Regular b-functions of D-modules

Toshinori Oaku

Tokyo Woman's Christian University, Suginami-ku, Tokyo 167-8585, Japan
e-mail: oaku@lab.twcu.ac.jp

Abstract

Let M be an algebraic D-module defined on an affine space X and Y be a linear submanifold of X. We give an algorithm to determine if M is regular specializable along Y, and to find, if so, its regular b-function. (M has a regular b-function by definition if and only if M is regular specializable.) We also prove that the A-hypergeometric system of Gelfand-Kapranov-Zelevinsky is always regular specializable along the origin.

Key words: D-module, b-function, A-hypergeometric system, Gröbner base
1991 MSC: 32C38; 33C70; 35N10; 13P10

1 Introduction

Let X be the affine space $\mathbb{C}^n$ with the coordinate system $x = (x_1, \dots, x_n)$ and Y be the linear subvariety defined by $x_1 = \cdots = x_d = 0$ with $1 \le d \le n$. We denote by $\mathcal{D}_X$ the sheaf on X of linear partial differential operators (of finite order) with holomorphic coefficients. A section P of $\mathcal{D}_X$ is written in a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial^\alpha = \sum_{\alpha_1,\dots,\alpha_n \in \mathbb{N}} a_{\alpha_1 \dots \alpha_n}(x_1, \dots, x_n)\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

with $a_\alpha(x)$ being holomorphic on an open set of X, where $\partial_i = \partial/\partial x_i$ denotes the partial derivation with respect to x_i, and $\mathbb{N}$ is the set of nonnegative integers. The order of P is defined to be

$$\operatorname{ord} P := \max\{|\alpha| = \alpha_1 + \cdots + \alpha_n \mid a_\alpha(x) \neq 0\}.$$

We have two filtrations on $\mathcal{D}_X$: The *order filtration* is defined by

$$F_k(\mathcal{D}_X) := \{P \in \mathcal{D}_X \mid \operatorname{ord} P \le k\} \quad (k \in \mathbb{Z}),$$

and the *V-filtration* with respect to Y is defined by

$$V_k(\mathcal{D}_X) := \left\{P \in \mathcal{D}_X \mid Pf \in \mathcal{J}^{i-k} \text{ for any } f \in \mathcal{J}^i \text{ and } i \in \mathbb{Z}\right\} \quad (k \in \mathbb{Z}),$$

where $\mathcal{J}$ is the defining ideal of Y in $\mathcal{O}_X$, the sheaf of holomorphic functions on X. (We set $\mathcal{J}^i = \mathcal{O}_X$ if $i \leq 0$.)

Definition 1 A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be *regular specializable* along Y at $p \in Y$ (cf. [13], [18]) if and only if for any germ u of $\mathcal{M}$ at p, there exist a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and an operator $Q \in \mathcal{D}_X$ defined on a neighborhood of p such that

$$(b(x_1\partial_1 + \cdots + x_d\partial_d) + Q)u = 0, \quad Q \in V_{-1}(\mathcal{D}_X), \quad \operatorname{ord} Q \leq \deg b(s). \tag{1}$$

A polynomial $b(s)$ satisfying (1) is called a *regular b-function* of u along Y at p. If $\mathcal{I}$ is a left ideal of $\mathcal{D}$, the stalk of $\mathcal{D}_X$ at $0 \in X$, and $b(s)$ is a regular b-function of the residue class of $1 \in \mathcal{D}$ in $\mathcal{D}/\mathcal{I}$, then we call $b(s)$ simply a regular b-function of $\mathcal{I}$ and say that $\mathcal{I}$ is regular specializable along Y.

The b-function of u is the monic polynomial $b(s)$ of the minimum degree which satisfies (1) without the condition $\operatorname{ord} Q \leq \deg b(s)$. Hence a regular b-function is a multiple of the b-function.

An equation of the form (1) is essential in proving the convergence of the power series solutions of $\mathcal{M}$. Let $\mathcal{O}_{X,0} = \mathbb{C}\{x_1, \ldots, x_n\}$ be the ring of convergent power series. The formal completion of $\mathcal{O}_{X,0}$ along Y is defined to be

$$\mathcal{O}_{\widehat{X|Y},0} = \Big\{ \sum_{\alpha_1,\ldots,\alpha_d \geq 0} a_{\alpha_1,\ldots,\alpha_d}(x_{d+1}, \ldots, x_n) x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mid a_{\alpha_1,\ldots,\alpha_d}(x_{d+1}, \ldots, x_n) \text{ are holomorphic on some neighborhood } U \subset Y \text{ of } 0 \Big\}.$$

Kashiwara and Kawai [11] proved that if $\mathcal{I}$ is regular specializable along Y, then one has

$$\operatorname{Ext}^k_{\mathcal{D}}(\mathcal{D}/\mathcal{I}, \mathcal{O}_{X,0}) = \operatorname{Ext}^k_{\mathcal{D}}(\mathcal{D}/\mathcal{I}, \mathcal{O}_{\widehat{X|Y},0}) \quad (\forall k \in \mathbb{Z}). \tag{2}$$

On the other hand, Laurent [13] (see also [14]) defined the (algebraic) slopes of $\mathcal{M}$ along Y at p. Laurent and Mebkhout [14] proved that (2) holds if and only if $\mathcal{I}$ has no slopes along Y. It was conjectured in [14] that $\mathcal{M}$ is regular specializable along Y if and only if there is no slope of $\mathcal{M}$ along Y. (As far as the author knows, this remains to be an open problem.) Assi et al. [2] (see also [6]) presented an algorithm for computing the slopes of $\mathcal{M}$ when $\mathcal{M}$ is algebraic and Y is a hyperplane.

We give some examples of regular b-functions for A-hypergeometric systems of Gelfand-Kapranov-Zelevinsky [8]. We prove that A-hypergeometric systems are regular specializable along the origin without assuming the homogeneity (Theorem 1). In particular, this implies (2) with $\mathcal{D}/\mathcal{I}$ being the A-hypergeometric system and $Y = \{0\}$.

For a left ideal I of the Weyl algebra D, we also give an algorithm for determining if $\mathcal{D}I$ is regular specializable along a linear submanifold of arbitrary codimension, and if so, finding a regular b-function of the minimum degree and an associated operator Q satisfying (1). For that purpose, we make use of the homogenization (or the Rees algebra) of D with respect to the order filtration, which we denote by $D^{(h)}$, and (a generalization of) the division algorithm of [9] in $D^{(h)}$.

Our method consists in calculating the b-function, or the indicial polynomial, in the homogenized ring $D^{(h)}$. Note that an algorithm for computing the usual (i.e. non-homogenized) b-function of a D-module was given in [16] for the case $d = 1$, and in [17] for the general case. We also generalize the notion of regular b-function and give an algorithm to compute it. Once a regular b-function is found, we can compute an associated operator Q by using the division algorithm.

Remark 1 A regular b-function of the minimum degree is not necessarily unique up to constant multiple. For example, set $n = 2$, $d = 1$ and let

$$\mathcal{I} := \mathcal{D} \cdot \{\partial_1^2, \partial_1 + \partial_2^2\}$$

be the left ideal of $\mathcal{D}$ generated by $\{\partial_1^2, \partial_1 + \partial_2^2\}$. Then

$$(x_1\partial_1)(x_1\partial_1 - 1 + c) + cx_1\partial_2^2 = x_1^2\partial_1^2 + cx_1(\partial_1 + \partial_2^2)$$

belongs to $\mathcal{I}$ and $cx_1\partial_2^2$ belongs to $V_{-1}(\mathcal{D}) \cap F_2(\mathcal{D})$. Hence $s(s-1+c)$ is a regular b-function (of the minimum degree) of $\mathcal{I}$ along $x_1 = 0$ for any $c \in \mathbb{C}$, while the b-function is s.

Remark 2 Let $f = f(x)$ be a polynomial in $x = (x_1, \ldots, x_n)$. Denote by $\mathcal{D}_n$ the ring of differential operators on the variables $x_1, \ldots, x_n$ with convergent power series coefficients. Introducing a new variable t, let $\mathcal{D}_{n+1}$ be the ring of differential operators on the variables x and t with convergent power series coefficients. Let $\mathcal{I}_f$ be the left ideal of $\mathcal{D}_{n+1}$ generated by

$$t - f(x), \quad \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t} \quad (i = 1, \ldots, n).$$

Then $b(s) \in \mathbb{C}[s]$ is a regular b-function of $\mathcal{I}_f$ along the hyperplane $t = 0$ if and only if there exists $Q(s) \in \mathcal{D}_n[s]$ such that the degree of $Q(s)$ in

$\partial/\partial x_1, \ldots, \partial/\partial x_n$ and s is less than or equal to $\deg b(s)$ and

$$Q(s) f^{s+1} = b(-s-1) f^s.$$

2 Regular b-functions of A-hypergeometric systems

Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{d1} & \cdots & a_{dn} \end{pmatrix} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$$

be an integer $d \times n$ matrix with rank $A = d$, and $\beta = (\beta_1, \ldots, \beta_d)$ be a complex d-dimensional vector. The toric ideal I_A is the left ideal of $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots \partial_n]$ generated by $\{\partial^u - \partial^v \mid u, v \in \mathbb{N}^n,\ Au = Av\}$. We denote by $\langle A\theta - \beta \rangle$ the ideal of $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n]$ generated by

$$\left\{ \sum_{j=1}^{n} a_{ij}\theta_j - \beta_i \mid i = 1, \ldots, d \right\}$$

with $\theta = (\theta_1, \ldots, \theta_n) = (x_1\partial_1, \ldots, x_n\partial_n)$. Then the *$A$-hypergeometric ideal* $H_A(\beta)$ is defined to be the left ideal of the Weyl algebra

$$D = D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

which are generated by I_A and $\langle A\theta - \beta \rangle$. The left D-module $D/H_A(\beta)$ is called the *A-hypergeometric system*, which was introduced by Gelfand, Kapranov, Zelevinsky (see e.g., [8]).

The following examples were computed by using algorithms to be presented in Section 7 with Kan/sm1 [21], a computer algebra system for algebraic analysis. In the following examples, regular b-functions $b_{\mathrm{reg}}(s)$ of the minimum degree are unique up to constant multiple and coincide with the b-functions.

Example 1 Set $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$. Then $I_A = \langle \partial_1\partial_2 - \partial_3^2 \rangle$, and the regular b-functions of $H_A(\beta)$ of minimum degree along coordinate submanifolds Y are as follows:

- $Y = \{x_1 = 0\}$: The regular b-function $b_{\mathrm{reg}}(s)$ along Y is $s(2s - \beta_1 + \beta_2)$.
- $Y = \{x_3 = 0\}$: $b_{\mathrm{reg}}(s) = s(s-1)$.
- $Y = \{x_1 = x_2 = 0\}$: $b_{\mathrm{reg}}(s) = (2s + \beta_1 - \beta_2)(2s - \beta_1 + \beta_2)$.
- $Y = \{x_1 = x_3 = 0\}$: $b_{\mathrm{reg}}(s) = (2s - \beta_1)(2s - \beta_1 - 1)$.

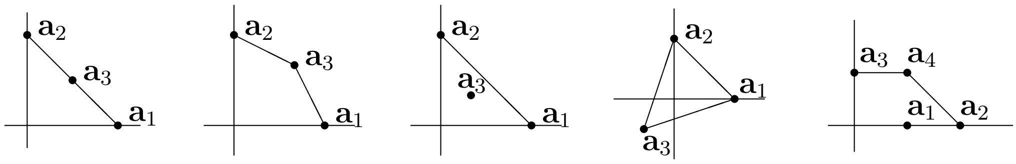

Fig. 1. Column vectors of A for Examples 1–5 (from left to right)

- $Y = \{0\}$: $b_{\mathrm{reg}}(s) = 2s - \beta_1 - \beta_2$.

Example 2 $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \end{pmatrix}$, $I_A = \langle \partial_1^2 \partial_2^2 - \partial_3^3 \rangle$.

- $Y = \{x_1 = 0\}$: $b_{\mathrm{reg}}(s) = s(s-1)(3s - \beta_1 + \beta_2)(3s - \beta_1 + \beta_2 - 3)$.
- $Y = \{x_3 = 0\}$: not regular specializable.
- $Y = \{x_1 = x_2 = 0\}$: $b_{\mathrm{reg}}(s) = (3s - \beta_1 + \beta_2)(3s - \beta_1 + \beta_2 - 6)(3s + \beta_1 - \beta_2)(3s + \beta_1 - \beta_2 - 6)$.
- $Y = \{x_1 = x_3 = 0\}$: not regular specializable.
- $Y = \{0\}$: $b_{\mathrm{reg}}(s) = (6s - 2\beta_1 - \beta_2)(6s - 2\beta_1 - \beta_2 - 3)(6s - \beta_1 - 2\beta_2)(6s - \beta_1 - 2\beta_2 - 3)$.

Example 3 $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, $I_A = \langle \partial_3^3 - \partial_1 \partial_2 \rangle$.

- $Y = \{x_1 = 0\}$: not regular specializable.
- $Y = \{x_3 = 0\}$: $b_{\mathrm{reg}}(s) = s(s-1)(s-2)$.
- $Y = \{x_1 = x_2 = 0\}$: not regular specializable.
- $Y = \{x_1 = x_3 = 0\}$: $b_{\mathrm{reg}}(s) = (3s - \beta_1)(3s - \beta_1 - 2)(3s - \beta_1 - 4)$.
- $Y = \{0\}$: $b_{\mathrm{reg}}(s) = (3s - \beta_1 - \beta_2)(3s - \beta_1 - \beta_2 - 1)(3s - \beta_1 - \beta_2 - 2)$.

Example 4 $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$, $I_A = \langle \partial_1 \partial_2 \partial_3^2 - 1 \rangle$.

- $Y = \{x_1 = 0\}$: $b_{\mathrm{reg}}(s) = s(2s - \beta_1)(2s - \beta_1 - 1)(2s - \beta_1 + \beta_2)$.
- $Y = \{x_3 = 0\}$: $b_{\mathrm{reg}}(s) = s(s-1)(s + \beta_1)(s + \beta_2)$.
- $Y = \{x_1 = x_2 = 0\}$: $b_{\mathrm{reg}}(s) = (2s - \beta_1 - \beta_2)(2s - \beta_1 - \beta_2 - 2)(2s - \beta_1 + \beta_2)(2s + \beta_1 - \beta_2)$.
- $Y = \{x_1 = x_3 = 0\}$: $b_{\mathrm{reg}}(s) = (s + \beta_1)(2s - \beta_1)(2s - \beta_1 - 3)(2s - \beta_1 + 3\beta_2)$.
- $Y = \{0\}$: $b_{\mathrm{reg}}(s) = (2s - \beta_1 - \beta_2)(2s - \beta_1 - \beta_2 - 4)(2s - \beta_1 + 3\beta_2)(2s + 3\beta_1 - \beta_2)$.

Example 5 $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $I_A = \langle \partial_1^2 - \partial_2,\ \partial_1 \partial_3 - \partial_4,\ \partial_2 \partial_3 - \partial_1 \partial_4 \rangle$.

- regular specializable along $x_1 = 0,\ x_3 = 0,$
  $x_1 = x_2 = 0,\ x_1 = x_3 = 0,\ x_1 = x_4 = 0,\ x_3 = x_4 = 0$
  $x_1 = x_2 = x_3 = 0,\ x_1 = x_3 = x_4 = 0,$
  $x_1 = x_2 = x_3 = x_4 = 0.$
- not regular specializable along $x_2 = 0,\ x_4 = 0,$
  $x_2 = x_3 = 0,\ x_2 = x_4 = 0,$
  $x_1 = x_2 = x_4 = 0,\ x_2 = x_3 = x_4 = 0.$
- $Y = \{0\}$: $b_{\mathrm{reg}}(s) = (s - \beta_2)(2s - \beta_1 - \beta_2)(2s - \beta_1 - \beta_2 - 1).$

For $Y = \{0\}$, an associated $Q \in V_{-1}(D) \cap F_3(D)$ such that

$$b_{\mathrm{reg}}(\theta_1 + \theta_2 + \theta_3 + \theta_4) - Q \in H_A(\beta)$$

is given by

$$\begin{aligned} Q = &-2x_1^2x_2\partial_2^2 + 6x_1^2x_3\partial_2\partial_3 + 4x_1x_2x_3\partial_2\partial_4 - 12x_1x_3^2\partial_3\partial_4 \\ &- 14x_1x_3x_4\partial_4^2 + 2x_2x_3^2\partial_4^2 + 2(\beta_1 - \beta_2)x_1^2\partial_2 + (14\beta_2 - 10)x_1x_3\partial_4. \end{aligned}$$

(In fact, $Q \in F_2(D)$.)

Theorem 1 *Assume* $\operatorname{rank} A = d$. *Then* $H_A(\beta)$ *is regular specializable along* $\{0\}$ *for any* $\beta \in \mathbb{C}^d$. *In particular,*

$$\mathrm{Ext}_D^k(D/H_A(\beta), \mathbb{C}\{x_1, \dots, x_n\}) = \mathrm{Ext}_D^k(D/H_A(\beta), \mathbb{C}[[x_1, \dots, x_n]]) \qquad (3)$$

holds for any integer k, *where* $\mathbb{C}[[x_1, \dots, x_n]]$ *denotes the formal power series ring.*

The proof of this theorem will be given in Section 4. Note that the dimensions of the cohomology groups of the right-hand side of (3) are computable (see Algorithm 5.4 of [17]). In particular, the cohomology groups of (3) all vanish if the b-function along the origin has no integral roots. That is the case with Examples 1–5 above for generic β. Note also that Schulze and Walther [20] described the slopes of $H_A(\beta)$ along coordinate subvarieties in terms of what they call (A, L)-umbrellas under the condition that the column vectors of A are contained in a proper convex cone with vertex at the origin.

3 Homogenization of the ring of differential operators

In order to prove Theorem 1 as well as to deduce algorithms for computing a regular b-function and an associated operator, we work in the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients $D = D_n$. The following constructions are also valid for the ring $\mathcal{D}$ of differential operators with convergent power series coefficients.

We introduce the homogenized ring $D^{(h)}$ of D with respect to the order filtration. That is, $D^{(h)}$ is a $\mathbb{C}$-algebra generated by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ and h with the commutation relations

$$\partial_i x_j = x_j \partial_i + \delta_{ij} h, \quad x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i h = h \partial_i, \quad x_i h = h x_i$$

for $1 \le i, j \le n$. Then $D^{(h)}$ is a (non-commutative) graded ring with respect to the following weights:

x_1	$\cdots$	x_n	∂_1	$\cdots$	∂_n	h
0	$\cdots$	0	1	$\cdots$	1	1

The homogeneous part of degree m of $D^{(h)}$ is the set $(D^{(h)})_m$ consisting of 0 and the homogeneous operators of $D^{(h)}$ of weight m.

The homogenization of an element $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha$ of D is defined to be

$$P^{(h)} := \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha h^{m - |\alpha|} \in D^{(h)} \quad (m := \operatorname{ord} P).$$

The homogenization $I^{(h)}$ of a left ideal I of D is the left ideal of $D^{(h)}$ generated by $\{P^{(h)} \mid P \in I\}$, which is homogeneous with respect to the above weights.

The V-filtration of $D^{(h)}$ with respect to Y is defined by

$$V_k(D^{(h)}) := \Big\{ P = \sum_{\alpha, \beta \in \mathbb{N}^n, \nu \in \mathbb{N}} a_{\alpha\beta\nu} x^\alpha \partial^\beta h^\nu \in D^{(h)} \;\mid\; a_{\alpha\beta\nu} \in \mathbb{C},$$
$$a_{\alpha\beta\nu} = 0 \text{ if } \beta_1 + \cdots + \beta_d - \alpha_1 - \cdots - \alpha_d > k \Big\}$$

For an element P of $D^{(h)}$, we define its V-order $\operatorname{ord}_V(P)$ to be the minimum integer k such that $P \in V_k(D^{(h)})$. For a left ideal I' of $D^{(h)}$, its V-graded ideal is

$$\operatorname{gr}_V(I') := \bigoplus_{k \in \mathbb{Z}} (V_k(D^{(h)}) \cap I') / (V_{k-1}(D^{(h)}) \cap I'),$$

which is a left ideal of the V-graded ring

$$\operatorname{gr}_V(D^{(h)}) := \bigoplus_{k \in \mathbb{Z}} V_k(D^{(h)}) / V_{k-1}(D^{(h)}) \simeq D^{(h)}.$$

For a nonzero element P of $D^{(h)}$ with $\operatorname{ord}_V(P) = k$, we denote by $\sigma_V(P)$ the residue class of P in $V_k(D^{(h)})/V_{k-1}(D^{(h)}) \subset \operatorname{gr}_V(D^{(h)})$. Note that $\sigma_V(P)$ can be regarded as an element of $D^{(h)}$ since $\operatorname{gr}_V(D^{(h)})$ is isomorphic to $D^{(h)}$ as graded ring with respect to the V-filtration.

4 Regular specializability of the A-hypergeometric system — proof of Theorem 1

We denote by $\{V_k(D)\}_{k\in\mathbb{Z}}$ and $\{V_k(D^{(h)})\}_{k\in\mathbb{Z}}$ the V-filtrations with respect to the origin on the Weyl algebra D and on its homogenization $D^{(h)}$ respectively. Restricted to the commutative subring $\mathbb{C}[\partial] = \mathbb{C}[\partial_1,\dots,\partial_n]$ of D, the V-filtration coincides with the order filtration, which we denote by $\{V_k(\mathbb{C}[\partial])\}_{k\in\mathbb{Z}}$. For an ideal I of $\mathbb{C}[\partial]$, we denote by

$$\mathrm{gr}_V(I) := \bigoplus_{k\geq 0}(I\cap V_k(\mathbb{C}[\partial]))/(I\cap V_{k-1}(\mathbb{C}[\partial]))$$

the graded ideal with respect to this filtration, which is an ideal of $\mathrm{gr}_V(\mathbb{C}[\partial]) \simeq \mathbb{C}[\partial]$. Its zero set $\mathbf{V}(\mathrm{gr}_V(I)) \subset \mathbb{C}^n$ is the characteristic variety of I regarded as a system of linear partial differential equations with constant coefficients.

Let $A = (\mathbf{a}_1,\dots,\mathbf{a}_n)$ be an integer $d\times n$ matrix with $\mathrm{rank}\,A = d$. We denote by Δ_A the convex hull of the set $\{0,\mathbf{a}_1,\dots,\mathbf{a}_n\}$ in $\mathbb{R}^d$, and by $\mathcal{F}_A$ the set of the facets of Δ_A which do not contain the origin.

Lemma 1 (Adolphson)

$$\mathbf{V}(\mathrm{gr}_V(I_A)) \subset \bigcup_{\gamma\in\mathcal{F}_A}\{\xi = (\xi_1,\dots,\xi_n)\in\mathbb{C}^n \mid \xi_j = 0 \ \textit{if}\ \mathbf{a}_j\notin\gamma\}.$$

Proof: This inclusion follows directly from (the proof of) Lemma 3.2 of Adolphson [1]. □

For example, if $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1\end{pmatrix}$, then the toric ideal I_A is generated by $\{\partial_1^2 - \partial_2,\ \partial_1\partial_3-\partial_4,\ \partial_2\partial_3-\partial_1\partial_4\}$, and $\mathrm{gr}_V(I_A)$ is generated by $\{\partial_1^2,\ \partial_1\partial_3,\ \partial_2\partial_3-\partial_1\partial_4\}$. Thus we have

$$\mathbf{V}(\mathrm{gr}_V(I_A)) = \{\xi_1=\xi_2=0\}\cup\{\xi_1=\xi_3=0\}.$$

Let $I_A^{(h)}$ be the homogenization of I_A in the commutative subring $\mathbb{C}[\partial,h]$ of $D^{(h)}$. We denote by $\mathrm{gr}_V(I_A^{(h)})$ the graded ideal of $I_A^{(h)}$ with respect to the V-filtration of $D^{(h)}$ restricted to $\mathbb{C}[\partial,h]$. Then from the definition of $I_A^{(h)}$, it follows that

$$\mathbf{V}(\mathrm{gr}_V(I_A^{(h)})) = \mathbf{V}(\mathrm{gr}_V(I_A))\times\mathbb{C}.$$

In general, for a homogeneous ideal I of $\mathbb{C}[\partial,h]$, we define its *distraction* to be

$$\mathrm{dist}(I) := D^{(h)}I \ \cap\ \mathbb{C}[\theta_1,\dots,\theta_n,h]$$

with $\theta_i = x_i\partial_i$, which is a homogeneous ideal of the commutative subring $\mathbb{C}[\theta, h] = \mathbb{C}[\theta_1, \ldots, \theta_n, h]$ of $D^{(h)}$. Note that this definition slightly differs from the one given in [19]. The following lemma is an immediate consequence of the definition:

Lemma 2 *Let I be an ideal of $\mathbb{C}[\partial, h]$ generated by $\{\partial^\alpha \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda\}$ with a finite subset Λ of $\mathbb{N}^n$. Then $\mathrm{dist}(I)$ is an ideal of $\mathbb{C}[\theta, h]$ generated by*

$$\left\{\prod_{i=1}^{n} \theta_i(\theta_i - h)\cdots(\theta_i - (\alpha_i - 1)h) \mid \alpha \in \Lambda\right\}.$$

The following is an immediate consequence of this lemma.

Corollary 1 *Let I_1 and I_2 be ideals of $\mathbb{C}[\theta, h]$ which are generated by monomials in ∂. Then $\mathrm{dist}(I_1 \cap I_2) = \mathrm{dist}(I_1) \cap \mathrm{dist}(I_2)$ holds.*

Now by lemma 1, we have

$$\mathbf{V}(\mathrm{gr}_V(I_A^{(h)})) \ \subset \ \bigcup_{\gamma \in \mathcal{F}_A} \{(\xi, h) \in \mathbb{C}^{n+1} \mid \xi_j = 0 \text{ if } \mathbf{a}_j \notin \gamma\}.$$

This implies

$$\sqrt{\mathrm{gr}_V(I_A^{(h)})} \ \supset \ \bigcap_{\gamma \in \mathcal{F}_A} \langle \partial_j \mid \mathbf{a}_j \notin \gamma \rangle,$$

and hence

$$\mathrm{gr}_V(I_A^{(h)}) \ \supset \ \bigcap_{\gamma \in \mathcal{F}_A} \langle \partial_j^{n_{\gamma,j}} \mid \mathbf{a}_j \notin \gamma \rangle$$

with some positive integers $n_{\gamma,j}$. It follows, in view of Lemma 2 and Corollary 1, that

$$\mathrm{dist}(\mathrm{gr}_V(I_A^{(h)})) \ \supset \ \bigcap_{\gamma \in \mathcal{F}_A} \langle \prod_{\nu=0}^{n_{\gamma,j}-1} (\theta_j - \nu h) \mid \mathbf{a}_j \notin \gamma \rangle.$$

Define an ideal J of $\mathbb{C}[\theta_1, \ldots, \theta_n, h]$ by

$$J := \mathrm{dist}(\mathrm{gr}_V(I_A^{(h)})) + \langle A\theta - \beta h \rangle,$$

where $\langle A\theta - \beta h \rangle$ denotes the ideal of $\mathbb{C}[\theta_1, \ldots, \theta_n, h]$ generated by

$$\left\{\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i h \mid i = 1, \ldots, d\right\}.$$

Then we have

$$J \ \subset \ \mathbb{C}[\theta_1, \ldots, \theta_n, h] \cap \mathrm{gr}_V(H_A(\beta)^{(h)})$$

and the set $\mathbf{V}(J)$ is contained in

$$\bigcup_{\gamma \in \mathcal{F}_A} \left\{(\theta, h) \mid \prod_{\nu=0}^{n_{\gamma,j}-1} (\theta_j - \nu h) = 0 \text{ if } \mathbf{a}_j \notin \gamma, \ \sum_{j=1}^{n} a_{ij}\theta_j = \beta_i h \ (i = 1, \ldots, d)\right\}.$$

For each γ, there exist $c_{\gamma,i} \in \mathbb{Q}$ such that

$$\sum_{i=1}^{d} c_{\gamma,i} a_{ij} = 1 \quad \text{if } \mathbf{a}_j \in \gamma$$

since γ is contained in a hyperplane of $\mathbb{R}^d$ which does not contain the origin. Hence we get an element

$$\sum_{i=1}^{d} c_{\gamma,i} \left(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i h \right) = \sum_{\mathbf{a}_j \in \gamma} \theta_j + \sum_{\mathbf{a}_j \notin \gamma} \mu_j \theta_j - \sum_{i=1}^{d} c_{\gamma,i}\beta_i h$$

of $\langle A\theta - \beta h \rangle$ with some $\mu_j \in \mathbb{C}$. This implies that $\mathbf{V}(J)$ is contained in

$$\bigcup_{\gamma \in \mathcal{F}_A} \left\{ (\theta, h) \mid \prod_{\nu=0}^{n_{\gamma,j}-1} (\theta_j - \nu h) = 0 \text{ if } \mathbf{a}_j \notin \gamma, \sum_{\mathbf{a}_j \in \gamma} \theta_j + \sum_{\mathbf{a}_j \notin \gamma} \mu_j \theta_j = \sum_{i=1}^{d} c_{\gamma,i}\beta_i h \right\}.$$

In particular, there exist $N_\gamma \in \mathbb{N}$ and $b_{\gamma,k} \in \mathbb{C}$ such that

$$\mathbf{V}(J) \subset \bigcup_{\gamma \in \mathcal{F}_A} \bigcup_{k=1}^{N_\gamma} \{(\theta, h) \mid \theta_1 + \cdots + \theta_n = b_{\gamma,k} h\}.$$

By Hilbert's Nullstellensatz,

$$b(\theta_1 + \cdots + \theta_n, h) := \prod_{\gamma \in \mathcal{F}_A} \prod_{k=1}^{N_\gamma} (\theta_1 + \cdots + \theta_n - b_{\gamma,k} h)^{m_{\gamma,k}}$$

belongs to $J \subset \mathrm{gr}_V(H_A(\beta)^{(h)})$ with some positive integers $m_{\gamma,k}$. In view of the definition of $\mathrm{gr}_V(H_A(\beta)^{(h)})$, there exists an element Q of $V_{-1}(D^{(h)})$ homogeneous of the same degree as $b(s,h)$ such that

$$b(\theta_1 + \cdots + \theta_n, h) + Q \in H_A(\beta)^{(h)}.$$

Substituting 1 for h, we conclude that $H_A(\beta)$ is regular specializable along the origin noting that $\mathrm{ord}(Q|_{h=1}) \leq \deg Q = \deg b(s,h) = \deg b(s,1)$.

5 Generalized regular b-function

Let us generalize the definition of regular b-function. Let $\mathcal{I}$ be a sheaf of left ideals of $\mathcal{D}_X$ and Y a submanifold of X which is defined by $Y = \{x = (x_1, \ldots, x_n) \mid x_1 = \cdots = x_d = 0\}$ in terms of a local coordinate x of X. Then a *generalized regular b-function* of $\mathcal{I}$ along Y at p is a monic polynomial $b(x', s) = s^m + c_1(x')s^{m-1} + \cdots + c_m(x')$ in s with coefficients $c_i(x')$ being analytic functions of $x' = (x_{d+1}, \dots, x_n)$ on a neighborhood of $p \in Y$ satisfying

$$b(x', x_1\partial_1 + \cdots + x_d\partial_d) + Q \in \mathcal{I}$$

with a germ Q of $V_{-1}(\mathcal{D}_X) \cap F_m(\mathcal{D}_X)$ at p. If there exists a generalized b-function, $\mathcal{I}$ is said to be regular specializable in a weak sense along Y at p.

This definition coincides with that of b-function if Y is a point. In case $d = 1$, $\mathcal{I}$ is regular specializable in a weak sense if and only if $\mathcal{I}$ is regular singular in a weak sense in the terminology of [12], or equivalently, it is Fuchsian along Y in the sense of [15] (or [4] for a single equation). For general d, $\mathcal{I}$ is Fuchsian along Y in the sense of [15] if it is regular specializable in a weak sense. It was proved in [15] that (2) holds if $\mathcal{I}$ is Fuchsian along Y.

6 Division algorithm and standard bases

Our algorithms for regular b-functions are based on the following division algorithm, which is a generalization of the one given in [9] (Algorithm 2.2) to general monomial orderings. In what follows we present only the scalar version for the sake of simplicity.

We make use of the second homogenization by introducing a new variable s in addition to h. For a homogeneous element $P = \sum_{\alpha,\beta,k} a_{\alpha\beta k} x^\alpha \partial^\beta h^k$ of $D^{(h)}$, we define its second homogenization to be

$$P^{(s)} := \sum_{\alpha,\beta,k} a_{\alpha\beta k} x^\alpha \partial^\beta h^k s^{m-|\alpha|+|\beta|} \ \in \ D^{(h)}[s]$$

with

$$m := \max\{|\alpha| - |\beta| \mid a_{\alpha\beta k} \neq 0 \text{ for some } k\}.$$

An element of $D^{(h)}[s]$ is said to be bihomogeneous if it is simultaneously homogeneous with respect to both weights defined by the rows of the following table:

x_1	$\cdots$	x_n	∂_1	$\cdots$	∂_n	h	s
0	$\cdots$	0	1	$\cdots$	1	1	0
1	$\cdots$	1	-1	$\cdots$	-1	0	1

Let $\prec$ be a *monomial ordering* for $D^{(h)}$, i.e., a total ordering on the set of monomials $\{x^\alpha \xi^\beta h^k \mid \alpha, \beta \in \mathbb{N}^n,\ k \in \mathbb{N}\}$ which is invariant under the multiplication by the same monomial on both sides, with the additional condition

$$h \prec x_i\xi_i \quad (i = 1, \dots, n). \tag{4}$$

For an element $P = \sum_{\alpha,\beta,k} a_{\alpha\beta k} x^\alpha \partial^\beta h^k$ of $D^{(h)}$, its *leading monomial* with respect to $\prec$ is

$$\mathrm{LM}_\prec(P) := \max{}_\prec \{x^\alpha \xi^\beta h^k \mid a_{\alpha\beta k} \neq 0\},$$

which is a monomial in the commutative polynomial ring $\mathbb{C}[x, \xi, h]$ with $\xi = (\xi_1, \dots, \xi_n)$ being the commutative variables corresponding to ∂. We call the element

$$P(x, \xi, h) = \sum_{\alpha,\beta,k} a_{\alpha\beta k} x^\alpha \xi^\beta h^k$$

of $\mathbb{C}[x, \xi, h]$ the *total symbol* of P.

A monomial ordering $\prec_s$ for $D^{(h)}[s]$ is defined so that $x^\alpha \xi^\beta h^i s^\mu \prec_s x^{\alpha'} \xi^{\beta'} h^j s^\nu$ if and only if

$$\begin{aligned}
&|\alpha| - |\beta| + \mu < |\alpha'| - |\beta'| + \nu \\
&\text{or } (|\alpha| - |\beta| + \mu = |\alpha'| - |\beta'| + \nu \text{ and } x^\alpha \xi^\beta h^i \prec x^{\alpha'} \xi^{\beta'} h^j).
\end{aligned}$$

Let P, Q be bihomogeneous elements of $D^{(h)}[s]$. If $\mathrm{LM}_{\prec_s}(P)$ devides $\mathrm{LM}_{\prec_s}(Q)$, then we set

$$\mathrm{Red}(P, Q) = (R, U) \quad \text{with } R := P - UQ,$$

where U is an element of $D^{(h)}[s]$ whose total symbol is $\mathrm{LM}_{\prec_s}(Q)/\mathrm{LM}_{\prec_s}(P)$.

Algorithm 1 (Division algorithm in $D^{(h)}$)

Input: homogeneous elements $P, P_1, \dots, P_m$ of $D^{(h)}$, a monomial ordering $\prec$ for $D^{(h)}$.

Output: homogeneous $Q_1, \dots, Q_m \in D^{(h)}$ and $a \in \mathbb{C}[x]$ such that

(1) $(1 + a)P = Q_1 P_1 + \cdots + Q_m P_m + R$,

(2) $\mathrm{LM}_\prec(a) \prec 1$ if $a \neq 0$,

(3) If $R \neq 0$, then $\mathrm{LM}_\prec(R)$ is not divisible by any $\mathrm{LM}_\prec(P_i)$,

(4) $\mathrm{LM}_\prec(Q_i P_i) \preceq \mathrm{LM}_\prec(P)$ if $Q_i \neq 0$.

$\mathcal{G} := (P_1^{(s)}, \dots, P_m^{(s)})$ (a list), $R := P^{(s)}$, $A := 1$
$Q = (Q_1, \dots, Q_m) := (0, \dots, 0) \in (D^{(h)})^m$
IF $R \neq 0$ THEN
    $\mathcal{F} := \{P' \in \mathcal{G} \mid \mathrm{LM}_{\prec_s}(P')$ divides $\mathrm{LM}_{\prec_s}(s^\ell R)$ for some $\ell \in \mathbb{N}\}$
ELSE $\mathcal{F} := \emptyset$ (an empty set)
$\mathcal{H} := (\,)$ (an empty list)
WHILE $(\mathcal{F} \neq \emptyset)$ DO
    Choose $P' \in \mathcal{F}$ with ℓ minimal, which is the i-th element of $\mathcal{G}$
    IF $\ell > 0$ THEN
        $\mathcal{G} := \mathcal{G} \cup (R)$ (append R to $\mathcal{G}$ as the last element)
        $\mathcal{H} := \mathcal{H} \cup ((A, Q))$ (append a list (A, Q) to $\mathcal{H}$ as the last element)
    $(R, U) := \mathrm{Red}(s^\ell R, P')$

IF $i \le m$ THEN $Q_i := Q_i + U$
IF $i > m$ THEN
$(A', Q') := \mathcal{H}[i-m]$ (the $(i-m)$-th element of $\mathcal{H}$)
$A := A - UA'$
FOR $j = 1, \dots, m$ DO $Q_j := Q_j - UQ'_j$
IF $R \neq 0$ THEN
$\nu :=$ the highest power of s dividing R
$R := R/s^\nu$
$\mathcal{F} := \{P' \in \mathcal{G} \mid \text{LM}_{\prec_s}(P') \text{ divides } \text{LM}_{\prec_s}(s^\ell R) \text{ for some } \ell \in \mathbb{N}\}$
ELSE $\mathcal{F} := \emptyset$
FOR $j = 1, \dots, m$ DO $Q_j := Q_j|_{s=1}$
$R := R|_{s=1}$, $a := A|_{s=1}$

The correctness of this algorithm can be proved in a way similar to [9]. See also [10, Chapter 2] for the commutative case.

By using this division in the Buchberger algorithm, we can compute a Gröbner (or a standard) base of a given homogeneous left ideal of $D_\prec^{(h)}$ with respect to an arbitrary monomial ordering $\prec$ for $D^{(h)}$. See [9] for details. Here $D_\prec^{(h)}$ is the localization with respect to the multiplicative subset

$$S_\prec := \{1\} \cup \{1 + a(x) \mid a(x) \in \mathbb{C}[x],\ a(x) \neq 0,\ \text{LM}_\prec(a(x)) \prec 1\}$$

of $D^{(h)}$. An element P of $D_\prec^{(h)}$ is expressed in a finite sum

$$P = \sum_{\alpha,k} \frac{a_{\alpha k}(x)}{b_{\alpha k}(x)} \partial^\alpha h^k \qquad (a_{\alpha k}(x) \in \mathbb{C}[x],\, b_{\alpha k}(x) \in S_\prec).$$

In fact, all computations can be done in $D^{(h)}$ not in $D_\prec^{(h)}$. For this purpose, let us introduce the following definition:

Definition 2 Let I be a left homogeneous ideal of $D^{(h)}$ and $\prec$ be a monomial ordering for $D^{(h)}$. Then a finite set $G \subset I$ consisting of homogeneous elements is called a *standard base* of I with respect to $\prec$ if the ideal $\langle \text{LM}_\prec(I) \rangle$ of $\mathbb{C}[x, \xi, h]$, which is generated by the leading monomials with respect to $\prec$ of the elements of I, coincides with the ideal $\langle \text{LM}_\prec(G) \rangle$ which is generated by the leading monomials of the elements of G. If, in addition, G generates I, then G is called a *Gröbner base* of I.

Lemma 3 *Let $G = \{P_1, \dots, P_r\}$ be a standard base of a homogeneous left ideal I of $D^{(h)}$ with respect to a monomial ordering $\prec$ for $D^{(h)}$. Then for any P, there exist homogeneous $Q_1, \dots, Q_r \in D^{(h)}$ and $a \in \mathbb{C}[x]$ such that*

$$(1 + a)P = Q_1 P_1 + \cdots + Q_r P_r,$$

$\mathrm{LM}_\prec(a) \prec 1$ *if* $a \neq 0$, $\mathrm{LM}_\prec(Q_i P_i) \preceq \mathrm{LM}_\prec(P)$ *if* $Q_i \neq 0$.

Proof: Applying Alogorithm 1, we get an expression

$$(1+a)P = Q_1 P_1 + \cdots + Q_r P_r + R$$

with the conditions (2),(3),(4) of Alogorithm 1. If $R \neq 0$, then $\mathrm{LM}_\prec(R)$ does not belong to $\langle \mathrm{LM}_\prec(G) \rangle$, which contradicts the fact that $R \in I$ and G is a standard base of I. □

Given a finite set of generators of a homogeneous left ideal I of $D^{(h)}$, we can compute a standard base of I with respect to an arbitrary monomial ordering $\prec$ by the Buchberger algorithm with the usual division replaced by Algorithm 1. Then by the preceding lemma, G generates $D_\prec^{(h)} I$ in $D_\prec^{(h)}$.

7 Algorithms for (generalized) regular b-functions

Let us describe the whole algorithm in several steps. The inputs are a finite set of generators of a left ideal I of the Weyl algebra D, and a linear submanifold $Y = \{x_1 = \cdots = x_d = 0\}$. The outputs are a (generalized) regular b-function of $\mathcal{I} := \mathcal{D}I$ along Y of the minimum degree, and an associated operator $Q \in \mathcal{D}$ satisfying (1), where $\mathcal{D}$ is the stalk of $\mathcal{D}_X$ at 0.

Step 1. Computation of generators of the homogenized ideal $I^{(h)}$. Let $\prec$ be a monomial ordering for $D^{(h)}$ such that

$$x^\alpha \xi^\beta h^i \prec x^{\alpha'} \xi^{\beta'} h^j \quad \text{if} \quad |\beta| < |\beta'|$$

and that 1 is the minimum monomial. Let $\{P_1, \ldots, P_r\}$ be a set of generators of a given left ideal I of D. Let $\{P_1', \ldots, P_k'\}$ be a Gröbner base with respect to $\prec$ of the ideal of $D^{(h)}$ generated by $\{P_1^{(h)}, \ldots, P_r^{(h)}\}$. Let ν_i be the maximum nonnegative integer such that h^{ν_i} divides P_i' and set $P_i'' := P_i'/h^{\nu_i}$. Then $G_1 := \{P_1'', \ldots, P_k''\}$ is a set of generators of the homogenized ideal $I^{(h)}$ of I.

In fact, let P be an arbitrary nonzero element of I. Then it is easy to see that there exists a nonnegative integer ν such that $h^\nu P^{(h)}$ belongs to the ideal generated by G_1. Hence by division we have

$$h^\nu P^{(h)} = Q_1 P_1'' + \cdots + Q_k P_k''$$

with homogeneous $Q_i \in D^{(h)}$ such that $\mathrm{LM}_\prec(Q_i P_i'') \preceq \mathrm{LM}_\prec(h^\nu P^{(h)})$ if $Q_i \neq 0$. This implies that h^ν divides each Q_i in view of the definition of $\prec$. Hence $P^{(h)}$ belongs to the ideal generated by G_1. Since $P_i''|_{h=1}$ belongs to I, it is also easy to see that G_1 is a subset of $I^{(h)}$.

Step 2. Computation of generators of $\mathrm{gr}_V(I^{(h)})$. Let $\prec$ be a monomial ordering for $D^{(h)}$ compatible with the V-filtration, i.e., satisfying

$$x^\alpha\xi^\beta h^i \prec x^{\alpha'}\xi^{\beta'}h^j \quad \text{if} \quad \beta_1+\cdots+\beta_d-\alpha_1-\cdots-\alpha_d < \beta'_1+\cdots+\beta'_d-\alpha'_1-\cdots-\alpha'_d,$$

and $1 \prec x_i$ for $d+1 \le i \le n$. Let $\{P_1,\ldots,P_r\}$ be a standard base of $I^{(h)}$ with respect to $\prec$. Then

$$G_2 := \{\mathrm{gr}_V(P_1),\ldots,\mathrm{gr}_V(P_r)\}$$

is a set of generators of $\mathrm{gr}_V(I^{(h)})$.

In fact, let P_0 be a homogeneous nonzero element of $\mathrm{gr}_V(I^{(h)})$ with $\mathrm{ord}_V(P_0) = m$. Then there exists $Q \in V_{m-1}(D^{(h)})$ such that $P_0 + Q \in I^{(h)}$. By Lemma 3, there exist homogeneous $Q_1,\ldots,Q_r \in D^{(h)}$ and a polynomial $a(x) \in \mathbb{C}[x]$ such that

$$(1+a(x))(P_0+Q) = Q_1P_1 + \cdots + Q_rP_r,$$

$\mathrm{LM}_\prec(Q_iP_i) \preceq \mathrm{LM}_\prec(P_0)$ if $Q_i \neq 0$, and $\mathrm{LM}_\prec(a(x)) \prec 1$ if $a(x) \neq 0$. The last condition implies that $a(x) \in V_{-1}(D^{(h)})$. In fact, if $a(x) \notin \langle x_1,\ldots,x_d\rangle$, we would have $\mathrm{LM}_\prec(a(x)) \succ 1$. From

$$P_0 = Q_1P_1 + \cdots + Q_rP_r - a(x)P_0 - (1+a(x))Q$$

it follows that

$$\sigma_V(P_0) = Q'_1\sigma_V(P_1) + \cdots + Q'_r\sigma_V(P_r),$$

where $Q'_i := \sigma_V(Q_i)$ if $\mathrm{ord}_V(Q_iP_i) = m$, and $Q'_i := 0$ otherwise. Hence G_2 is a set of generators of $\mathrm{gr}_V(I^{(h)})$. We may assume that each element of G_2 is bihomogeneous, i.e., homogeneous with respect to the graded structure of $\mathrm{gr}_V(D^{(h)})$ as well as to the one coming from $D^{(h)}$.

Step 3. Computation of generators of the ideal

$$J := \mathrm{gr}_V(I^{(h)}) \cap \mathbb{C}[x_{d+1},\ldots,x_n,\theta_1+\cdots+\theta_d,h]$$

with $\theta_i = x_i\theta_i$. Introducing new commutative variables u_i, v_i with $i = 1,\ldots,d$, we work in the ring $D^{(h)}[u_1,\ldots,u_d,v_1,\ldots,v_d]$. For an element

$$P = \sum_{\alpha,\beta,k} a_{\alpha\beta k}x^\alpha\partial^\beta h^k$$

of $D^{(h)}$, we define its multi-homogenization to be

$$\mathrm{mh}(P) = \sum_{\alpha,\beta,k} a_{\alpha\beta k}x^\alpha\partial^\beta h^k u_1^{\kappa_1-\alpha_1+\beta_1}\cdots u_d^{\kappa_d-\alpha_d+\beta_d} \;\in D^{(h)}[u_1,\ldots,u_d,v_1,\ldots,v_d]$$

with $\kappa_i := \max\{\alpha_i - \beta_i \mid a_{\alpha\beta k} \neq 0 \text{ for some } k\}$.

Let $\{P_1,\ldots,P_r\}$ be a set of bihomogeneous generators of $\mathrm{gr}_V(I^{(h)})$. (Here we identify $\mathrm{gr}_V(D^{(h)})$ with $D^{(h)}$.) Let $\tilde{I}$ be the left ideal generated by

$$\{\mathrm{mh}(P_1),\ldots,\mathrm{mh}(P_r)\} \cup \{u_iv_i - 1 \mid i = 1,\ldots,d\}$$

in $D^{(h)}[u_1, \dots, u_d, v_1, \dots, v_d]$, and compute a Gröbner base $\tilde{G}$ of $\tilde{I}$ with respect to a monomial ordering for eliminating u_i's and v_i's. Set $\tilde{G}_0 := \tilde{G} \cap D^{(h)}$. For each element P of $\tilde{G}_0$, there exist unique $\mu = (\mu_1, \dots, \mu_d)$ and $\nu = (\nu_1, \dots, \nu_d)$ in $\mathbb{N}^d$ with $\mu_i \nu_i = 0$ $(i = 1, \dots, d)$ so that there exists a $P' \in \mathbb{C}[x_{d+1}, \dots, x_n, \theta_1, \dots, \theta_d, h]$ satisfying

$$x_1^{\mu_1} \cdots x_d^{\mu_d} \partial_1^{\nu_1} \cdots \partial_d^{\nu_d} P = P'(x_{d+1}, \dots, x_n, \theta_1, \dots, \theta_d, h).$$

Let us denote this P' by $\psi(P)$. Then one can prove that the set $\psi(\tilde{G}_0) := \{\psi(P) \mid P \in \tilde{G}_0\}$ generates

$$\tilde{J} := \mathrm{gr}_V(I^{(h)}) \cap \mathbb{C}[x_{d+1}, \dots, x_n, \theta_1, \dots, \theta_d, h]$$

in the same way as the proof of Proposition 4.3 of [17].

Next compute a set G_3 of generators of the ideal

$$J := \tilde{J} \cap \mathbb{C}[x_{d+1}, \dots, x_n, s, h]$$

with $s = \theta_1 + \cdots + \theta_d$. This can be done by computing the intersection

$$\Big(\tilde{J} + \langle s - \theta_1 - \cdots - \theta_d \rangle\Big) \cap \mathbb{C}[x_{d+1}, \dots, x_n, s, h]$$

through a Gröbner base.

Step 4A. Computation of a generalized regular b-function of the minimum degree. Let J be as in Step 3. We denote $x' = (x_{d+1}, \dots, x_n)$.

(1) Set $J|_{h=0} := \{f|_{h=0} \mid f \in J\}$, which is an ideal of $\mathbb{C}[x', s]$. Compute a set G of generators of
$$\mathbb{C}[x'] \cap ((J|_{h=0}) : s^\infty)$$
by a Gröbner base (see e.g., [7]).

(2) If there exists $a(x') \in G$ such that $a(0) \neq 0$, then find the minimum integer $m \geq 0$ so that
$$\mathbb{C}[x'] \cap ((J|_{h=0}) : s^m)$$
contains an element $a(x')$ with $a(0) \neq 0$. If there is no $a(x') \in G$ such that $a(0) \neq 0$, then quit (there is no generalized b-function). In view of the homogeneity of J, m gives us the minimum degree in s of generalized b-functions.

(3) Let $\prec$ be a monomial ordering for $\mathbb{C}[x', s, h]$ such that
$$x_{d+1}^{\alpha_{d+1}} \cdots x_n^{\alpha_n} s^\mu h^i \prec x_{d+1}^{\beta_{d+1}} \cdots x_n^{\beta_n} s^\nu h^j \quad \text{if } \mu < \nu$$
and $x_i \prec 1$ for $i = d+1, \dots, n$. Let $\{f_1, \dots, f_k\}$ be a standard base of J with respect to $\prec$ consisting of homogeneous elements. By applying

Algorithm 1 to the commutative subring $\mathbb{C}[x', s, h]$ of $D^{(h)}$, we can find $a(x') \in \mathbb{C}[x']$ and $q_1, \dots, q_k, r \in \mathbb{C}[x', s, h]$ such that

$$(1 + a(x'))s^m = q_1 f_1 + \cdots + q_k f_k + r,$$

$a(x') \prec 1$ if $a(x') \neq 0$, $\mathrm{LM}_\prec(r)$ is not divisible by any $\mathrm{LM}_\prec(f_i)$ with $\mathrm{LM}_\prec(r) \preceq s^m$ if $r \neq 0$. In fact, we have $\mathrm{LM}_\prec(r) \prec s^m$ since s^m belongs to the monomial ideal generated by $\{\mathrm{LM}_\prec(f) \mid f \in J\}$ in view of the definition of m. Hence r can be written in a form

$$r = c_0(x')s^m + c_1(x')s^{m-1}h + \cdots + c_m(x')h^m$$

with $c_j(x') \in \mathbb{C}[x']$ and $c_0(0) = 0$. This implies that

$$b'(x', s, h) := (1 + a(x') - c_0(x'))s^m - c_1(x')s^{m-1}h - \cdots - c_m(x')h^m$$

belongs to J. Thus $b(x', s) := (1 + a(x') - c_0(x'))^{-1} b'(x', s, 1)$ is a (monic) generalized b-function of $\mathcal{I}$ of the minimum degree.

Step 4B. Computation of a regular b-function of the minimum degree. Compute a primary decomposition of J:

$$J = Q_1 \cap \cdots \cap Q_l.$$

Set

$$K := \{k \in \{1, \dots, l\} \mid a(0) = 0 \text{ for any } a(x') \in Q_k \cap \mathbb{C}[x']\}.$$

Compute a Gröbener base G_4 of of the intersection

$$B := \bigcap_{k \in K} (Q_k \cap \mathbb{C}[s, h])$$

with respect to a monomial ordering such that 1 is the minimum monomial. Choose, if any, an element $b'(s, h)$ of G_4 of the minimum degree such that $b'(s, 0) \neq 0$. Then $b'(s, 1)$ is a regular b-function of $\mathcal{I}$ of the minimum degree. If there is no such $b'(s, h)$, then $\mathcal{I}$ is not regular specializable along Y at 0.

In fact, we can prove that

$$B = \{b(s, h) \in \mathbb{C}[s, h] \mid \exists a(x') \in \mathbb{C}[x'] \,:\, a(0) \neq 0,\ a(x')b(s, h) \in J\}$$

in the same way as the proof of Lemma 4.4 of [17].

Step 5. Computation of an associated operator Q. Let $b(x', s)$ be a (generalized) regular b-function of $\mathcal{I}$ computed in Step 4A or 4B. Let $\prec$ be a monomial ordering for $D^{(h)}$ which is compatible with the V-filtration and satisfies $x_i \prec 1$ for $i = d + 1, \dots, n$. Let $G = \{P_1, \dots, P_r\}$ be a standard base of $I^{(h)}$ w.r.t. $\prec$. Take a homogeneous polynomial $b'(x', s, h)$ such that $b'(x', s, 1) = b(x', s)$

and $b'(x', s, 0) \neq 0$. Dividing $b'(x', \theta_1 + \cdots + \theta_d, h)$ by $P_1, \dots, P_r$, we get an expression

$$(1 + a)b'(x', \theta_1 + \cdots + \theta_d, h) = Q_1 P_1 + \cdots + Q_r P_r + R$$

with the conditions (2),(3),(4) in Algorithm 1. Then R belongs to $(D^{(h)})_m \cap V_{-1}(D^{(h)})$ with m being the degree of $b(x', s)$ in s, and

$$b'(x', \theta_1 + \cdots + \theta_d) - (1 + a)^{-1} R|_{h=1} \ \in \ \mathcal{I}.$$

In fact, it is easy to see that $\sigma_V(G) = \{\sigma_V(P_1), \dots, \sigma_V(P_r)\}$ is a standard base of $\mathrm{gr}_V(I^{(h)})$ in view of the definition of $\prec$. In particular, we have $\langle \mathrm{LM}_\prec(G) \rangle = \langle \mathrm{LM}_\prec(\sigma_V(G)) \rangle$. Note that $\mathrm{ord}_V(R) \leq 0$. Since there exists a $c(x) \in \mathbb{C}[x]$ such that $c(0) = 0$ and

$$(1 + c(x))b'(x', \theta_1 + \cdots + \theta_d, h) \ \in \ \mathrm{gr}_V(I^{(h)}),$$

it follows that $\sigma_V((1 + c)R)$ would also belong to $\mathrm{gr}_V(I^{(h)})$ if $\mathrm{ord}_V(R) = 0$. Hence $\mathrm{LM}_\prec(\sigma_V(R)) = \mathrm{LM}_\prec(R)$ would belong to $\langle \mathrm{LM}_\prec(\sigma_V(G)) \rangle = \langle \mathrm{LM}_\prec(G) \rangle$. This contradicts the property (3) of Algorithm 1. Thus we have $\mathrm{ord}_V(R) \leq -1$.

This completes the description of the algorithms. The proof of the correctness of the above algorithms will be completed in the next section.

8 Analytic versus algebraic regular b-functions

We denote by $\mathcal{D}$ the stalk of $\mathcal{D}_X$ at the origin, i.e., the ring of differential operators with convergent power series coefficients. As was introduced in [3], the homogenized ring $\mathcal{D}^{(h)}$ of $\mathcal{D}$ is defined to be the set of operators P expressed in a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n, k \geq 0} a_{\alpha k}(x) \partial^\alpha h^k \quad (a_{\alpha k}(x) \in \mathbb{C}\{x\})$$

with the commutation relations

$$\partial_i a = a \partial_i + \frac{\partial a}{\partial x_i} h, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i h = h \partial_i, \quad ah = ha$$

for $a \in \mathbb{C}\{x\}$ and $1 \leq i, j \leq n$. This is a graded ring with respect to the total degree in $\partial_1, \dots, \partial_n, h$. Then $D^{(h)}$ is a graded subring of $\mathcal{D}^{(h)}$.

For an operator $P = \sum_\alpha a_\alpha(x) \partial^\alpha$ of $\mathcal{D}$, its homogenization is defined to be

$$P^{(h)} := \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha h^{m - |\alpha|} \ \in \mathcal{D}^{(h)}$$

with $m := \operatorname{ord} P$. The homogenization $\mathcal{I}^{(h)}$ of a left ideal $\mathcal{I}$ of $\mathcal{D}$ is the left ideal of $\mathcal{D}^{(h)}$ generated by the homogenizations of the elements of $\mathcal{I}$.

Lemma 4 *Let I be a left ideal of D and $\mathcal{I} = \mathcal{D}I$ be the left ideal of $\mathcal{D}$ generated by I. Then $\mathcal{I}^{(h)}$ is generated by $I^{(h)}$ over $\mathcal{D}^{(h)}$.*

Proof: Let $\prec$ be a monomial ordering for D such that

$$x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'} \quad \text{if} \quad |\beta| < |\beta'|$$

and $x_i \prec 1$ for $i = 1, \ldots, n$. Let $G = \{P_1, \ldots, P_r\}$ be a standard base of I with respect to $\prec$. Applying the Buchberger criterion to $\mathcal{D}$, we see that G is also a standard base of $\mathcal{I}$. Hence by using the division in $\mathcal{D}$ (see [5]), or by the flatness of $\mathcal{D}$ over D, we see that for any $P \in \mathcal{I}$, there exist $Q_1, \ldots, Q_r \in \mathcal{D}$ such that

$$P = Q_1 P_1 + \cdots + Q_r P_r$$

and $\operatorname{ord}(Q_i P_i) \le \operatorname{ord} P$. Then by homogenization we get

$$P^{(h)} = h^{\nu_1} Q_1^{(h)} P_1^{(h)} + \cdots + h^{\nu_r} Q_r^{(h)} P_r^{(h)}$$

with $\nu_i = \operatorname{ord} P - \operatorname{ord}(Q_i P_i)$. Hence $\mathcal{I}^{(h)}$ is generated by $P_1^{(h)}, \ldots, P_r^{(h)} \in I^{(h)}$. $\square$

Lemma 5 *Let I be a left ideal of D and set $\mathcal{I} = \mathcal{D}I$. Then $\operatorname{gr}_V(\mathcal{I}^{(h)})$ is generated by $\operatorname{gr}_V(I^{(h)})$ over $\operatorname{gr}_V(\mathcal{D}^{(h)})$.*

Proof: Let $\prec$ be a monomial ordering for $D^{(h)}$ adapted to the V-filtration such that $x_i \prec 1$ for $i = d+1, \ldots, n$. Let $G = \{P_1, \ldots, P_r\}$ be a standard base of $I^{(h)}$ with respect to $\prec$. Then G is also a standard base of $\mathcal{I}^{(h)} = \mathcal{D}^{(h)} I^{(h)}$ with respect to $\prec$ (see Theorem 3.2 of [9]). Let P be an arbitrary element of $\mathcal{I}^{(h)}$. Then by the division algorithm of Assi-Castro-Granger [3] for $\mathcal{D}^{(h)}$, there exist $Q_1, \ldots, Q_r \in \mathcal{D}^{(h)}$ such that

$$P = Q_1 P_1 + \cdots + Q_r P_r$$

with $\operatorname{ord}_V(Q_i P_i) \le \operatorname{ord}_V(P)$. Hence $\sigma_V(P)$ belongs to the left ideal of $\mathcal{D}^{(h)}$ generated by $\{\sigma_V(P_1), \ldots, \sigma_V(P_r)\}$. This completes the proof. $\square$

Lemma 6 *Let I be a left ideal of D and set $\mathcal{I} = \mathcal{D}I$. Then*

$$\tilde{J}^{\mathrm{an}} := \operatorname{gr}_V(\mathcal{I}^{(h)}) \cap \mathbb{C}\{x'\}[\theta_1, \ldots, \theta_d, h]$$

is generated by

$$\tilde{J} := \operatorname{gr}_V(I^{(h)}) \cap \mathbb{C}[x', \theta_1, \ldots, \theta_d, h]$$

over $\mathbb{C}\{x'\}[\theta_1, \ldots, \theta_d, h]$.

Proof: Let $\prec$ be a monomial ordering for $D^{(h)}[u,v]$ such that

$$x^\alpha \partial^\beta h^j u^\mu v^\nu \prec x^{\alpha'} \partial^{\beta'} h^k u^{\mu'} v^{\nu'} \quad \text{if } |\mu| + |\nu| < |\mu'| + |\nu'|$$

and $x_i \prec 1$ for $i = 1, \dots, n$, where we denote $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$. Let $\tilde{G} = \{P_1, \dots, P_r\}$ be a standard base of the left ideal $\tilde{I}$ of $D^{(h)}[u,v]$ defined in Step 3. (Note that $\tilde{G}$ is not necessarily the same as in Step 3.) Then $\tilde{G}$ is also a standard base of $\mathrm{gr}_V(\mathcal{D}^{(h)})[u,v]\tilde{I}$ with respect to $\prec$. We can show that $\tilde{J}^{\mathrm{an}}$ is generated by $\psi(\tilde{G}_0)$ with

$$\tilde{G}_0 := \tilde{G} \cap \mathrm{gr}_V(\mathcal{D}^{(h)}) = \tilde{G} \cap D^{(h)}$$

in the same way as the proof of Proposition 4.3 of [17]. By substituting 1 for each u_i and v_i, we see that $\psi(\tilde{G}_0)$ is contained in $\mathrm{gr}_V(I^{(h)})$. This completes the proof. □

Now we are ready to prove the following theorem, which implies the correctness of the algorithms given in the preceding section.

Theorem 2 *Let I be a left ideal of D and set $\mathcal{I} = \mathcal{D}I$.*

(1) If $b(x', s) \in \mathbb{C}\{x'\}[s]$ is a generalized regular b-function of $\mathcal{I}$ along Y, then there exists a generalized regular b-function $\tilde{b}(x', s)$ along Y which belongs to $\mathbb{C}[x', s]$ and is of the same degree (in s) as $b(x', s)$.

(2) A polynomial $b(x', s) \in \mathbb{C}[x', s]$ is a generalized regular b-function of $\mathcal{I}$ along Y if and only if there exist a homogeneous $b'(x', s, h) \in \mathbb{C}[x', s, h]$ with $b'(x', s, 0) \neq 0$ and $b'(x', s, 1) = b(x', s)$, and a polynomial $c(x') \in \mathbb{C}[x']$ with $c(0) \neq 0$ such that

$$c(x')b'(x', x_1\partial_1 + \cdots + x_d\partial_d, h) \ \in \ \mathrm{gr}_V(I^{(h)}).$$

Proof: Let J be as in Step 3 of the preceding section and set

$$J^{\mathrm{an}} := \mathbb{C}\{x'\}[s,h] \cap \mathrm{gr}_V(\mathcal{I}^{(h)})$$

with $s = \theta_1 + \cdots + \theta_d$. Then by Lemmas 4,5,6, it is easy to see that J^{an} is generated by J over $\mathbb{C}\{x'\}[s,h]$. Let $\prec$ be the same monomial ordering for $\mathbb{C}[x', s, h]$ as in (3) of Step 4A and $G = \{f_1, \dots, f_k\}$ be a standard base of J with respect to $\prec$. Then G is also a standard base of J^{an} with respect to $\prec$. Set m be as in (2) of Step 4A. It follows that m is also the minimum integer such that $\mathbb{C}\{x'\} \cap ((J^{\mathrm{an}}|_{h=0}) : s^m)$ contains a $c(x') \in \mathbb{C}\{x'\}$ with $c(0) \neq 0$, or equivalently, that s^m is contained in $\langle \mathrm{LM}_\prec(G) \rangle$. This completes the proof of (1).

Now in order to prove (2), let $b(x', s) \in \mathbb{C}[x', s]$ be a (generalized) regular b-function of $\mathcal{I}$ of degree m in s. Then there eixsts $Q \in V_{-1}(\mathcal{D}) \cap F_m(\mathcal{D})$ such that

$$b(x', \theta_1 + \cdots + \theta_d) + Q \in \mathcal{I}.$$

By homogenization, there eixst a homogeneous $b'(x', s, h) \in \mathbb{C}[x', s, h]$ with $b'(x', s, 1) = b(x', s)$ and $b(x', s, 0) \neq 0$, and a $\nu \in \mathbb{N}$ such that

$$b'(x', \theta_1 + \cdots + \theta_d, h) + h^\nu Q^{(h)} \in \mathcal{I}^{(h)}.$$

This implies that

$$b'(x', \theta_1 + \cdots + \theta_d, h) \in \mathrm{gr}_V(\mathcal{I}^{(h)})$$

and hence $b'(x', s, h)$ belongs to J^{an}. By division in $\mathbb{C}[x', s, h]$, there exist $q_1, \dots, q_k, r \in \mathbb{C}[x', s, h]$ and $a(x') \in \mathbb{C}[x']$ such that $a(0) = 0$,

$$(1 + a(x'))b'(x', s, h) = q_1 f_1 + \cdots + q_k f_k + r,$$

$\mathrm{LM}_\prec(q_i f_i) \preceq s^m$ if $q_i \neq 0$, and $\mathrm{LM}_\prec(r)$ is not divisible by $\mathrm{LM}_\prec(f_i)$ for any $i = 1, \dots, k$ if $r \neq 0$. Since r belongs to J^{an} and G is a standard base of J^{an}, it follows that $r = 0$. Thus $(1 + a(x'))b'(x', s, h)$ belongs to $\mathrm{gr}_V(I^{(h)})$. The converse implication of the statement (2) is obvious. □

References

[1] A. Adolphson, Hypergeometric functions and rings generated by monomials, Duke Math. J. 73 (1994) 269–290.

[2] A. Assi, F.J. Castro-Jiménez, M. Granger, How to compute the slopes of a $\mathcal{D}$-module, Compositio Math. 104 (1996) 107–123.

[3] A. Assi, F.J. Castro-Jiménez, M. Granger, The analytic standard fan of a $\mathcal{D}$-module, J. Pure Appl. Algebra 164 (2001) 3–21.

[4] M.S. Baouendi, C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. 26 (1973) 455–475.

[5] F. Castro, Calcul effectifs pour les idéaux d'opérateurs différentiels, in: J.-M. Aroca, T. Sanchez-Giralda, J.-L. Vicente (Eds.), Géométrie algébrique et applications, III, Hermann, Paris, 1987, pp. 53–98.

[6] F.J. Castro-Jiménez, N. Takayama, Singularities of the hypergeometric system associated with a monomial curve, Trans. Amer. Math. Soc. 355 (2003) 3761–3775.

[7] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms, Springer-Verlag, New York, 1991.

[8] I.M. Gel'fand, A.V. Zelevinsky, M.M. Kapranov, Hypergeometric functions and toric varieties, Funct. Anal. Appl. 23 (1989) 94–106.

[9] M. Granger, T. Oaku, N. Takayama, Tangent cone algorithm for homogenized differential operators, J. Symbolic Comput. 39 (2005) 417–431.

[10] G.M. Greuel, G. Pfister, A Singular Introduction to Commutative Algebra, Springer, Berlin Heidelberg, 2002.

[11] M. Kashiwara, T. Kawai, On holonomic systems of micro-differential operators III, Publ. Res. Inst. Math. Sci. 17 (1981) 813–979.

[12] M. Kashiwara, T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math. 106 (1977) 145–200.

[13] Y. Laurent, Y., Polygône de Newton et b-fonctions pour les modules microdifférentiels, Ann. Sci. École Norm. Sup. 20 (1987) 391–441.

[14] Y. Laurent, Z. Mebkhout, Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module, Ann. Sci. École Norm. Sup. 32 (1999) 39–69.

[15] Y. Laurent, T. Monteiro Fernandes, Systèmes différentiels fuchsiens le long d'une sous-variété, Publ. Res. Inst. Math. Sci. 24 (1988) 397–431.

[16] T. Oaku, An algorithm of computing b-functions, Duke Math. J. 87 (1997) 115–132.

[17] T. Oaku, N. Takayama, Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups, J. Pure Appl. Algebra 156 (2001) 267–308.

[18] C. Sabbah, D-modules et cycles évanescents, in: J.-M. Aroca, T. Sanchez-Giralda, J.-L. Vicente (Eds.), Géométrie algébrique et applications, III, Hermann, Paris, 1987, pp. 53–98.

[19] M. Saito, B. Sturmfels, N. Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Springer, Berlin Heidelberg, 2000.

[20] M. Schulze, U. Walther, Irregularity of hypergeometric systems via slopes along coordinate subspaces. Preprint (2006–2008), arXiv:math/0608668v3

[21] N. Takayama, Kan/sm1, `http://www.math.kobe-u.ac.jp/KAN/`.