n} \left(\sum_{i=0}^{m_\nu} a_{i,j}^{(\nu)} x^i \right) (d_\nu(x) + y)^j, \end{aligned}$$

where

$$a^{(\nu+1)}_{i_{\nu+1},0} = a^{(\nu)}_{i_{\nu+1},0} + a^{(\nu)}_{i_{\nu+1}-i_\nu,1}\left(-\frac{a^{(\nu)}_{i_\nu,0}}{a^{(1)}_{0,1}}\right).$$

Thus,

$$d_{\nu+1}(x) = -\frac{a^{(\nu+1)}_{i_{\nu+1},0}}{a^{(1)}_{0,1}} x^{i_{\nu+1}}.$$

The coefficient of $x^{i_{\nu+1}}$ in $f_{\nu+1}(x, d_{\nu+1}(x))$ is

$$a^{(\nu+1)}_{i_{\nu+1},0} + a^{(\nu)}_{0,1}\left(-\frac{a^{(\nu+1)}_{i_{\nu+1},0}}{a^{(1)}_{0,1}}\right) = 0.$$

Therefore,

$$\begin{aligned} f(x, y_{\nu+1}(x)) &= f_1(x, d_1(x) + \cdots + d_{\nu+1}(x)) = f_2(x, d_2(x) + \cdots + d_{\nu+1}(x)) = \cdots \\ &= f_{\nu+1}(x, d_{\nu+1}(x)) = o(x^{i_{\nu+1}}) \quad \text{as } x \to 0. \end{aligned}$$

This theorem has been proved by mathematical induction. □

The core of Algorithm 2.1 is to take $d_\nu(x)$ so that

$$a^{(\nu)}_{i_\nu,0} x^{i_\nu} + a^{(1)}_{0,1} d_\nu(x) = 0,$$

thereby increasing the order $i_{\nu+1}$ of $f_{\nu+1}(x, 0)$. The equation

$$a^{(\nu)}_{i_\nu,0} x^{i_\nu} + a^{(1)}_{0,1} y = 0$$

corresponds with "in which p and x separately are of least dimension and suppose them nearly equal to zero."

By the implicit function theorem, there exist open intervals $I = (-\delta, \delta), (\delta > 0)$, $J = (c - \eta, c + \eta), (\eta > 0)$ and the unique function $\phi : I \to J$ such that

(i) $\phi(0) = c$,

(ii) $f(x, \phi(x)) = 0$ for $\forall x \in I$,

(iii) $\phi(x)$ is of class $C^\infty(I)$.

Since $y_\nu(0) = c$, the asymptotic formula (2.2) means that $y_\nu(x)$ asymptotically converges to $\phi(x)$ as $\nu \to \infty$. We can say that Newton gave an algorithm to construct the implicit function $\phi(x)$ of an algebraic equation $f(x, y) = 0$ when $f(0, c) = 0$ with $\frac{\partial}{\partial y} f(0, c) \neq 0$.

The next theorem shows that Algorithm 2.1 (iii) is valid.

Theorem 2.4. *Let $f(x,y)$ be a polynomial of the form*

$$f(x,y)=\sum_{i=0}^{l/\mu} a_{i\mu,0}x^{i\mu}+\sum_{j=1}^{n}\left(\sum_{i=0}^{m/\mu} a_{i\mu,j}x^{i\mu}\right)y^j,\quad \mu\in\mathbb{N}.$$

Suppose $f(0,y)=0$ has a root c with $\frac{\partial}{\partial y}f(0,c)\neq 0$. Let integers ν, N, i_ν, l_ν, and m_ν be the same as Algorithm 2.1. Let polynomials or monomials $f_\nu(x,y), f_N(x,y), d_\nu(x), \tilde{d}_N(x)$, $y_\nu(x), \tilde{y}_N(x), \mathcal{N}(x)$, and $\mathcal{D}(x)$ be the same as Algorithm 2.1. Then, the following asymptotic formulas hold:

$$\begin{aligned} f_N(x,\tilde{d}_N(x)) =& O(x^{i_N+N\mu}) \quad \text{as } x\to 0,\\ f(x,\tilde{y}_N(x)) =& O(x^{i_N+N\mu}) \quad \text{as } x\to 0.\end{aligned}$$

Proof. As in the proof of Theorem 2.3, it holds that

$$f_\nu(x,y)=\sum_{i=i_\nu/\mu}^{l_\nu/\mu} a^{(\nu)}_{i\mu,0}x^{i\mu}+\sum_{j=1}^{n}\left(\sum_{i=0}^{m_\nu/\mu} a^{(\nu)}_{i\mu,j}x^{i\mu}\right)y^j,\quad a^{(\nu)}_{i_\nu,0}\neq 0.$$

Thus, $\mathcal{N}(x)$ and $\mathcal{D}(x)$ can be written as

$$\begin{aligned}\mathcal{N}(x)&=a^{(N)}_{i_N,0}x^{i_N}+a^{(N)}_{i_N+\mu,0}x^{i_N+\mu}+\cdots+a^{(N)}_{i_N+(N-1)\mu,0}x^{i_N+(N-1)\mu},\\ \mathcal{D}(x)&=a^{(N)}_{0,1}+a^{(N)}_{\mu,1}x^{\mu}+\cdots+a^{(N)}_{(N-1)\mu,1}x^{(N-1)\mu},\end{aligned}$$

respectively. Then, $\mathcal{N}(x)/\mathcal{D}(x)$ can be written as

$$\begin{aligned}\frac{\mathcal{N}(x)}{\mathcal{D}(x)}&=e_{i_N}x^{i_N}+e_{i_N+\mu}x^{i_N+\mu}+\cdots+e_{i_N+(N-1)\mu}x^{i_N+(N-1)\mu}\\ &\qquad+O(x^{i_N+N\mu})\quad\text{as } x\to 0.\end{aligned}\tag{2.3}$$

Thus,

$$\begin{aligned} f_N(x,y)=&\,\mathcal{N}(x)+\sum_{i=i_N/\mu+N}^{l_N/\mu} a^{(N)}_{i\mu,0}x^{i\mu}+\mathcal{D}(x)y+\left(\sum_{i=N}^{m_N/\mu} a^{(N)}_{i\mu,1}x^{i\mu}\right)y\\ &+\sum_{j=2}^{n}\left(\sum_{i=0}^{m_N/\mu} a^{(N)}_{i\mu,j}x^{i\mu}\right)y^j,\\ \tilde{d}_N(x)=&-(e_{i_N}x^{i_N}+e_{i_N+\mu}x^{i_N+\mu}+\cdots+e_{i_N+(N-1)\mu}x^{i_N+(N-1)\mu}).\end{aligned}$$

Since $\mathcal{D}(x)=O(1)$ and $\mathcal{N}(x)/\mathcal{D}(x)=-\tilde{d}_N(x)+O(x^{i_N+N\mu})$ as $x\to 0$, we have

$$f_N(x,\tilde{d}_N(x))=O(x^{i_N+N\mu})\quad\text{as } x\to 0.$$

This completes the proof. □

Example 2.5. We apply Algorithm 2.1 with $N = 3$ to $f(x,y) = x^2 - 2ay + y^2 = 0$ which satisfies (2.1) with $\mu = 2$.

The roots of $f(0,y) = -2ay + y^2 = 0$ are $2a$ and 0. We take $c = 0$ as the root of the equation. Then, $\frac{\partial}{\partial y} f(0,0) = -2a \neq 0$. The implicit function $\phi(x)$ of $f(x,y) = 0$ with $\phi(0) = 0$ is $\phi(x) = a - \sqrt{a^2 - x^2}$.

$\nu = 0$ Put $f_0(x,y) = x^2 - 2ay + y^2$ and $d_0(x) = 0$.

$\nu = 1$ Since $f_1(x,y) = f_0(x, 0+y) = x^2 - 2ay + y^2$, we have $i_1 = 2, a^{(1)}_{2,0} = 1, a^{(1)}_{0,1} = -2a$, and thus $d_1(x) = \frac{1}{2a}x^2$.

$\nu = 2$ Since $f_2(x,y) = \frac{1}{4a^2}x^4 + \left(-2a + \frac{1}{a}x^2\right) y + y^2$, we have $i_2 = 4, a^{(2)}_{4,0} = \frac{1}{4a^2}, a^{(2)}_{0,1} = -2a$, and thus, $d_2(x) = \frac{1}{8a^3}x^4$.

$\nu = 3$ Since $f_3(x,y) = \dfrac{1}{8a^4}x^6 + \dfrac{1}{64a^6}x^8 + \left(-2a + \dfrac{1}{a}x^2 + \dfrac{1}{4a^3}x^4\right) y + y^2$, we have $i_3 = 6$.

$$\tilde{d}_3(x) = -\frac{\frac{1}{8a^4}x^6 + \frac{1}{64a^6}x^8}{-2a + \frac{1}{a}x^2 + \frac{1}{4a^3}x^4} = \frac{1}{16a^5}x^6 + \frac{5}{128a^7}x^8 + \frac{7}{256a^9}x^{10} + O(x^{12}).$$

Therefore,

$$\tilde{y}_3(x) = 0 + \frac{1}{2a}x^2 + \frac{1}{8a^3}x^4 + \frac{1}{16a^5}x^6 + \frac{5}{128a^7}x^8 + \frac{7}{256a^9}x^{10} + O(x^{12}),$$
$$f(x, \tilde{y}_3(x)) = f_3(x, \tilde{d}_3(x)) = O(x^{12}).$$

§ 3. The asymptotic expansion of the implicit function as $x \to \infty$

Up to this point, Newton elucidated the literal resolution when x is close to zero, but from here elucidated when x is sufficiently large.

> But if you wish that the value of the area should approach nearer the truth greater x is, take this an example:$y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Accordingly, ready to resolve this, I take out the terms $y^3 + x^2y - 2x^3$ in which x and y either separately or multiplied together are of the most and equal dimensions everywhere. From these, set as it were equal to zero, I elicit the root, finding it to be x, and write it in the quotient: or, what comes to the same thing, on substituting unity for x from $y^3 + y - 2$ I extract the root 1, multiply it by x and write the product x in the quotient. Then I suppose $x + p = y$ and so proceed as in the former example until I have the quotient $x - \dfrac{a}{4} + \dfrac{a^2}{64x} + \dfrac{131a^3}{512x^2} + \dfrac{509a^4}{16384x^3}$&c,[...]
>
> MP II, pp.226-227

We formulate the literal resolution of the affected equation when x is sufficiently large as a modern algorithm faithful to Newton's elucidation.

For a function

$$f(x,y) = \sum_{i,j} a_{i,j} x^{q_{i,j}} y^j, \quad q_{i,j} \in \mathbb{Q},$$

the set of all exponents of x is defined by

$$P(f,\alpha) = \{ \, q_{i,j} + \alpha j \mid a_{i,j} \neq 0 \, \},$$

when $y \sim cx^\alpha$ as $x \to \infty$ for some $c \in \mathbb{R}$.

Algorithm 3.1. **(The asymptotic expansion as $x \to \infty$.)** Let

$$f(x,y) = \sum_{j=0}^{n} \left(\sum_{i=0}^{m} a_{i,j} x^i \right) y^j = 0,$$

be an algebraic equation.

(i) Put $f_0(x,y) = f(x,y)$.

(ii) Find a rational number α_0 such that there are two or more terms in

$$g_0(x,y;\alpha_0) = \sum_{i+\alpha_0 j = \max P(f,\alpha_0)} a_{i,j} x^i y^j.$$

(iii) Take a root $v = c_0$ of the equation $g_0(1,v;\alpha_0) = 0$.

(iv) Put $d_0(x) = c_0 x^{\alpha_0}$.

(v) Repeat (1), (2), (3) and (4) below for $\nu = 1, 2, \ldots, N$:

(1) calculate $f_\nu(x,y) = f_{\nu-1}(x, d_{\nu-1}(x) + y)$, say

$$f_\nu(x,y) = \sum_{j=0}^{n} \left(\sum_{i} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} \right) y^j,$$

(2) find a rational number α_ν with $\max P(f_\nu, \alpha_\nu) < \max P(f_{\nu-1}, \alpha_{\nu-1})$ such that there are two or more terms in

$$g_\nu(x,y;\alpha_\nu) = \sum_{q_{i,j}^{(\nu)} + \alpha_\nu j = \max P(f_\nu,\alpha_\nu)} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} y^j,$$

(3) take a root $v = c_\nu$ of the equation $g_\nu(1,v;\alpha_\nu) = 0$,

(4) put $d_\nu(x) = c_\nu x^{\alpha_\nu}$.

Then, the function $y_N(x) = \sum_{\nu=0}^{N} d_\nu(x)$ satisfies

$$f(x, y_N(x)) = o(x^{\max P(f_N, \alpha_N)}) \quad \text{as } x \to \infty.$$

Theorem 3.2. *Keep the notation in Algorithm 3.1. Put* $\max P(f_{-1}, \alpha_{-1}) = \infty$. *Suppose there exist* α_ν *and* c_ν *in Algorithm 3.1, for* $\nu = 0, 1, \ldots, N$. *Then,*

1. *for* $\nu = 0, 1, \ldots, N$,

$$\begin{aligned} &g_\nu(x, d_\nu(x); \alpha_\nu) = 0, \\ &f_\nu(x, d_\nu(x)) = o(x^{\max P(f_\nu, \alpha_\nu)}) \quad \text{as } x \to \infty, \\ &\max P(f_\nu, \alpha_\nu) < \max P(f_{\nu-1}, \alpha_{\nu-1}); \end{aligned}$$

2. *the function* $y_N(x) = \sum_{\nu=0}^{N} d_\nu(x)$ *satisfies*

$$f(x, y_N(x)) = o(x^{\max P(f_N, \alpha_N)}) \quad \text{as } x \to \infty.$$

Proof.

1. Assume there exists a rational number α_ν such that there are two or more terms in

$$g_\nu(x, y; \alpha_\nu) = \sum_{q_{i,j}^{(\nu)} + \alpha_\nu j = \max P(f_\nu, \alpha_\nu)} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} y^j.$$

Let c_ν be a non-zero root of

$$g_\nu(1, v; \alpha_\nu) = \sum_{q_{i,j}^{(\nu)} + \alpha_\nu j = \max P(f_\nu, \alpha_\nu)} a_{i,j}^{(\nu)} v^j = 0.$$

Then,

$$\begin{aligned} g_\nu(x, c_\nu x^{\alpha_\nu}; \alpha_\nu) = &\sum_{q_{i,j}^{(\nu)} + \alpha_\nu j = \max P(f_\nu, \alpha_\nu)} a_{i,j}^{(\nu)} c_\nu^j x^{\max P(f_\nu, \alpha_\nu)} \\ = &g_\nu(1, c_\nu; \alpha_\nu) x^{\max P(f_\nu, \alpha_\nu)} = 0. \end{aligned}$$

Since

$$f_\nu(x, c_\nu x^{\alpha_\nu}) = g_\nu(x, c_\nu x^{\alpha_\nu}; \alpha_\nu) + o(x^{\max P(f_\nu, \alpha_\nu)}) \quad \text{as } x \to \infty,$$

we have

$$f_\nu(x, d_\nu(x)) = o(x^{\max P(f_\nu, \alpha_\nu)}) \quad \text{as } x \to \infty.$$

From the way of deciding α_ν, it holds that

$$\max P(f_\nu, \alpha_\nu) < \max P(f_{\nu-1}, \alpha_{\nu-1}).$$

2. By the definition of f_ν,

$$f(x, d_0(x) + d_1(x) + \cdots + d_N(x)) = f_1(x, d_1(x) + \cdots + d_N(x)) = \cdots$$
$$= f_N(x, d_N(x)) = o(x^{\max P(f_N, \alpha_N)}) \quad \text{as } x \to \infty.$$

□

The function $g_\nu(x, y; \alpha_\nu)$ is the sum of the terms in which "x and y either separately or multiplied together are of the most and equal dimensions everywhere." The equation $g_\nu(1, v; \alpha_\nu) = 0$ corresponds with "on substituting unity for x."

Example 3.3. We apply Algorithm 3.1 to Newton's example $f(x, y) = y^3 + axy + x^2y - a^3 - 2x^3 = 0$.

- $\boldsymbol{\nu = 0}$ Two or more exponents of $\{3\alpha, 1 + \alpha, 2 + \alpha, 0, 3\}$ are equal to each other and maximized when $\alpha = 1$. Thus, $g_0(x, y; 1) = y^3 + x^2y - 2x^3$, and the root of $g_0(1, v; 1) = v^3 + v - 2 = 0$ is $c = 1$. Therefore, $d_0(x) = x$ and $f(x, x) = ax^2 - a^3 = o(x^3)$ as $x \to \infty$.
- $\boldsymbol{\nu = 1}$ Since $f_1(x, y) = f(x, x + y) = ax^2 - a^3 + (4x^2 + ax)y + 3xy^2 + y^3$, two or more exponents of $\{2, 0, 2 + \alpha, 1 + \alpha, 1 + 2\alpha, 3\alpha\}$ are equal to each other and maximized when $\alpha = 0$. Thus, $g_1(x, y; 0) = ax^2 + 4x^2y, c = -\frac{a}{4}$, and $d_1(x) = -\frac{a}{4}$. Therefore, $f_1(x, -\frac{a}{4}) = -\frac{a^2}{16}x - \frac{65}{64}a^3 = o(x^2)$ as $x \to \infty$.
- $\boldsymbol{\nu = 2}$ Since $f_2(x, y) = -\dfrac{1}{16}a^2x - \dfrac{65}{64}a^3 + \left(4x^2 - \dfrac{1}{2}ax + \dfrac{3a^2}{16}\right)y + (3x - \dfrac{3}{4}a)y^2 + y^3$, two or more exponents of $\{1, 0, 2 + \alpha, 1 + \alpha, \alpha, 1 + 2\alpha, 2\alpha, 3\alpha\}$ are equal to each other and maximized when $\alpha = -1$. Thus, $g_2(x, y; -1) = -\frac{1}{16}a^2 + 4x^2y, c = \frac{a^2}{64}, d_2(x) = \frac{a^2}{64x}$. Therefore $f_2(x, \frac{a^2}{64x}) = -\frac{131a^3}{128} + \frac{15a^4}{4096x} - \frac{3a^5}{16384x^2} + \frac{a^6}{262144x^3} = o(x)$ as $x \to \infty$.

Put $y_2(x) = x - \frac{a}{4} + \frac{a^2}{64x}$. Then

$$f(x, y_2(x)) = f_1(x, -\frac{a}{4} + \frac{a^2}{64x}) = f_2(x, \frac{a^2}{64x}) = o(x) \quad \text{as } x \to \infty.$$

By Theorem 3.2, the infinite series

$$x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3} \cdots$$

asymptotically converges to the implicit function of $y^3 + axy + x^2y - a^3 - 2x^3 = 0$ as $x \to \infty$.

§ 4. The Newton diagram in *De Methodis*

The explanation of the literal resolution of affected equations in *De Analysi* is insufficient. Even if $f(0, y) = 0$ has a root c, the initial quotient can not always be found when $\frac{\partial}{\partial y} f(0, c) = 0$. Also, when x is large, we cannot obtain the terms of the maximum dimension unless we decide α of $y \sim cx^{\alpha}$. In *De Methodis*, Newton solved these points by using the Newton diagram which he called the parallelogram.

> However, to make this rule still more evident, I thought it fitting to expound it in addition with the aid of the following diagram. Describing the right angle BAC, I divide its sides BA, AC into equal segments and from these raise normals distributing the space between the angle into equal squares or rectangles: these I conceive to be denominated by the powers of the variables x and y, as you see them entered in figure 1. Next, when some equation is proposed, I mark the rectangles corresponding to each of its terms with some sign and apply a ruler to two or maybe several of the rectangles so marked, one of which it to be the lowest in the left-hand column alongside AB, a second to the right touching the ruler, and all the rest not in contact with the ruler should lie above it. I then choose the terms of the equation which are marked out by the rectangles in contact with the ruler and thence seek the quantity to be added to the quotient.

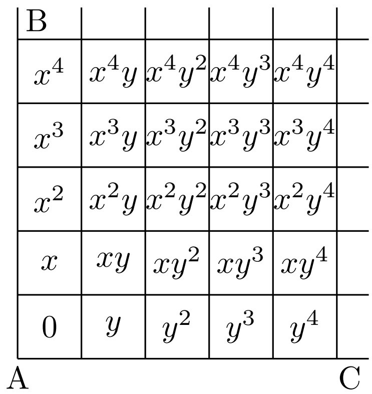


fig 1

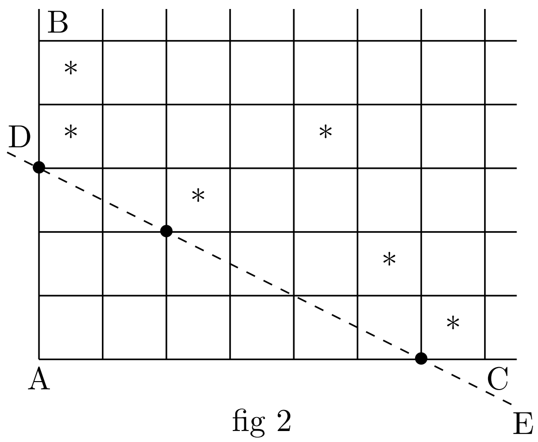


fig 2

> So to extract the root y from
>
> $$y^6 - 5xy^5 + (x^3/a)y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0,$$
>
> I mark the rectangles answering to its terms with some sign $*$, as you see done in the second illustration. I then apply the ruler DE to the lower corner of the places marked out in the left-hand column and make it swing to the right from bottom to top until in like fashion it begins to touch a second or maybe

several together of the other marked places. Those so touched I see to be x^3, x^2y^2 and y^6. Hence from the terms $y^6 - 7a^2x^2y^2 + 6a^3x^3$ as through set equal to nothing (and in addition, if it places, reduced to $v^6 - 7v^2 + 6 = 0$ by supposing $y = v \times \sqrt{ax}$) I seek the value of y and find it to be fourfold, $+\sqrt{ax}, -\sqrt{ax}, +\sqrt{2ax}$ and $-\sqrt{2ax}$. Any of these may be acceptable as an initial term in the quotient depending on whether the decision is made to extract one or other of the roots.

MP III, pp.48-53

Modern expression of how to extract the root y from $f(x, y) = 0$ when x is close to 0 is as follows. Let $f(x, y) = \sum_{j=0}^{n} \sum_i a_{i,j}x^iy^j = 0$ be an algebraic equation. Each monomial $a_{i,j}x^iy^j$ is represented by a lattice point (j, i) in the j-i plane with j on the horizontal axis and i on the vertical axis. The set of all lattice points of $f(x, y)$ is denoted by $L(f) = \{(j, i) \mid a_{i,j} \neq 0\}$. Let $S(f)$ be the set of line segments whose end points are the points of the set $L(f)$. Take the lowest point $(0, m)$ of $L(f)$ and the line segment $\ell \in S(f)$ with the equation

$$\ell : i + \alpha j = m \tag{4.1}$$

such that $\bar{i} + \alpha\bar{j} \geq m \ \forall(\bar{j}, \bar{i}) \in L(f)$. The line segment ℓ corresponds with "the ruler". Let $(j_1, i_1), \ldots, (j_r, i_r)$ be all points of $L(f)$ on ℓ, then $i_k + \alpha j_k = m, (k = 1, \ldots, r)$. Put

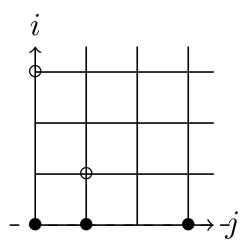
$$g(x, y) = \sum_{k=1}^{r} a_{i_kj_k}x^{i_k}y^{j_k}.$$

Let c be a root of $g(1, v) = 0$. Take $d(x) = cx^\alpha$ as a quotient.

The j-i plane plotted the lattice points in $L(f)$ and drawn the lowest line segment is called the Newton diagram of $f(x, y) = \sum_{j=0}^{n} \sum_i a_{i,j}x^iy^j = 0$, and the above method is called the Newton diagram method. At the time of illustration, the lattice points on the line segment are represented by black circles $\bullet$, and the lattice points outside the line segment are represented by white circles $\circ$.

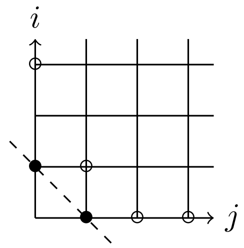
Example 4.1. We apply the Newton diagram method to Newton's example $f(x, y) = y^3 + a^2y - 2a^3 + axy - x^3 = 0$, when x is close to 0.

$\boldsymbol{\nu = 0}$ The Newton diagram of $f_0(x, y) = y^3 + a^2y - 2a^3 + axy - x^3 = 0$ is as below.



The lowest line is $\ell : i = 0$; thus, $\alpha = 0$ and $g(x, y) = y^3 + a^2y - 2a^3$. Thus, the first term of the quotient is $y = a$, which is the root of $y^3 + a^2y - 2a^3 = 0$.

$\boldsymbol{\nu = 1}$ The Newton diagram of $f_1(x, y) = f(x, a + y) = a^2x - x^3 + (4a^2 + ax)y + 3ay^2 + y^3$ is as below.



The lowest line is $\ell : i + j = 1$; thus, $\alpha = 1$ and $g(x, y) = a^2x + 4a^2y$. Thus, the second term of the quotient is $y = -\frac{x}{4}$ which is the root of $a^2x + 4a^2y = 0$.

We formulate the Newton diagram method in *De Methodis* as an algorithm similar to Algorithm 3.1. Therefore, the order of a function

$$f(x) = \sum_i a_i x^{q_i}, \quad q_i \in \mathbb{Q},$$

is defined by

$$\operatorname{ord} f = \min\{\ q_i \mid a_i \neq 0\ \}.$$

Algorithm 4.2. (**The asymptotic expansion as $\boldsymbol{x \to 0}$.**) For an algebraic equation

$$f(x, y) = \sum_{j=0}^{n} \left(\sum_{i=0}^{m} a_{i,j} x^i \right) y^j = 0,$$

put $f_0(x, y) = f(x, y)$.

(i) Find a rational number α_0 such that there are two or more terms in

$$g_0(x, y; \alpha_0) = \sum_{i + \alpha_0 j = \operatorname{ord} f_0(x,0)} a_{i,j}^{(0)} x^i y^j.$$

(ii) Take a root $v = c_0$ of the equation $g_0(1, v; \alpha_0) = 0$.

(iii) Put $d_0(x) = c_0 x^{\alpha_0}$.

(iv) Repeat (1), (2), (3) and (4) below for $\nu = 1, 2, \ldots, N$:

(1) calculate $f_\nu(x,y) = f_{\nu-1}(x, d_{\nu-1}(x)+y)$, say

$$f_\nu(x,y) = \sum_{j=0}^{n} \left(\sum_i a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} \right) y^j,$$

(2) find a rational number α_ν such that there are two or more terms in

$$g_\nu(x,y;\alpha_\nu) = \sum_{q_{i,j}^{(\nu)}+\alpha_\nu j = \operatorname{ord} f_\nu(x,0)} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} y^j,$$

(3) take a root $v = c_\nu$ of the equation $g_\nu(1,v;\alpha_\nu) = 0$,

(4) put $d_\nu(x) = c_\nu x^{\alpha_\nu}$.

Then, the function $y_N(x) = \sum_{\nu=0}^{N} d_\nu(x)$ satisfies

$$f(x, y_N(x)) = o(x^{\operatorname{ord} f_N(x,0)}) \quad \text{as } x \to 0.$$

Example 4.3. We apply Algorithm 4.2 to Newton's example $f(x,y) = y^3 + a^2y - 2a^3 + axy - x^3 = 0$.

$\nu = 0$ Two or more exponents of $\{3\alpha, \alpha, 0, 1+\alpha, 3\}$ are equal to $\operatorname{ord} f(x,0) = 0$ and minimized when $\alpha = 0$. Thus, $g_0(x,y;0) = y^3 + a^2y - 2a^3$. The root of $g_0(x,y;0) = 0$ is $y = a$, and thus, $d_0(x) = a$. Therefore, $f(x,a) = a^2x - x^3 = o(1) \quad \text{as } x \to 0$.

$\nu = 1$ Since $f_1(x,y) = f(x, a+y) = a^2x - x^3 + (4a^2 + ax)y + 3ay^2 + y^3$, two or more exponents of $\{1, 3, \alpha, 1+\alpha, 2\alpha, 3\alpha\}$ are equal to $\operatorname{ord} f_1(x,0) = 1$ and minimized when $\alpha = 1$. Thus, $g_1(x,y;1) = a^2x + 4a^2y$, and $d_1(x) = -\frac{1}{4}x$. Therefore, $f_1(x, -\frac{1}{4}x) = -\frac{1}{16}ax^2 - \frac{65}{64}x^3 = o(x) \quad \text{as } x \to 0$.

$\nu = 2$ Thereafter, it can be executed in the same way.

Krantz and Parks [6, pp.15-20] determine α by different method from Algorithm 4.2. They assume $y(0) = 0$ and $y(x) = x^\alpha \tilde{y}(x)$ with $\tilde{y}(x)$ a continuous function that does not vanish when $x = 0$. They substitute $y = x^\alpha \tilde{y}(x)$ in $f(x,y) = 0$, and explain "To be able to determine $\tilde{y}(0)$ from" $f(x, x^\alpha \tilde{y}(x)) = 0$, "there must be two or more monomials in" $f(x, x^\alpha \tilde{y}(x)) = 0$ "which have the same power of x and all other monomials must have a large power of x."

§ 5. An improved algorithm as $x \to 0$

By rewriting Algorithm 3.1 dually, even if $(0,c)$ is a singular point, infinite series expansion (Puiseux expansion) of one of the branches can be obtained.

Algorithm 5.1. (**An improved algorithm as** $x \to 0$.) For an algebraic equation

$$f(x,y) = \sum_{j=0}^{n} \left(\sum_{i=0}^{m} a_{i,j} x^i \right) y^j = 0,$$

put $f_0(x,y) = f(x,y)$.

(i) Find a rational number α_0 such that there are two or more terms in

$$g_0(x,y;\alpha_0) = \sum_{i+\alpha_0 j = \min P(f_0,\alpha_0)} a_{i,j}^{(0)} x^i y^j.$$

(ii) Take a root $v = c_0$ of the equation $g_0(1,v;\alpha_0) = 0$.

(iii) Put $d_0(x) = c_0 x^{\alpha_0}$.

(iv) Repeat (1), (2), (3) and (4) below for $\nu = 1, 2, \ldots, N$:

(1) calculate $f_\nu(x,y) = f_{\nu-1}(x, d_{\nu-1}(x) + y)$, say

$$f_\nu(x,y) = \sum_{j=0}^{n} \left(\sum_{i} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} \right) y^j,$$

(2) find a rational number α_ν with $\min P(f_\nu, \alpha_\nu) > \min P(f_{\nu-1}, \alpha_{\nu-1})$ such that there are two or more terms in

$$g_\nu(x,y;\alpha_\nu) = \sum_{q_{i,j}^{(\nu)} + \alpha_\nu j = \min P(f_\nu,\alpha_\nu)} a_{i,j}^{(\nu)} x^{q_{i,j}^{(\nu)}} y^j,$$

(3) take a root $v = c_\nu$ of the equation $g_\nu(1,v;\alpha_\nu) = 0$,

(4) put $d_\nu(x) = c_\nu x^{\alpha_\nu}$.

Then, the function $y_N(x) = \sum_{\nu=0}^{N} d_\nu(x)$ satisfies

$$f(x, y_N(x)) = o(x^{\min P(f_N,\alpha_N)}) \quad \text{as } x \to 0.$$

The Newton polygon of an algebraic equation

$$f(x,y) = \sum_{i,j} a_{i,j} x^i y^j = 0$$

is defined by the convex hull of the set $\{\ (j+x, i+y)\ |a_{i,j} \neq 0; x, y \in \mathbb{R}^+\}\}$ in the j-i plane. Algorithm 5.1 is equivalent to modern Newton polygon method [6, pp.15-20] [10, pp.58-61].

Theorem 5.2. *Keep the notation in Algorithm 5.1. Put* $\min P(f_{-1},\alpha_{-1}) = -\infty$. *Suppose there exist* α_ν *and* c_ν *in Algorithm 5.1, for* $\nu = 0,1,\dots,N$. *Then*

1. *For* $\nu = 0,1,\dots,N$,

$$g_\nu(x,d_\nu(x);\alpha_\nu) = 0,$$
$$f_\nu(x,d_\nu(x)) = o(x^{\min P(f_\nu,\alpha_\nu)}) \quad \textit{as } x\to 0,$$
$$\min P(f_\nu,\alpha_\nu) > \min P(f_{\nu-1},\alpha_{\nu-1}).$$

2. *The function* $y_N(x) = \sum_{\nu=0}^{N} d_\nu(x)$ *satisfies*

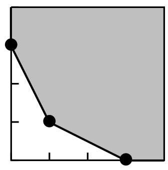
$$f(x,y_N(x)) = o(x^{\min P(f_N,\alpha_N)}) \quad \textit{as } x\to 0.$$

Proof. The proof can be shown dually with that of Theorem 3.2. □

Example 5.3. We apply Algorithm 5.1 to the Descartes folium

$$f(x,y) = y^3 - 3axy + x^3 = 0,$$

which have two branches at $(0,0)$. The Newton polygon of $f(x,y)=0$ is shown below.



$\nu = 0$ Two or more exponents of $\{3\alpha, 1+\alpha, 3\}$ are equal to each other and minimized if $\alpha = \frac{1}{2}$ and $\alpha = 2$. When $\alpha_0 = \frac{1}{2}$, $g_0(x,y;\frac{1}{2}) = y^3 - 3axy$. By solving $y^3 - 3axy = 0$, we take $d_0(x) = \pm\sqrt{3ax}$. When $\alpha_0 = 2$, $g_0(x,y;2) = -3axy + x^3$. By solving $-3axy + x^3 = 0$, we take $d_0(x) = \frac{1}{3a}x^2$.

$\nu = 1$ When $\alpha_0 = \frac{1}{2}$, $f_1(x,y) = f(x, \pm\sqrt{3ax} + y) = y^3 \pm 3\sqrt{3ax}y^2 + 6axy + x^3$. Two or more exponents of $\{3\alpha, \frac{1}{2} + 2\alpha, 1+\alpha, 3\}$ are equal to each other and minimized and $\min P(f_1,\alpha) > \min P(f_0,\frac{1}{2}) = \frac{3}{2}$, when $\alpha = 2$. Thus, $\alpha_1 = 2$ and $g_1(x,y;2) = 6axy + x^3$. By solving $6axy + x^3 = 0$, we have $d_1(x) = -\frac{1}{6a}x^2$. When $\alpha_0 = 2$, $f_1(x,y) = f(x, \frac{1}{3a}x^2 + y) = y^3 + \frac{1}{a}x^2y^2 - 3axy + \frac{1}{3a^2}x^4y + \frac{1}{27a^3}x^6$. Two or more exponents of $\{3\alpha, 2+2\alpha, 1+\alpha, 4+\alpha, 6\}$ are equal to each other and minimized and $\min P(f_1,\alpha) > \min P(f_0,2) = 3$, when $\alpha = 5$. Thus, $\alpha_1 = 5$ and $g_1(x,y;5) = -3axy + \frac{1}{27a^3}x^6$. By solving $-3axy + \frac{1}{27a^3}x^6 = 0$, we have $d_1(x) = \frac{1}{81a^4}x^5$.

Thus, we have the Puiseux expansions of the two branches

$$y = \pm\sqrt{3a}x^{\frac{1}{2}} - \frac{1}{6a}x^2 \mp \cdots, \tag{5.1}$$
$$y = \frac{1}{3a}x^2 + \frac{1}{81a^4}x^5 + \cdots.$$

The Newton diagram method cannot yield (5.1).

§ 6. Conclusion

In this paper we proved the following.

1. When $f(0, c) = 0, \frac{\partial}{\partial y} f(0, c) \neq 0$, the power series given by the algorithm in *De Analysi* is well defined and asymptotically converges to the implicit function of $f(x, y) = 0$.

2. When x is sufficiently large, an infinite series can be constructed by the algorithm in *De Analysi*, and the series asymptotically converges to the implicit function of $f(x, y) = 0$.

3. When $(0, c)$ is a singular point of $f(x, y) = 0$, and an infinite series can be constructed by the Newton diagram method given in *De Methodis*, the series asymptotically converges to one of the branches of $f(x, y) = 0$.

References

[1] S. S. Abyhyankar, Historical Ramblings in Algebraic Geometry and Related Algebra, *The American Mathematical Monthly*, **83** (1976) 409-448.
[2] E. Casas-Alvero, Singularities of Plane Curves, London Mathematical Society Lecture Note Series 276, Cambridge University Press, 2000.
[3] J. L. Chabert, A History of Algorithms, Springer, 1999.
[4] N. Guicciardini, Isaac Newton on Mathematical Certainty and Method, The MIT Press, 2009.
[5] K. Konno, Riemann Surface and Algebraic Curves (in Japanese), 今野一宏, リーマン面と代数曲線, 共立出版, 2015.
[6] S. G. Krantz and H. R. Parks, The Implicit Function Theorem — History, Theory, and Applications, Birkhäuser, 2013.
[7] V. Puiseux, Recherches sur les fonctions algébriques, *Journal de mathématiques pures et appliquées 1re série*, tome **15** (1850), 365-480.
http://portail.mathdoc.fr/JMPA/
[8] E. D. Solomentshev, Analytic Function Theory, in Mathematics of the 19th Century, edited by A. N. Kolmogorov and A. P. Yushkevich, Translated by R. Cooke, Birkhäuser, 1998. https://books.google.co.jp
[9] H. W. Turnbull, The Correspondences of Isaac Newton, Vol. II. Cambridge University Press, 1960.
[10] K. Ueno, Algebraic Geometry (in Japanese), 上野健爾, 代数幾何, 岩波講座応用数学, 1993.
[11] J. Wallis, A Treatise of Algebra, both Historical and Practical, 1685.
http://echo.mpiwg-berlin.mpg.de/
[12] D. T. Whiteside, The Mathematical Papers of Isaac Newton, Vol. I-VIII, Cambridge University Press, 1967-1981.