Corrections to ‘Algorithms for D-modules, integration, and generalized functions with applications to statistics’

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- Proof of Lemma 2.6 (p. 265):

First, suppose $w_i \geq 0$ for all i. Then for any positive integer k, $F_k^w(D_n)$ is generated over $F_0^w(D_n)$ by the finite set

$$\{x^\alpha \partial^\beta \mid \langle w, (\alpha, \beta)\rangle \leq k,\ \ \alpha_i = 0 \text{ if } w_i = 0,\ \beta_i = 0 \text{ if } w_{n+i} = 0\}.$$

Now suppose $|w_i| \leq 1$ and $w_i + w_{n+i} = 0$ for $i = 1, \dots, n$. We may assume $w_i \geq 0$ for $1 \leq i \leq n$ by exchanging x_i and ∂_i if necessary. Each element of D_n is expressed as a linear combination of a finite set of ‘monomials’ of the form $x^\alpha \partial^\beta$. Assume

$$\mathrm{ord}_w(x^\alpha \partial^\beta) = \langle (w_1, \dots, w_n), \alpha\rangle - \langle (w_1, \dots, w_n), \beta\rangle > 0.$$

Then there exists $\gamma \in \mathbb{N}^n$ such that $\alpha - \gamma \in \mathbb{N}^n$ and

$$\langle (w_1, \dots, w_n), \alpha - \gamma\rangle = \langle (w_1, \dots, w_n), \beta\rangle$$

since w_i is 0 or 1 for $i = 1, \dots, n$. Hence $x^\alpha \partial^\beta$ belongs to $x^\gamma F_0^w(D_n)$ with $\langle (w_1, \dots, w_n), \gamma\rangle = \mathrm{ord}_w(x^\alpha \partial^\beta)$. Thus $F_k^w(D_n)$ is generated by the finite set

$$\{x^\gamma \mid \langle (w_1, \dots, w_n), \gamma\rangle \leq k,\ \ \gamma_i = 0 \text{ if } w_i = 0\}$$

over $F_0^w(D_n)$ if $k > 0$.

On the other hand, if

$$\mathrm{ord}_w(\partial^\beta x^\alpha) = \langle (w_1, \dots, w_n), \alpha\rangle - \langle (w_1, \dots, w_n), \beta\rangle \leq k < 0,$$

there exists $\gamma \in \mathbb{N}^n$ such that $\beta - \gamma \in \mathbb{N}^n$ and

$$\langle (w_1, \dots, w_n), \beta - \gamma\rangle = \langle (w_1, \dots, w_n), \alpha\rangle + k - \mathrm{ord}_w(\partial^\beta x^\alpha) \geq 0.$$

Hence $\partial^\beta x^\alpha$ belongs to $\partial^\gamma F_0^w(D_n)$ with $\langle (w_1, \dots, w_n), \gamma\rangle = -k$. Thus $F_k^w(D_n)$ is generated by a finite set

$$\{\partial^\gamma \mid \langle (w_1, \dots, w_n), \gamma\rangle = -k,\ \ \gamma_i = 0 \text{ if } w_{n+i} = 0\}$$

over $F_0^w(D_n)$ if $k<0$ since D_n is spanned by $\partial^\beta x^\alpha$.

• Proof of Lemma 2.7 (p. 266):

Let us show $F_{k+1}^w(D_n) = F_1^w(D_n)F_k^w(D_n)$ for $k \geq 0$. We show that $x^\alpha\partial^\beta \in F_{k+1}^w(D_n)$ belongs to $F_1^w(D_n)F_k^w(D_n)$ by induction on $|\alpha|+|\beta|$. It is trivial for $\alpha=\beta=0$.

There exists $i \in \{1,\ldots,n\}$ such that $w_i = 1$ and $\alpha_i > 0$, or else $w_{n+i} = 1$ and $\beta_i > 0$. If the former condition holds, then one has

$$x^\alpha\partial^\beta = x_i x^{\alpha-e_i}\partial^\beta$$

and $x^{\alpha-e_i}\partial^\beta$ belongs to $F_k^w(D_n)$, where e_i is the i-th unit vector in $\mathbb{Z}^n$. If the latter condition holds, then one has

$$x^\alpha\partial^\beta = \partial_i x^\alpha\partial^{\beta-e_i} - \alpha_i x^{\alpha-e_i}\partial^{\beta-e_i}$$

and $x^\alpha\partial^{\beta-e_i}$ belongs to $F_k^w(D_n)$ and $x^{\alpha-e_i}\partial^{\beta-e_i} \in F_{k+1}^w(D_n)$ belongs to $F_1^w(D_n)F_k^w(D_n)$ by the induction hypothesis.

Next let us show $F_{-k-1}^w(D_n) = F_{-1}^w(D_n)F_{-k}^w(D_n)$ for $k \geq 0$. We show that $x^\alpha\partial^\beta \in F_{-k-1}^w(D_n)$ belongs to $F_{-1}^w(D_n)F_{-k}^w(D_n)$ by induction on $|\alpha|+|\beta|$. It is trivial for $\alpha=\beta=0$ since $1 \notin F_{-k-1}^w(D_n)$. Assume $\langle w,(\alpha,\beta)\rangle \leq -k-1$. Then there exists $i \in \{1,\ldots,n\}$ such that $w_i = -1$ and $\alpha_i > 0$, or else $w_{n+i} = -1$ and $\beta_i > 0$. If the former condition holds, then one has

$$x^\alpha\partial^\beta = x_i x^{\alpha-e_i}\partial^\beta$$

and $x^{\alpha-e_i}\partial^\beta$ belongs to $F_{-k}^w(D_n)$. If the latter condition holds, then one has

$$x^\alpha\partial^\beta = \partial_i x^\alpha\partial^{\beta-e_i} - \alpha_i x^{\alpha-e_i}\partial^{\beta-e_i}.$$

Here $x^\alpha\partial^{\beta-e_i}$ belongs to $F_{-k}^w(D_n)$, and $x^{\alpha-e_i}\partial^{\beta-e_i}$ belongs to $F_{-k-1}^w(D_n)$ since $w_i = 1$, hence to $F_{-1}^w(D_n)F_{-k}^w(D_n)$ by the induction hypothesis.