Plan of the course

1st lecture	**Introduction:** Aim and an example **Chapter 1:** Basics of D-modules
2nd lecture	**Chapter 2:** Gröbner bases in the ring of differential operators **Chapter 3:** Distributions as generalized functions
3rd lecture	**Chapter 4:** D-module theoretic integration algorithm **Chapter 5:** Integration over the domain defined by polynomial inequalities

5. Integration over the domain defined by polynomial inequalities

5.1 Powers of polynomials and tensor products

Let K be a field of characteristic zero and $f_1, \dots, f_p \in K[x] = K[x_1, \dots, x_n]$ be nonzero polynomials.
Let us consider a 'function' $f_1^{s_1} \cdots f_p^{s_p}$ with indeterminates (as parameters) $s = (s_1, \dots, s_p)$. More precisely, set

$$\mathcal{L} := K[x, (f_1 \cdots f_p)^{-1}, s] f_1^{s_1} \cdots f_p^{s_p},$$

which is regarded as a free $K[x, (f_1 \cdots f_p)^{-1}, s]$-module generated by the 'symbol' $f_1^{s_1} \cdots f_p^{s_p}$. Then $\mathcal{L}$ is a left $D_n[s]$-module with the natural derivations

$$\partial_{x_i}(f_1^{s_1} \cdots f_p^{s_p}) = \sum_{j=1}^{p} s_j \frac{\partial f_j}{\partial x_i} f_j^{-1} f_1^{s_1} \cdots f_p^{s_p} \qquad (i = 1, \dots, n).$$

Tensor product with a holonomic module

Denote $f^s = f_1^{s_1} \cdots f_p^{s_p}$.
Let $M = D_n u = M/I$ be a holonomic left D_n-module generated by an element $u \in M$ with the left ideal $I = \mathrm{Ann}_{D_n} u$.
Let us consider the tensor product

$$M \otimes_{K[x]} \mathcal{L},$$

which has a natural strucutre of left $D_n[s]$-module with the derivations

$$\partial_{x_i}(u' \otimes v) = (\partial_{x_i} u') \otimes v + u' \otimes (\partial_{x_i} v) \quad (u' \in M,\ v \in \mathcal{L},\ i = 1, \dots, n).$$

Our aim is to compute the annihilator (in $D_n[s]$) of $u \otimes f^s \in M \otimes_{K[x]} \mathcal{L}$.

For this purpose, define shift (difference) operators E_j by

$$E_j : \mathcal{L} \ni a(x, s_1, \ldots, s_p) f^s \longmapsto a(x, s_1, \ldots, s_j + 1, \ldots, s_p) f_j f^s \in \mathcal{L}$$

for $j = 1, \ldots, p$, which are bijective with the inverse shifts $E_j^{-1} : \mathcal{L} \to \mathcal{L}$.

Mellin transform

Let $D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra generated by $s = (s_1, \ldots, s_p)$, $E = (E_1, \ldots, E_p)$, and $E^{-1} = (E_1^{-1}, \ldots, E_p^{-1})$.

We introduce new variables $t = (t_1, \ldots, t_p)$ and the associated derivations $\partial_t = (\partial_{t_1}, \ldots, \partial_{t_p})$. Let D_{n+p} be the ring of differential operators with respect to the variables $(x, t) = (x_1, \ldots, x_n, t_1, \ldots, t_p)$.

Let $\mu : D_{n+p} \to D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra homomorphism (Mellin transform) of D_n defined by

$$\mu(t_j) = E_j, \quad \mu(\partial_{t_j}) = -s_j E_j^{-1}.$$

This homomorphism is well-defined since

$$\begin{aligned}\mu(\partial_{t_i} t_i - t_i \partial_{t_i}) &= \mu(\partial_{t_i})\mu(t_i) - \mu(t_i)\mu(\partial_{t_i}) \\ &= -s_i E_{s_i}^{-1} E_i - E_{s_i}(-s_i)E_i^{-1} = 1.\end{aligned}$$

Since μ is injective, we can regard $E\langle s, E, E^{-1}\rangle$ as a subring of D_{n+p} through μ. With this identification, we have

$$t_j = E_j, \qquad \partial_{t_j} = -s_j E_j^{-1}, \qquad s_j = -\partial_{t_j} t_j = -t_j \partial_{t_j} - 1.$$

Hence we have inclusions

$$D_n[s] \subset D_n\langle s, E\rangle \subset D_{n+p} \subset D_n\langle s, E, E^{-1}\rangle$$

of rings. We will be mostly concerned with $D_n[s]$ and D_{n+p}.

• $M \otimes_{K[x]} \mathcal{L}$ is a left $D_n\langle s, E, E^{-1}\rangle$-module, and cosequently left modules over the subrings above. (s and E act only on $\mathcal{L}$.)

Algorithm (a holonomic D_{n+p}-module for $u \otimes f^s$)

Input: A set G_0 of generators of I with $M = D_n/I$ and nonzero polynomials $f_1, \ldots, f_p \in K[x]$.

For $P = P(x, \partial_{x_1}, \ldots, \partial_{x_n}) \in G_0$, set

$$\tau(P) := P\left(x, \partial_{x_1} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\partial_{t_j}, \ldots, \partial_{x_n} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_n}\partial_{t_j}\right).$$

This substitution is well-defined in the ring D_{n+p} since the operators which are substituted for $\partial_{x_1}, \ldots, \partial_{x_n}$ commute with each other.
Output: $G := \{\tau(P) \mid P \in G_0\} \cup \{t_j - f_j(x) \mid j = 1, \ldots, p\}$ generates a left ideal J of D_{n+p} such that $J \subset \mathrm{Ann}_{D_{n+p}} u \otimes f^s$ and D_{n+p}/J is holonomic.

Case $M = K[x]$

In particular, setting $M = K[x]$ with $u = 1$, this gives the left ideal J of D_{n+p} generated by

$$\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j} \quad (i = 1, \dots, n), \quad t_j - f_j \quad (j = 1, \dots, m),$$

which annihilates f^s in $\mathcal{L}$.

Sketch of the proof of the correctness

In view of the equality

$$\left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}\partial_{t_j}\right)(u \otimes f^s)$$
$$= (\partial_{x_i} u) \otimes f^s + u \otimes \left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}\partial_{t_j}\right) f^s$$
$$= (\partial_{x_i} u) \otimes f^s + u \otimes \left(\partial_{x_i} + \sum_{j=1}^{p}(-s_j) f_j^{-1} \frac{\partial f_j}{\partial x_i}\right) f^s$$
$$= (\partial_{x_i} u) \otimes f^s$$

in $M \otimes_{K[x]} \mathcal{L}$, we have, for $j = 1, \dots, p$,

$$\tau(P)(u \otimes f^s) = (Pu) \otimes f^s = 0, \quad (t_j - f_j)(u \otimes f^s) = u \otimes (t_j - f_j) f^s = 0.$$

Hence J annihilates $u \otimes f^s$ in $M \otimes_{K[x]} \mathcal{L}$.

Let us show that D_{n+p}/J is holonomic. Since D_n/I is holonomic, its characteristic variety $\mathrm{Char}(D_n/I)$ is an n-dimensional algebraic set of K^{2n}. By the definition, we have

$$\begin{aligned}
&\mathrm{Char}(D_{n+p}/J) \\
&\subset \Big\{(x,t,\xi,\tau) \in K^{2(n+p)} \mid \sigma(P)\Big(x, \xi_1 + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\tau_j, \dots\Big) = 0 \\
&(\forall P \in I), \quad t_j = f_j(x)\ (j = 1, \dots, p)\Big\} \\
&= \Big\{(x,t,\xi,\tau) \in K^{2(n+p)} \mid \Big(x, \xi_1 + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\tau_j, \dots\Big) \in \mathrm{Char}(D_n/I), \\
&\quad t_j = f_j(x)\ (j = 1, \dots, p)\Big\}.
\end{aligned}$$

Since the set on the last line is in one-to-one correspondence with the set $\mathrm{Char}(D_n/I) \times \mathbb{C}^p$, the dimension of $\mathrm{Char}(D_{n+p}/J)$ is $n+p$, which implies that D_{n+p}/J is a holonomic module. This completes the proof.

Next aim is to compute a $D_n[s]$-module for $u \otimes f^s$.

Algorithm (intersection with the subring $D_n[s]$)

Input: A set G_0 of generators of a left ideal J of D_{n+p}.
Output: A set G of generators of the left ideal $J \cap D_n[s]$ of $D_n[s]$.

1. Introducing new variables u_j, v_j for $j = 1, \ldots, p$, let $h(P) \in D_{n+p}[u]$ be the multi-homogenization of $P \in D_{n+p}$; i.e., $h(P)$ is homogeneous with respect to the weight -1 for t_j and u_j, and 1 for ∂_{t_j}, for each j.

2. Let N be the left ideal of $D_{n+p}[u, v]$ generated by the set

$$\{h(P) \mid P \in G_0\} \cup \{1 - u_j v_j \mid j = 1, \ldots, p\}.$$

3. Compute a set G_1 of generators of the ideal $N \cap D_{n+p}$ by eliminating u, v via an appropriate Gröbner basis.

4. Since each element P of G_1 is multi-homogeneous without u, v, there exist a monomial S in t, ∂_t and an operator $Q(s) \in D_n[s]$ such that

$$SP = Q(-\partial_{t_1} t_1, \ldots, -\partial_{t_p} t_p).$$

Let G be the set of such Q for each $P \in G_1$.

By using the two algorithms above, we get a left $D_n[s]$-module for $u \otimes f^s$ (especially for $f^s \in \mathcal{L}$).

Powers of polynomials as distributions

From now on, let us assume $K = \mathbb{C}$ and $f_1, \dots, f_p \in \mathbb{R}[x]$. Set

$$\Omega := \{(z_1, \dots, z_p) \in \mathbb{C}^p \mid \mathrm{Re}\, z_1 > 0, \dots, \mathrm{Re}\, z_p > 0\}.$$

We define the local integrable function $(f_j)_+^{\lambda_j}$ on $\mathbb{R}^n$ by

$$f_j(x)_+^{\lambda_j} = \begin{cases} f_j(x)^{\lambda_j} & \text{if } f_j(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda_j \in \mathbb{C}$ with nonnegative real part $\mathrm{Re}\, \lambda_j \geq 0$. Their product $f_+^s := (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$ is a locally integrable function on $\mathbb{R}^n$ if $(\lambda_1, \dots, \lambda_p)$ belongs to the closure $\overline{\Omega}$ of Ω.

Let $v = v(x)$ be a distribution defined on an open set U of $\mathbb{R}^n$. Let I be a left ideal of D_n which annihilates $v(x)$ such that $M := D_n/I$ is holonomic. Set $M = M/I = D_n u$ with $u = \overline{1}$ and $\mathcal{L} = K[x, (f_1 \cdots f_p)^{-1}, s]f^s$ as before with $K = \mathbb{C}$.

Theorem

Let $v(x)$ be a complex-valued C^∞ function on an open set $U \subset \mathbb{R}^n$ such that

$$U \supset \{x \in \mathbb{R}^n \mid f_(x) \geq 0 \quad (j = 1, \ldots, p)\}.$$

Assume that $v(x)$ is holonomic; i.e, there exists a left ideal of I such that I annihilates $v(x)$ and M/I is holonomic. Let J be the left ideal of $D_n[s]$ obtained by the preceeding two algorithms. Then for any $\lambda = (\lambda_1, \ldots, \lambda) \in \overline{\Omega}$, $J(\lambda) := \{P(\lambda) \mid P(s) \in J\}$ annihilates $v(x)f_+^\lambda$ and $D_n/J(\lambda)$ is holonomic.

Especially, $D_n/J(0)$ is a holonomic system for $v(x)Y(f_1(x))\cdots Y(f_p(x))$. Combined with the integration algorithm, this gives us an algorithm to compute a holonomic system for

$$v(x_1,\dots,x_{n-d}) = \int_{D(x_1,\dots,x_{n-d})} u(x_1,\dots,x_n)\,dx_{n-d+1}\cdots dx_n,$$

$$D(x_1,\dots,x_{n-d}) := \{(x_{n-d+1},\dots,x_n)\in\mathbb{R}^d \mid f_j(x_1,\dots,x_n)\geq 0 \quad (1\leq j\leq p)\}.$$

Sketch of the proof of Theorem

Let $P(s)\in J$. Then $P(s)(u\otimes f^s)=0$ holds in $M\otimes\mathcal{L}$. This does not necessarily imlies $P(\lambda)(v(x)f_+^\lambda)=0$ since we used the inverse shift E_j^{-1} with $E_j^{-1}f^s = f_j^{-1}f^s$ in order to derive a holonomic system for $u\otimes f^s$. However, we can deduce $P(\lambda)(v(x)f_+^\lambda)=0$ for any $\lambda\in\overline{\Omega}$ by using the unique continuation property with respect to the holomorphic parameters λ.

For example, let $u(x) = u(x_1, \dots, x_n)$ be a C^∞ holonomic function on $V \times \mathbb{R}$ with an open set V of $\mathbb{R}^{n-1}$. Set $x' = (x_1, \dots, x_{n-1})$. Then the indefinite integral

$$v(x) := \int_0^{x_n} u(x', t)\,dt = \int_{\mathbb{R}} u(x', t) Y(t) Y(x_n - t)\,dt$$

is a holonomic function.

Example (posed by A. Takemura)

Set $D(t) := \{(x,y) \in \mathbb{R}^2 \mid x^3 + y^3 \leq t\}$, then the integral

$$v(t) = \int_{D(t)} e^{-x^2-y^2}\,dxdy = \int_{\mathbb{R}^2} e^{-x^2-y^2}\,Y(t - x^3 - y - 3)\,dxdy.$$

satisfies the ordinary differential equation $Pv(t) = 0$ with

$$P = 729t^3\partial_t^7 + 6561t^2\partial_t^6 + 12555t\partial_t^5 + (648t^2 + 3240)\partial_t^4 + 1944t\partial_t^3 + 480\partial_t^2 + 128t\partial_t.$$

Exercise 1 (for beginners)

(1) Find a holonomic system for the function $e^{tx-t^3} = \exp(tx - t^3)$. Confirm that it is holonomic. (Hint: Differentiate the function with respect to x and t. See pages 43–44 of Oaku 1 for the characterisic variety.)

(2) Find a holonomic system for the distribution $e^{tx-t^3}Y(t)$. Confirm that it is holonomic. (Hint: One method is to apply the operators of (1) to the distribution and kill the delta function as in the example in Introduction (Oaku 1).)

(3) Find a holonomic system, i.e., a linear ordinary differential equation for

$$v(x) := \int_0^\infty e^{tx-t^3}\,dt = \int_{-\infty}^\infty e^{tx-t^3}\,Y(t)\,dt.$$

(Hint: Mimic the example on page 7 of Oaku 4.)

Exercise 2 (for specialists)

Deduce a linear differential equation (in x) for

$$v(x; a, b) := \int_0^1 e^{tx} t^a (1-t)^b \, dt$$

regarding a, b as parameters.

Beginners do not stay beginners forever.

(from the preface of a book by D. Eisenbud and J. Harris)