Algebraic and algorithmic study of some generalized functions associated with a real polynomial (or a real analytic function)

Toshinori Oaku

Tokyo Woman's Christian University

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Distributions (generalized functions)

Definition

Let $C_0^\infty(U)$ be the set of the C^∞ functions on an open set U of $\mathbb{R}^n$ with compact support. A distribution u on U is a linear mapping

$$u : C_0^\infty(U) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

such that $\lim_{j\to\infty}\langle u, \varphi_j\rangle = 0$ holds for a sequence $\{\varphi_j\}$ of $C_0^\infty(U)$ if there is a compact set $K \subset U$ such that $\varphi_j = 0$ on $U \setminus K$ and

$$\lim_{j\to\infty} \sup_{x\in U} |\partial^\alpha \varphi_j(x)| = 0 \quad \text{for any } \alpha \in \mathbb{N}^n,$$

where $x = (x_1, \dots, x_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$. The set of the distributions on U is denoted by $\mathcal{D}'(U)$.

Differential operators

Let $\mathcal{D}_X$ be the sheaf of linear differential operators (of finite order) with holomorphic coefficients on $X := \mathbb{C}^n$, and $\mathcal{D}_M := \mathcal{D}_X|_M$ be its sheaf-theoretic restriction to $M := \mathbb{R}^n$. These are coherent sheaves of rings on X and on M respectively. A section P of $\mathcal{D}_M$ on an open set $U \subset M$ is written in a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial^\alpha \quad (a_\alpha \in \mathcal{A}_M(U)),$$

where $\mathcal{A}_M := \mathcal{O}_X|_M$ denotes the sheaf of real analytic functions on M.

The derivative $\partial_k u$ of a distribution u on U with respect to x_k is defined by

$$\langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

For a C^∞ function a on U, the product au is defined by

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

In particular, by these actions of the derivations and the polynomial multiplications, the sheaf $\mathcal{D}'$ of distributions has a natural structure of left $\mathcal{D}_M$-module.

Example: Dirac's delta function $\delta(x)$ is the distribution defined by

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0) \quad (\forall \varphi \in C_0^\infty(\mathbb{R})).$$

$\delta(x)$ satisfies a holonomic system $x\delta(x) = 0$.

Power product of real analytic functions as distribution

Let $f_1, \ldots, f_p$ be real-valued real analytic functions defined on an open set $U \subset M$. We assume that the set $\{x \in U \mid f_i(x) > 0 \, (i = 1, \ldots, p)\}$ is not empty. Then the distribution $v = (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$ on U is defined to be

$$\langle v, \varphi \rangle = \int_{U_+} f_1(x)^{\lambda_1} \cdots f_p(x)^{\lambda_p} \varphi(x) \, dx$$

with $U_+ = \{x \in U \mid f_j(x) \leq 0 \quad (1 \leq j \leq p)\}$ for $\varphi \in C_0^\infty(U)$ if $\mathrm{Re}\ \lambda_i \geq 0$ for each i.

Moreover, v, that is, $\langle v, \varphi \rangle$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, is holomorphic in $(\lambda_1, \dots, \lambda_p)$ on the domain

$$\Omega_+ := \{(\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p \mid \mathrm{Re}\ \lambda_i > 0 \quad (i = 1, \dots, p)\}$$

and is continuous in $(\lambda_1, \dots, \lambda_p)$ on the closure of Ω_+.
In particular,

$$(f_1)_+^0 \cdots (f_p)_+^0 = Y(f_1) \cdots Y(f_p),$$

where $Y(t)$ is the Heaviside function; i.e., $Y(t) = 1$ for $t > 0$ and $Y(t) = 0$ for $t \leq 0$.

Functional equations

Theorem (Kashiwara)

Let f be a holomorphic function defined on an open neighborhood of $x_0 \in X$. Then there exist a germ $P(s)$ of $\mathcal{D}_X[s]$ at x_0, and $b_{f,x_0}(s) \in \mathbb{C}[s]$ such that

$$P(s)f^{s+1} = b_{f,x_0}(s)f^s$$

holds formally and $b_{f,x_0}(s) \neq 0$ is of minimum degree (the **Bernstein-Sato polynomial**, or the b-**function** of f at x_0).

Then $P(\lambda)f_+^{\lambda+1} = b_{f,x_0}(\lambda)f_+^{\lambda}$ holds on a neighborhood of x_0 in M.

Theorem (Kashiwara)

The roots of $b_f(s)$ are negative rational numbers.

Laurent coefficients of f_+^λ

Let f be a real-valued real analytic funciton on an open set $U \subset M$. Then by using the functional equation $b(\lambda)f_+^\lambda = P(\lambda)f_+^{\lambda+1}$, the distribution f_+^λ is extended to a $\mathcal{D}'(U)$-valued meromorphic function on $\mathbb{C}$. Let $\lambda = \lambda_0$ be a (possible) pole of f_+^λ. Then f_+^λ can be expressed as a Laurent series

$$f_+^\lambda = \sum_{k=-l}^{\infty} (\lambda - \lambda_0)^j u_k$$

with $u_k \in \mathcal{D}'(U)$ and $l \in \mathbb{N}$. In particular, u_{-1} is called the *residue* of f_+^λ at λ_0, which we denote by $\mathrm{Res}_{\lambda=\lambda_0} f_+^\lambda$.

Non-singular case

• If $f = 0$ is non-singular, then f_+^λ has only simple poles at negative integers with

$$\mathrm{Res}_{\lambda=-k-1}\, f_+^\lambda = \frac{(-1)^k}{k!}\delta^{(k)}(f) \quad (k = 0, 1, 2, \dots).$$

$\delta(f)$ represents the layer (the Dirac delta function) concentrated on the hypersurface $f = 0$,
$\delta^{(1)}(f) = \delta'(f)$ represents the double layer (dipole),...
Cf. Gelfand-Shilov: 'Generalized Functions, Vol. 1'

Singular case

Definition

For a non-negative integer k, set

$$\delta_+^{(k)}(f) := (-1)^k k! \operatorname{Res}_{\lambda=-k-1} f_+^\lambda,$$
$$\delta_-^{(k)}(f) := k! \operatorname{Res}_{\lambda=-k-1} f_-^\lambda = k! \operatorname{Res}_{\lambda=-k-1} (-f)_+^\lambda = (-1)^k \delta_+^{(k)}(-f).$$

Then we have

Proposition

(1) $f^{k+1}\delta_\pm^{(k)}(f) = 0 \quad (k \geq 0)$.

(2) $\dfrac{\partial}{\partial x_i} Y(\pm f) = \dfrac{\partial f}{\partial x_i}\delta_\pm(f)$ for $i = 1, \ldots, n$.

(3) $f\delta_\pm^{(k)}(f) = -k\delta_\pm^{(k-1)}(f) \quad (k \geq 1)$.

Theorem (well-known?)

Each Laurent coefficient u_k satisfies a holonomic left $\mathcal{D}_M$-module.

Problems:

- Determine the annihilator $\mathrm{Ann}_{\mathcal{D}_M} u_k = \{P \in \mathcal{D}_M \mid P u_k = 0\}$.
- Is it a coherent left ideal of $\mathcal{D}_M$?
- If so, what is its characteristic cycle?

Remark: Set $X = \mathbb{C}$ and $M = \mathbb{R}$. Then

$$\mathrm{Ann}_{(\mathcal{D}_M)_{x_0}} Y(x) = \begin{cases} (\mathcal{D}_M)_{x_0} \partial_x & \text{if } x_0 > 0 \\ (\mathcal{D}_M)_{x_0} x \partial_x & \text{if } x_0 = 0 \\ (\mathcal{D}_M)_{x_0} & \text{if } x_0 < 0 \end{cases}$$

Hence $\mathrm{Ann}_{\mathcal{D}_M} Y(x)$ is not coherent as sheaf of left ideals of $\mathcal{D}_M$.

Normal crossing case

Let $f_1, \ldots, f_m$ be (real-valued) real anlytic functions defined on a neighborhood of $x_0 \in M$ such that $df_1 \wedge \cdots \wedge df_m \neq 0$ at x_0. Let

$$(f_1 \cdots f_m)_+^\lambda = (\lambda + 1)^{-m} u_{-m} + (\lambda + 1)^{-m+1} u_{-m+k} + \cdots + (\lambda + 1)^{-1} u_1 + u_0 + (\lambda + 1) u_1 + \cdots$$

be the Laurent expansion about $\lambda = -1$. Let $v_1, \ldots, v_n$ be real analytic vector fields defined on a neighborhood of x_0 which are linearly independent at x_0 and satisfy

$$v_i(f_j) = \begin{cases} 1 & (\text{if } i = j \leq m) \\ 0 & (\text{otherwise}) \end{cases}$$

Theorem

For $k = 0, 1, \ldots, m-1$, *the annihilator* $\mathrm{Ann}_{(\mathcal{D}_X)_{x_0}} u_{-m+k} = \{P \in (\mathcal{D}_X)_{x_0} \mid Pu = 0\}$ *is generated by*

$$f_{j_1} \cdots f_{j_{k+1}} \quad (1 \le j_1 < \cdots < j_{k+1} \le m),$$
$$f_1 v_1 - f_i v_i \quad (2 \le i \le m), \quad v_j \quad (m+1 \le j \le n).$$

Corollary

The sheaf $\mathrm{Ann}_{\mathcal{D}_M} u_{-n+k}$ *of left ideals of* $\mathcal{D}_M$ *is coherent on a neighborhood of* $x_0 \in M$ *for each* $k = 0, 1, \ldots, n-1$.

The theorem above follows from the special case below:

Theorem

Let

$$(x_1 \cdots x_n)_+^\lambda = (\lambda+1)^{-n} u_{-n} + (\lambda+1)^{-n+1} u_{-n+1} + \cdots + (\lambda+1)^{-1} u_{-1} + u_0 + (\lambda+1) u_1 + \cdots$$

be the Laurent expansion of the distribution $(x_1 \cdots x_n)_+^\lambda$ *with respect to the holomorphic parameter* λ *about* $\lambda = -1$*. Then for* $k = 0, 1, \ldots, n-1$*, the annihilator of* u_{-n+k}

$$\mathrm{Ann}_{(\mathcal{D}_M)_0} u_{-n+k} = \{P \in (\mathcal{D}_M)_0 \mid P u_{-n+k} = 0\}$$

is generated by

$$x_{j_1} \cdots x_{j_{k+1}} \quad (1 \le j_1 < \cdots < j_{k+1} \le n), \quad x_1 \partial_1 - x_i \partial_i \quad (2 \le i \le n).$$

Proof

We set $\mathcal{D}_0 := (\mathcal{D}_X)_0$. In one variable t, we have

$$
\begin{aligned}
t_+^\lambda &= (\lambda+1)^{-1}\partial_t t_+^{\lambda+1} \\
&= (\lambda+1)^{-1}\partial_t \left\{ Y(t) + (\lambda+1)\log t_+ + \frac{1}{2}(\lambda+1)^2(\log t_+)^2 + \cdots \right\} \\
&= (\lambda+1)^{-1}\delta(t) + \partial_t \log t_+ + \frac{1}{2}(\lambda+1)\partial_t(\log t_+)^2 + \cdots ,
\end{aligned}
$$

where $(\log t_+)^m$ is the distribution defined by the pairing

$$
\langle (\log t_+)^m, \varphi\rangle = \int_0^\infty (\log t)^m \varphi(t)\,dt
$$

for $\varphi \in C_0^\infty(\mathbb{R})$ and $m = 1, 2, 3, \ldots$.

Let us introduce the following notations:

- For a nonnegative integer j, we set

$$h_j(t) = \begin{cases} \delta(t) & (j = 0), \\ \frac{1}{j!}\partial_t(\log t_+)^j & (j \geq 1) \end{cases}$$

  with $\partial_t = \partial/\partial_t$ and

$$h_\alpha(x) = h_{\alpha_1}(x_1)\cdots h_{\alpha_n}(x_n)$$

  for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad [\alpha] = \max\{\alpha_i \mid 1 \leq i \leq n\}.$$

- $S(n) = \{\sigma = (\sigma_1, \ldots, \sigma_n) \in \{1, -1\}^n \mid \sigma_1 \cdots \sigma_n = 1\}.$

Since

$$(x_1 \cdots x_n)_+^\lambda = \sum_{\sigma \in S(n)} (\sigma_1 x_1)_+^\lambda \cdots (\sigma_n x_n)_+^\lambda,$$

we have

$$u_{-n+k}(x) = \sum_{\sigma \in S(n)} \sum_{|\alpha|=k} h_\alpha(\sigma x),$$

and in particular,

$$u_{-n}(x) = \sum_{\sigma \in S(n)} \delta(\sigma_1 x_1) \cdots \delta(\sigma_n x_n) = 2^{n-1} \delta(x_1) \cdots \delta(x_n).$$

It follows that $\mathrm{Ann}_{\mathcal{D}} u_{-n}$ is generated by $x_1, \ldots, x_n$. This proves the assertion for $k = 0$ since $x_1 \partial_1 - x_i \partial_i = \partial_1 x_1 - \partial_i x_i$ belongs to the left ideal of $\mathcal{D}_0$ generated by $x_1, \ldots, x_n$.

We shall prove the assertion by induction on k. Assume $k \geq 1$ and $P \in \mathcal{D}_0$ annihilates u_{-n+k}, that is, $Pu_{-n+k} = 0$. By division, there exist $Q_1, \ldots, Q_r, R \in \mathcal{D}_0$ such that

$$P = Q_1 \partial_1 x_1 + \cdots + Q_n \partial_n x_n + R,$$
$$R = \sum_{\alpha_1 \beta_1 = \cdots = \alpha_n \beta_n = 0} a_{\alpha,\beta} x^\alpha \partial^\beta \qquad (a_{\alpha,\beta} \in \mathbb{C}).$$

Since

$$u_{-n+k}(x) = \sum_{\sigma \in S(n)} \sum_{|\alpha|=k,\, [\alpha]=1} h_\alpha(\sigma x) + \sum_{\sigma \in S(n)} \sum_{|\alpha|=k,\, [\alpha] \geq 2} h_\alpha(\sigma x), \quad (1)$$

we have

$$u_{-n+k}(x) = 2^{n-k-1}\delta(x_1)\cdots\delta(x_{n-k})h_1(x_{n-k+1})\cdots h_1(x_n)$$
$$= 2^{n-k-1}\delta(x_1)\cdots\delta(x_{n-k})\frac{1}{x_{n-k+1}}\cdots\frac{1}{x_n}$$

on the domain $x_{n-k+1} > 0, \ldots, x_n > 0$. Hence

$$0 = Pu_{-n+k} = Ru_{-n+k}$$
$$= \sum_{\alpha_1=\cdots=\alpha_{n-k}=0,\alpha_{n-k+1}\beta_{n-k+1}=\cdots\alpha_n\beta_n=0} (-1)^{\beta_{n-k+1}+\cdots+\beta_n}$$
$$\times \beta_{n-k+1}!\cdots\beta_n! a_{\alpha,\beta}$$
$$\times \delta^{(\beta_1)}(x_1)\cdots\delta^{(\beta_{n-k})}(x_{n-k})x_{n-k+1}^{\alpha_{n-k+1}-\beta_{n-k+1}-1}\cdots x_n^{\alpha_n-\beta_n-1}$$

holds there.

This implies $a_{\alpha,\beta} = 0$ if $\alpha_1 = \cdots = \alpha_{n-k} = 0$. In the same way, we know that $a_{\alpha,\beta} = 0$ if the components of α are zero except at most k components. This implies that R is contained in the left ideal generated by $x_{j_1} \cdots x_{j_{k+1}}$ with $1 \leq j_1 < \cdots < j_{k+1} \leq n$.
In the right-hand-side of (1), each term contains the product of at least $n-k$ delta functions. Hence $x_{j_1} \cdots x_{j_{k+1}}$ with $1 \leq j_1 < \cdots < j_{k+1} \leq n$, and consequently R also, annihilates $u_{-n+k}(x)$. Hence we have

$$0 = Pu_{-n+k} = \sum_{i=1}^{n} Q_i \partial_i x_i u_{-n+k}.$$

On the other hand, since

$$\partial_i x_i (x_1 \cdots x_n)_+^\lambda = (x_i \partial_i + 1)(x_1 \cdots x_n)_+^\lambda = (\lambda + 1)(x_1 \cdots x_n)_+^\lambda,$$

we have

$$\partial_i x_i u_{-k} = u_{-k-1} \qquad (k \leq n-1,\ 1 \leq i \leq n)$$

and consequently

$$0 = \sum_{i=1}^{n} Q_i \partial_i x_i u_{-n+k} = \sum_{i=1}^{n} Q_i u_{-n+k-1}.$$

By the induction hypothesis, $\sum_{i=1}^{n} Q_i$ belongs to the left ideal of $\mathcal{D}_0$ generated by

$$x_{j_1} \cdots x_{j_k} \quad (1 \leq j_1 < \cdots < j_k \leq n), \quad x_1\partial_1 - x_i\partial_i \quad (2 \leq i \leq n).$$

Then we have

$$P = \sum_{i=1}^{n} Q_i \partial_1 x_1 + \sum_{i=2}^{n} Q_i(\partial_i x_i - \partial_1 x_1) + R.$$

If $j_1 > 1$, we have

$$x_{j_1} \cdots x_{j_k} \partial_1 x_1 = \partial_1 x_1 x_{j_1} \cdots x_{j_k}.$$

If $j_1 = 1$, let l be an integer with $2 \leq l \leq n$ such that $l \neq j_2, \dots, l \neq j_k$. Then we have

$$\begin{aligned} x_{j_1} \cdots x_{j_k} \partial_1 x_1 &= x_{j_2} \cdots x_{j_k} x_1 \partial_1 x_1 \\ &= x_{j_2} \cdots x_{j_k} x_1 (\partial_1 x_1 - \partial_l x_l) + \partial_l x_{j_2} \cdots x_{j_k} x_1 x_l. \end{aligned}$$

We conclude that P belongs to the left ideal generated by

$$x_{j_1} \cdots x_{j_{k+1}} \quad (1 \leq j_1 < \cdots < j_{k+1} \leq n), \quad x_1\partial_1 - x_i\partial_i \quad (2 \leq i \leq n).$$

Conversely it is easy to see that these generators annihilate u_{-n+k} since

$$x_1\partial_1(x_1 \cdots x_n)_+^\lambda = x_i\partial_i(x_1 \cdots x_n)_+^\lambda = \lambda(x_1 \cdots x_n)_+^\lambda.$$

The characteristic cycle

For a subset J of $\{1, \ldots, n\}$, set

$$X_J := \{x = (x_1, \ldots, x_n) \in X = \mathbb{C}^n \mid x_j = 0 \text{ for any } j \in J\}$$

and let $T^*_{X_J}X$ be its conormal bundle.

Theorem

Under the same assumptions as the theorem above, the characteristic cycle of $\mathcal{D}_M u_{-n+k} = \mathcal{D}_M / \mathrm{Ann}_{\mathcal{D}_M} u_{-n+k}$ is

$$\sum_{|J| \geq n-k} (k + 1 - n + |J|) T^*_{X_J} X$$

on a neighborhood of $M \times_X T^*_{X_J} X$.

Comparison with local cohomology

Let $f(x)$ be holomorphic on an open set $\tilde{U}$ of $X = \mathbb{C}^n$. The (algebraic) local cohomology group supported by $f = 0$ is defined to be the sheaf

$$\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X,$$

which consists of residue classes $[af^{-k}]$ modulo $\mathcal{O}_X$ with an analytic function a and a non-negative integer k.
Set $U = \tilde{U} \cap M$. We define an $\mathcal{A}_M$-homomorphism

$$\rho \ : \ \mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_U \ni [af^{-k}] \longmapsto \mathrm{Res}_{\lambda=0}\, af_+^{\lambda-k} \in \mathcal{D}'_M|_U$$

for $a(x) \in \mathcal{A}_M$ and $k \in \mathbb{N}$. Note that $\mathrm{supp}\,\rho(u) \subset \{f = 0\}$.

Theorem

Assume

(A) *For any negative integer* $-k$, $\lambda = -k$ *is at most a simple pole of* f_+^λ.

Then ρ *is a homomorphism of sheaves of left* $\mathcal{D}_M|_U$*-modules. In particular,*

$$\mathrm{Ann}_{\mathcal{D}_M} u \subset \mathrm{Ann}_{\mathcal{D}_M} \rho(u)$$

holds for any $u \in \mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_U$, *where* $\mathcal{D}_M := \mathcal{D}_X|_M$.

Corollary

Assume

(A′) $\tilde{b}_{f,y_0}(-k)$ *does not vanish for any negative integer* $-k$ *and for any point* y_0 *of* U *such that* $f(y_0) = 0$.

Then ρ *is a homomorphism of sheaves of left* $\mathcal{D}_M$*-modules. In particular,*

$$\mathrm{Ann}_{\mathcal{D}_M} u \subset \mathrm{Ann}_{\mathcal{D}_M} \rho(u)$$

holds for any $u \in \mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_U$.

Now let us introduce the following condition:

Condition (B):
Let $f(x)$ be real analytic on a neighborhood of $x_0 \in M$. By a real analytic local coordinate transformation, $f(x)$ can be written in the form

$$f(x) = c(x)(x_1^m + a_1(x')x_1^{m-1} + \cdots + a_m(x'))$$

with $m \geq 1$ and real-valued real analytic functions $c(x)$ and $a_j(x')$ with $x' = (x_2, \ldots, x_n)$ which are defined on a neighborhood of $x_0 = (0, x_0')$ such that $c(x_0) \neq 0$ and $a_j(x_0') = 0$ for $1 \leq j \leq m$. Moreover, for any neighborhood V of x_0 in M there exists $y_0' \in \mathbb{R}^{n-1}$ such that $(0, y_0') \in V$ and the equation

$$x_1^m + a_1(y_0')x_1^{m-1} + \cdots + a_m(y_0') = 0$$

in x_1 has m distinct real roots.

Theorem

Assume (B). Then

$$\mathrm{Ann}_{(\mathcal{D}_X)_{x_0}} \rho(u) \subset \mathrm{Ann}_{(\mathcal{D}_X)_{x_0}} u$$

holds for any germ $u \in \mathcal{H}^1_{[f=0]}(\mathcal{O}_X)_{x_0}$.

Corollary

- *(B)* $\Rightarrow$ ρ *is an injective* $\mathcal{A}_M$*-homomorphism.*
- *(A) and (B)* $\Rightarrow$ ρ *is an injective* $\mathcal{D}_M$*-homomorphism.*

Examples

Example 1
Let $f_1, \ldots, f_m$ ($m \geq 2$) be real analytic functions such that $df_1 \wedge \cdots \wedge df_m \neq 0$ at $x_0 \in M$. Then $f = f_1 \cdots f_m$ satisfies (B) (but not (A)). In fact $\mathrm{Ann}_{\mathcal{D}_{x_0}} \mathrm{Res}_{\lambda=-1} f_+^\lambda$ is generated by

$$f, \quad f_1 v_1 - f_i v_i \quad (2 \leq i \leq m), \quad v_j \quad (m+1 \leq j \leq n),$$

while $\mathrm{Ann}_{\mathcal{D}_{x_0}}[1/f]$ is generated by

$$f, \quad v_i f_i = f_i v_i + 1 \quad (1 \leq i \leq m), \quad v_j \quad (m+1 \leq j \leq n).$$

In particular, $\mathrm{Ann}_{\mathcal{D}_M} \rho([1/f]) \subsetneqq \mathrm{Ann}_{\mathcal{D}_M}[1/f]$.

Example 2

$f = x_1^2 x_2^2 + x_3^p$ with $n = 3$ and an odd integer $p \geq 3$ satisfies (A) and (B). In fact, the reduced b-function $b_{f,0}(s)/(s+1)$ of f at the origin does not have integral roots (T. Yano).

By a coordinate transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$, $y_3 = x_3$, f takes the form

$$f = (y_1^2 - y_2^2)^2 + y_3^p = y_1^4 - 2y_1^2 y_2^2 + y_2^4 + y_3^p.$$

Hence the equation $f = 0$ in y_1 has four distinct real roots if and only if $y_3 < 0$ and $y_2^4 + y_3^p > 0$.

Hence we have $\mathrm{Ann}_{\mathcal{D}_M} u = \mathrm{Ann}_{\mathcal{D}_M} \rho(u)$ for any section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_M$.

For example, if $p = 3$, the characteristic cycle of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{D}_X[f^{-1}]$ is given by

$$2\,T^*_{\{x_1=x_2=x_3=0\}}\mathbb{C}^3 + T^*_{\{x_1=x_3=0\}\setminus\{0\}}\mathbb{C}^3 + T^*_{\{x_2=x_3=0\}\setminus\{0\}}\mathbb{C}^3 + T^*_{Y'}\mathbb{C}^3$$

with
$Y' := \{(x_1, x_2, x_3) \mid x_1^2 x_2^2 + x_3^3 = 0\} \setminus \{(x_1, x_2, x_3) \mid x_1 x_2 = x_3 = 0\}$.

Example 3

$f = x_1(x_2^2 + x_3^2 + x_4^2)$ with $n = 4$ and $u := [f^{-1}]$. Then f^s satisfies a functional equation

$$\frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)f^{s+1} = (s+1)^2\left(s + \frac{3}{2}\right)f^s.$$

Let

$$f_+^\lambda = (\lambda+1)^{-2}v_{-2}(x) + (\lambda+1)^{-1}v_{-1}(x) + v_0(x) + \cdots$$

be the Laurent expansion around $\lambda = -1$. Then we have

$$
\begin{aligned}
v_{-2}(x) &= \frac{1}{2}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)Y(x_1) = 0, \\
v_{-1}(x) &= \frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)\left\{\lim_{\lambda\to -1}\frac{\partial}{\partial\lambda}\left(\left(\lambda + \frac{3}{2}\right)^{-1} f_+^{\lambda+1}\right)\right\} \\
&= \frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)\{-4Y(x_1) \\
&+ 2Y(x_1)(\log x_1 + \log(x_2^2 + x_3^2 + x_4^2)\} \\
&= \delta(x_1)(x_2^2 + x_3^2 + x_4^2)^{-1}.
\end{aligned}
$$

Thus $\lambda = -k$ is a simple pole of f_+^λ for any positive integer k. Hence (A) is satisfied with $U = M = \mathbb{R}^4$.
$\mathrm{Ann}_{\mathcal{D}_X} u$ is generated by

$$x_1(x_2^2 + x_3^2 + x_4^2), \quad x_1\partial_1 + 1, \quad x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + 2,$$
$$x_2\partial_3 - x_3\partial_2, \quad x_2\partial_4 - x_4\partial_2, \quad x_3\partial_4 - x_4\partial_3.$$

$\mathrm{Ann}_{\mathcal{D}_M} \rho(u)$ is generated by

$$x_1, \quad x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + 2, \quad x_2\partial_3 - x_3\partial_2,$$
$$x_2\partial_4 - x_4\partial_2, \quad x_3\partial_4 - x_4\partial_3.$$

Hence $\mathrm{Ann}_{\mathcal{D}_M} u \subsetneqq \mathrm{Ann}_{\mathcal{D}_M} \rho(u)$.

The characteristic cycle of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{D}_X u$ is

$$T^*_{\{0\}}\mathbb{C}^4 + T^*_{\{x_2=x_3=x_4=0\}\setminus\{0\}}\mathbb{C}^4 + T^*_{\{x_1=x_2^2+x_3^2+x_4^2=0\}\setminus\{0\}}\mathbb{C}^4 + T^*_{\{x_1=0,x_2^2+x_3^2+x_4^2\neq 0\}}\mathbb{C}^4 + T^*_{\{x_2^2+x_3^2+x_4^2=0,x_1\neq 0,(x_2,x_3,x_4)\neq(0,0,0)\}}\mathbb{C}^4,$$

while that of $\mathcal{D}_M\rho(u)$ is

$$T^*_{\{0\}}\mathbb{C}^4 + T^*_{\{x_1=x_2^2+x_3^2+x_4^2=0\}\setminus\{0\}}\mathbb{C}^4 + T^*_{\{x_1=0,x_2^2+x_3^2+x_4^2\neq 0\}}\mathbb{C}^4.$$

Normal forms satisfying (B) at 0

Among the normal forms of real hypersurface singularities in $M = \mathbb{R}^n$, at least the following ones satisfy the condition (B) at the origin, where $q(x_k, \dots, x_n)$ denotes a non-degenerate quadratic form in the variables $x_k, \dots, x_n$ and a is a real constant:

- $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 \quad (1 \le p \le n-1)$,
- $D_4^- : \; x_1^2 x_2 - x_2^3 + q(x_3, \dots, x_n)$,
- $E_7 : \; x_1^3 + x_1 x_2^3 + q(x_3, \dots, x_n)$,
- $P_8^\pm : \; x_1^3 + a x_1^2 x_3 \pm x_1 x_3^2 + x_2^2 x_3 + q(x_4, \dots, x_n)$ with $-a^2 \pm 4 < 0$,
- $J_{10}^\pm : \; x_1^3 + a x_1^2 x_2^2 \pm x_1 x_2^4 + q(x_3, \dots, x_n)$ with $-a^2 \pm 4 < 0$,
- $J_{10+k}^\pm : \; x_1^3 \pm x_1^2 x_2^2 + a x_2^{6+k} + q(x_3, \dots, x_n)$ with $k \ge 1$ and ($\pm a < 0$ or k: odd),

- $P^{\pm}_{8+k}$: $x_1^3 \pm x_1^2 x_3 + x_2^2 x_3 + a x_3^{k+3} + q(x_4, \ldots, x_n)$ with $k \geq 1$ and $a \neq 0$ and ($\pm a < 0$ or k: odd),
- $R_{l,m}$: $x_1(x_1^2 + x_2 x_3) \pm x_2^l \pm a x_3^m + q(x_4, \ldots, x_n)$ with $a \neq 0$, $m \geq l \geq 5$,
- $\tilde{R}^-_m$: $x_1(-x_1^2 + x_2^2 + x_3^2) + a x_2^m + q(x_4, \ldots, x_n)$ with $a \neq 0$, $m \geq 5$,
- E_{12} : $x_1^3 + x_2^7 \pm x_3^2 + a x_1 x_2^5 + q(x_4, \ldots, x_n)$,
- E_{13} : $x_1^3 + x_1 x_2^5 \pm x_3^2 + a x_2^8 + q(x_4, \ldots, x_n)$,
- E_{14} : $x_1^3 \pm x_2^8 \pm x_3^2 + a x_1 x_2^6 + q(x_4, \ldots, x_n)$,
- Z_{11} : $x_1^3 x_2 + x_2^5 \pm x_3^2 + a x_1 x_2^4 + q(x_4, \ldots, x_n)$,
- Z_{12} : $x_1^3 x_2 + x_1 x_2^4 \pm x_3^2 + a x_1^2 x_2^3 + q(x_4, \ldots, x_n)$,
- Z_{13} : $x_1^3 x_2 \pm x_2^6 \pm x_3^2 + a x_1 x_2^5 + q(x_4, \ldots, x_n)$,
- W_{12} : $\pm x_1^4 + x_2^5 \pm x_3^2 + a x_1^2 x_2^3 + q(x_4, \ldots, x_n)$,
- W_{13} : $\pm x_1^4 + x_1 x_2^4 \pm x_3^2 + a x_2^6 + q(x_4, \ldots, x_n)$,
- Q_{11} : $x_1^3 + x_2^2 x_3 \pm x_1 x_3^3 + a x_3^5 + q(x_4, \ldots, x_n)$.

Algorithm

Let f be a real polynomial in $x = (x_1, \dots, x_n)$ and D_n be the n-th Weyl algebra; i.e., the ring of differential operators with polynomial coefficients.

Aim

Compute a holonomic system for the Laurent coefficient u_k $(k \in \mathbb{Z})$ for f_+^λ about λ_0. (i.e. to find a left ideal $I \subset \mathrm{Ann}_{D_n} u_k$ such that D_n/I is holonomic.)

Step 1

(1) Take $m \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that $\mathrm{Re}\,\lambda_0 + m \geq 0$.

(2) Find a functional equation $b_f(s) f^s = P(s) f^{s+1}$.

(3) $Q(s) := P(s)P(s+1)\cdots P(s+m-1)$,
$b(s) := b_f(s) b_f(s+1) \cdots b_f(s+m-1)$.
Then we have $b(\lambda) f_+^\lambda = Q(\lambda) f_+^{\lambda+m}$.

Step 2

Factorize $b(s)$ as $b(s) = c(s)(s-\lambda_0)^l$ with $c(\lambda_0) \neq 0$ and $l \in \mathbb{N}$. Then we have

$$f_+^\lambda = (\lambda-\lambda_0)^{-l} c(\lambda)^{-1} Q(\lambda) f_+^{\lambda+m} = \sum_{k=-l}^{\infty} (\lambda-\lambda_0)^k u_k(x),$$

where $u_k(x) \in \mathcal{D}'(\mathbb{R}^n)$ are given by

$$u_k(x) = \frac{1}{(l+k)!} \left[\left(\frac{\partial}{\partial\lambda}\right)^{l+k} (c(\lambda)^{-1} Q(\lambda) f_+^{\lambda+m}) \right]_{\lambda=\lambda_0}$$
$$= \sum_{j=0}^{l+k} Q_j (f_+^{\lambda_0+m} (\log f)^j)$$

with $\quad Q_j := \dfrac{1}{j!(l+k-j)!} \left[\left(\dfrac{\partial}{\partial\lambda}\right)^{l+k-j} (c(\lambda)^{-1} Q(\lambda)) \right]_{\lambda=\lambda_0}.$

Algorithm (continued)

Step 3

Compute a holonomic system for $(f_+^\lambda, \dots, f_+^\lambda(\log f)^{k+l})$ as follows:

(1) Compute a set G_0 of generators of the annihilator $\mathrm{Ann}_{D_n[s]} f^s$.

(2) Let $e_1 = (1, 0, \dots, 0), \cdots, e_{k+l} = (0, \dots, 0, 1)$ be the canonical basis of $\mathbb{Z}^{k+l+1}$. For each $P(s) \in G_0$ and an integer j with $0 \le j \le k + l$, set

$$P^{(j)}(s) := \sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} P(s)}{\partial s^{j-i}} e_{i+1} \in (D_n[s])^{k+l+1}.$$

(3) Set $G_1 := \{P^{(j)}(\lambda_0 + m) \mid P(s) \in G_0,\ 0 \le j \le k + l\}$.

The output G_1 of Step 3 generates a left D_n-module N such that $(D_n)^{k+l+1}/N$ is holonomic and

$$P_0 f_+^{\lambda_0+m} + P_1(f_+^{\lambda_0+m}\log f) + \cdots + P_{k+l}(f_+^{\lambda_0+m}(\log f)^{k+l}) = 0$$

holds for any $P = (P_0, \ldots, P_{k+l}) \in G_1$.

Remark Step 3 is essentially differentiation of the equations

$$P(s)f_+^s = 0 \quad (P(s) \in \mathrm{Ann}_{D_n[s]} f^s)$$

with respect to s.

Algorithm (the final step)

Step 4

Let N be the left D_n-submodule of $(D_n)^{l+k+1}$ generated by the output G_1 of Step 3 and let $Q_0, Q_1, \ldots, Q_{l+k}$ be the operators computed in Step 2. Compute a set G_2 of generators of the left ideal

$$I := \{P \in D_n \mid (PQ_0, PQ_1, \ldots, PQ_{l+k}) \in N\}$$

by using quotient or syzygy computation.

Output

The ideal I annihilates the distribution u_k and D_n/I is holonomic.

Holonomicity of the output

Theorem

Let I be the left ideal of D_n computed by the preceding algorithm. Then D_n/I is holonomic.

Sketch of the proof:
(1) The left D_n-module $(D_n)^{k+l+1}/N$ is holonomic. In fact, set

$$N_j := \{(P_0, \ldots, P_j, 0, \ldots, 0) \in N\}.$$

Then $N_j/N_{j-1} \simeq \mathrm{Ann}_{D_n[s]} f^s/(s - \lambda_0 - m)\mathrm{Ann}_{D_n[s]} f^s$ is holonomic.

(2) D_n/I with $I := \{P \in D_n \mid (PQ_0, PQ_1, \ldots, PQ_{l+k}) \in N\}$ is holonomic since the map $h : D_n/I \to (D_n)^{k+l+1}/N$ defined by $h([P]) = (PQ_0, \ldots, PQ_{k+l+1})$ is an injective homomorphism of left D_n-modules.

An example: $f = x_1^2 - x_2^2$

- The functional equation is $(\lambda+1)^2 f_+^\lambda = \frac{1}{4}(\partial_1^2 - \partial_2^2) f_+^{\lambda+1}$

$\Rightarrow$ f_+^λ has poles (of order at most 2) only at $\lambda = -1, -2, -3, \dots$.

- The Laurent expansion around $\lambda = -1$ is

$$f_+^\lambda = (\lambda+1)^{-2} u_{-2}(x) + (\lambda+1)^{-1} u_{-1}(x) + u_0(x) + (\lambda+1) u_1(x) + \cdots$$

with

$$u_{-2}(x) = \frac{1}{4}(\partial_1^2 - \partial_2^2) f_+^0 = \frac{1}{4}(\partial_1^2 - \partial_2^2) Y(f),$$

$$u_{-1}(x) = \frac{1}{4}(\partial_1^2 - \partial_2^2)(Y(f) \log f).$$

Differentiating

$$(x_2\partial_1 + x_1\partial_2)f_+^s = (x_1\partial_1 + x_2\partial_2 - 2s)f_+^s = 0$$

with respect to s, we get

$$\begin{aligned}
&(x_2\partial_1 + x_1\partial_2)f_+^s = 0, \quad (x_2\partial_1 + x_1\partial_2)(f_+^s \log f) = 0,\\
&2f^s + (x_1\partial_1 + x_2\partial_2 - 2s)(f_+^s \log f) = 0,\\
&(x_1\partial_1 + x_2\partial_2 - 2s)f_+^s = 0.
\end{aligned}$$

Hence $(Y(f),\ Y(f)\log f)$ satisfies a holonomic system

$$\begin{aligned}
&(x_2\partial_1 + x_1\partial_2)Y(f) = 0, \quad (x_2\partial_1 + x_1\partial_2)(Y(f)\log f) = 0,\\
&2Y(f) + (x_1\partial_1 + x_2\partial_2)(Y(f)\log f) = 0,\\
&(x_1\partial_1 + x_2\partial_2)Y(f) = 0.
\end{aligned}$$

Let N be the left D_2-submodule of D_2^2 genererated by these vectors of differential operators. Then

$$P \cdot (\partial_1^2 - \partial_2^2, 0) \in N \quad \Rightarrow \quad Pu_{-2} = 0,$$
$$P \cdot (0, \partial_1^2 - \partial_2^2) \in N \quad \Rightarrow \quad Pu_{-1} = 0.$$

By module quotient (via intersection or syzygy computation in D_2)

- u_{-2} satisfies
$$x_1 u_{-2}(x) = x_2 u_{-2}(x) = 0$$
Hence $u_{-2}(x) = c\delta(x)$ $(\exists c \in \mathbb{C})$.
- u_{-1} satisfies
$$(x_2\partial_1 + x_1\partial_2)u_{-1}(x) = (x_1^2 - x_2^2)u_{-1}(x) = 0.$$
(This coincides with $\mathrm{Ann}_{D_2} u_{-1}$.)