Plan of the course

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Chapter 1: Basics of D-modules

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Chapter 5: Integration over the domain defined by polynomial inequalities

4. D-module theoretic integration algorithm
4.1. Integration as an operation on D-modules

Let D_{n+p} be the ring of differential operators in the variables (x, t) with $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_d)$. Let $\pi : K^{n+d} \ni (x, t) \longmapsto x \in K^n$ be the projection.

The *integral* of a left D_{n+d}-module M along the fibers of π, or the *direct image* by π, is defined by

$$\pi_* M := M/(\partial_{t_1} M + \cdots + \partial_{t_d} M).$$

This is a left D_n-module since any element of D_n commutes with ∂_{t_j}.

For the sake of simplicity, let us assume that M is generated by a single element $u \in M$ as left D_{n+d}-module. Let $[u]$ be the residue class of u in $\pi_* M$. Then $\pi_* M$ is generated by $\{t^\gamma [u] \mid \gamma \in \mathbb{N}^d\}$ over D_n.

Now assume $K = \mathbb{C}$ and let φ be an element of $\mathrm{Hom}_{D_{n+d}}(M, \mathcal{D}'\mathcal{E}'(U))$. Then $f := \varphi(u)$ belongs to $\mathcal{D}'\mathcal{E}'(U)$.

Define a $\mathbb{C}$-homomorphism $\varphi' : M \to \mathcal{D}'(U)$ by

$$\varphi'(Pu) = \int_{\mathbb{R}^d} Pf(x,t)\, dt \qquad (\forall P \in D_{n+d}).$$

This is well-defined since $Pu = 0$ in M implies $Pf = 0$ in $\mathcal{D}'\mathcal{E}'(U)$.

Note that φ' is D_n-linear by differentiation under the integral sign.

For any $P \in D_n$, $\varphi \in C_0^\infty(U)$, and $1 \le j \le d$, we have

$$\left\langle \int_{\mathbb{R}^d} \partial_{t_j} Pf(x,t)\,dt,\ \varphi(x) \right\rangle = \langle \partial_{t_j} Pf(x,t),\ \varphi(x)1(t)\rangle$$
$$= -\langle Pf(x,t),\ \partial_{t_j}(\varphi(x)1(t))\rangle = 0.$$

Hence φ' induces

$$\pi_*(\varphi) \in \mathrm{Hom}_{D_n}(\pi_* M, \mathcal{D}'(U)).$$

In conclusion, we have a $\mathbb{C}$-linear map

$$\pi_* \ :\ \mathrm{Hom}_{D_{n+d}}(M, \mathcal{D}'\mathcal{E}'(U)) \longrightarrow \mathrm{Hom}_{D_n}(\pi_* M, \mathcal{D}'(U)).$$

The argument so far works with $\mathcal{D}'\mathcal{E}'(U)$ replaced by $\mathcal{ES}(U)$, and also by $\mathcal{ES}(U) + \mathcal{D}'\mathcal{E}'(U)$ giving a $\mathbb{C}$-linear map

$$\pi_* \;:\; \mathrm{Hom}_{D_n}(M, \mathcal{ES}(U) + \mathcal{D}'\mathcal{E}'(U)) \longrightarrow \mathrm{Hom}_{D_{n-d}}(\pi_*(M), \mathcal{D}'(U)).$$

This means that for a solution in $\mathcal{DS}(U) + \mathcal{D}'\mathcal{E}'(U)$ of a system M of differential equations, its integral with respect to t satisfies the system $\pi_* M$.

The generators $t^\gamma[u]$ of $\pi_* M$ with $\gamma' \in \mathbb{N}^d$ are sent by $\pi_*(\varphi)$ to

$$\pi_*(\varphi)(t^\gamma[u]) = \int_{\mathbb{R}^d} t^\gamma f(x,t)\,dt \;\in \mathcal{D}'(U).$$

Theorem (Bernstein, Kashiwara)

If M is a holonomic D_n-module, then $\pi_* M$ is a holonomic D_n-module. In particular, $\pi_* M$ is finitely generated over D_n.

Hence $\pi_* M$ is generated by a finite subset of $\{t^\gamma[u] \mid \gamma \in \mathbb{N}^d\}$, and $\pi_* M$ represents the relations among these generators.

Example

Set $n = d = 1$ and write $x = x_1$, $t = t_1$. Consider

$$M := \frac{D_2}{D_2 t(t-1)(\partial_t - x) + D_2(\partial_x - t)}.$$

Set $u = \overline{1} \in M$. Then

$$t(t-1)(\partial_t - x)u = (\partial_x - t)u = 0 \text{ in } M.$$

Let $[u]$ be the residue class of u in $\pi_* M = M/\partial_t M$.

$\pi_* M$ is generated by $[u]$ since $tu = \partial_x u$ by the 2nd equation for u.
From $\partial_t t(t-1) = t(t-1)\partial_t + 2t - 1$, it follows

$$\{t(t-1)\partial_t + 2t - 1\}[u] = 0.$$

This gives, combined with $t(t-1)\partial_t u = xt(t-1)u = x\partial_x(\partial_x - 1)u$,

$$\{x\partial_x^2 - (x-2)\partial_x - 1\}[u] = 0.$$

4.2 An algorithm for integration

Let M be a left D_{n+d}-module generated by $u \in M$. Set

$$D_\pi := D_{n+d}/(\partial_{t_1} D_{n+d} + \cdots + \partial_{t_d} D_{n+d}).$$

Then D_π has a structure of (D_n, D_{n+d})-bimodule and we have

$$\pi_* M = M/(\partial_{t_1} M + \cdots + \partial_{t_d} M) = D_\pi \otimes_{D_{n+d}} M$$

as left D_n-module. Set

$$\theta := \partial_{t_1} t_1 + \cdots + \partial_{t_d} t_d = t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d} + d.$$

Now let us fix the weight vector

$$w := (0, \ldots, 0, 1, \ldots, 1; 0, \ldots, 0, -1, \ldots, -1) \in \mathbb{Z}^{2(n+d)}.$$

That is, we define the weight of x_i and ∂_{x_i} to be 0, while the weight of t_j and ∂_{t_j} are 1 and -1 respectively. Set

$$F_k(M) := F_k^w(D_{n+d})u, \qquad \mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M) \qquad (k \in \mathbb{Z}).$$

Then $\{F_k(M)\}$ is a good w-filtration of M.

Theorem

If $M = D_{n+d}$ is holonomic, then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ in s such that $b(\theta)\mathrm{gr}_0(M) = 0$. Such $b(s)$ of minimum degree is called the b-function of M with respect to the weight vector w.

The b-function can be comptuted by using a w-involutive basis of $I := \mathrm{Ann}_{D_{n+d}} u$.

Lemma

Let $b(s)$ be the *b-function* of M with respect to w. Then $b(\theta - k)\mathrm{gr}_k(M) = 0$ holds for any $k \in \mathbb{Z}$.

Proof: Note that $\theta \cdot t_j = t_j(\theta + 1)$ and $\theta \cdot \partial_j = \partial_j(\theta - 1)$. Hence $b(\theta)P = Pb(\theta + k)$ holds if P is homogeneous of order k with respect to w. This proves the lemma.

The proof of the following proposition also provides us with an algorithm to compute the D_n-module structure of $\pi_* M$.

Proposition

Suppose that a left D_{n+d}-module $M = D_{n+d}u = D_{n+d}/I$ has a b-function $b(s)$. Let $-k_1$ be the smallest integral root, if any, of $b(s)$. Set $k_1 = -1$ if $b(s)$ has no integral root. Then as a left D_n-module, $\pi_* M$ is generated by the set

$$\{t^\gamma[u] \mid \gamma \in \mathbb{N}^d, |\gamma| \le k_1\}$$

In particular, $\pi_* M = 0$ if $k_1 < 0$.

Proof

Since $\pi_* M$ is the cokernel of $M^d \xrightarrow{(\partial_{t_1},\dots,\partial_{t_d})} M$, we have an exact sequence

$$M^d \xrightarrow{(\partial_{t_1},\dots,\partial_{t_d})} M \longrightarrow \pi_* M \longrightarrow 0$$

of left D_n-modules. First let us show that the induced sequence

$$F_{k_1+1}(M)^d \xrightarrow{(\partial_{t_1},\dots,\partial_{t_d})} F_{k_1}(M) \longrightarrow \pi_* M \longrightarrow 0$$

is also exact. Let $k > k_1$ and $u_k \in F_k(M)$ with nonzero modulo class $\overline{u}_k \in \mathrm{gr}_k(M)$. We have $b(\theta - k)\overline{u}_k = 0$. There exists $c(s) \in \mathbb{C}[s]$ such that $b(\theta - k) - b(\theta) = \theta c(\theta)$. Then $b(-k)\overline{u}_k = \theta c(\theta)\overline{u}_k$ holds. Since $b(-k) \neq 0$, this implies that there exist $v_1, \dots, v_d \in F_{k+1}(M)$ such that

$$u_k - (\partial_{t_1} v_1 + \cdots + \partial_{t_d} v_d) \in F_{k-1}(M).$$

Continuing this argument, we conclude that there exist $v_1, \dots, v_d \in F_{k+1}$ such that

$$u_k - (\partial_{t_1} v_1 + \cdots + \partial_{t_d} v_d) \in F_{k_1}(M).$$

Now assume that $v_1, \dots, v_d \in F_k(M)$ with $k > k_1 + 1$. Let $\overline{v}_j$ be the modulo class of v_j in $\mathrm{gr}_k(M)$. Then we have

$$\partial_{t_1} \overline{v}_1 + \cdots + \partial_{t_d} \overline{v}_d = 0 \in \mathrm{gr}_{k_1}(M).$$

We want to show that $\overline{v}_j = 0$ for any j. Since this is rather technical, let us show only in case $d = 1$. Since $0 = b(\partial_1 t_1 - k)\overline{v}_1 = 0$ and $\partial_1 t_1 - k = t_1 \partial_1 - k + 1$, we get $b(-k+1)\overline{v}_1 = c(t_1 \partial_{t_1})\partial_t \overline{v}_1 = 0$ with some $c(s) \in \mathbb{C}[s]$. Since $b(-k+1) \neq 0$ by the assumption, $\overline{v}_1 = 0$.

Now let

$$(D_{n+d})^r \xrightarrow{\psi} D_{n+d} \xrightarrow{\varphi} M \longrightarrow 0$$

be a presentation of M, where

$$\varphi(P) = Pu \quad (\forall P \in D_{n+d}),$$
$$\psi((Q_1, \dots, Q_r)) = Q_1 P_1 + \cdots + Q_r P_r \quad (\forall Q_1, \dots, Q_r \in D_{n+d}).$$

Here we assume that $P_1, \dots, P_r$ are a w-involutive basis of $I = \mathrm{Ann}_{D_{n+d}} u$ with $\mathrm{ord}_w(P_i) = m_i$. This implies that the sequence

$$\oplus_{i=1}^r F_{k-m_i}(D_{n+d}) \xrightarrow{\psi} F_k(D_{n+d}) \xrightarrow{\varphi} F_k(M) \longrightarrow 0$$

is exact. Set $F_k[\mathbf{m}]((D_\pi)^r) := \oplus_{i=1}^r F_{k-m_i}(D_\pi)$ with $\mathbf{m} = (m_1, \dots, m_r)$, and so on.

Then ψ induces homomorphisms

$$\overline{\psi} : (D_\pi)^r \longrightarrow D_\pi,$$
$$\overline{\psi} : F_k[\mathbf{m}]((D_\pi)^r) := \oplus_{i=1}^r F_{k-m_i}(D_\pi) \longrightarrow F_k(D_\pi),$$

where $\{F_k(D_\pi)\}$ denotes the filtration induced by $\{F_k^w(D_{n+d})\}$.

We have a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
F_{k_1+1}[\mathbf{m}]((D_{n+d})^r)^d & \longrightarrow & F_{k_1}[\mathbf{m}]((D_{n+d})^r) & \longrightarrow & F_{k_1}[\mathbf{m}]((D_\pi)^r) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\psi,\dots,\psi)} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\
F_{k_1+1}(D_{n+d})^d & \xrightarrow{(\partial_{t_1},\dots,\partial_{t_d})} & F_{k_1}(D_{n+d}) & \longrightarrow & F_{k_1}(D_\pi) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\varphi,\dots,\varphi)} & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
F_{k_1+1}(M)^d & \xrightarrow{(\partial_{t_1},\dots,\partial_{t_d})} & F_{k_1}(M) & \longrightarrow & \pi_* M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

exact exact

where the upper leftmost morphisms send

$$\begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{d1} & \cdots & Q_{dr} \end{pmatrix} \in F_{k_1+1}[\mathbf{m}]((D_{n+d})^r)^d$$

to

$$\begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{d1} & \cdots & Q_{dr} \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_r \end{pmatrix} \in F_{k_1+1}(D_{n+d})^d,$$

$$\begin{pmatrix} \partial_{t_1} & \cdots & \partial_{t_d} \end{pmatrix} \begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{d1} & \cdots & Q_{dr} \end{pmatrix} \in F_{k_1}[\mathbf{m}]((D_{n+d})^r)$$

respectively.

In the commutaive diagram, the three horizontal sequences and the two vertical sequences except the rightmost one are exact. This implies that the rightmost vertical sequence is also exact; i.e.,

$$\pi_* M = \operatorname{coker}(\overline{\psi} : F_{k_1}[\mathbf{m}]((D_\pi)^r) \longrightarrow F_{k_1}(D_\pi)).$$

Note that

$$F_{k_1}(D_\pi) = \bigoplus_{|\gamma| \le k_1} t^\gamma D_n, \qquad F_{k_1}[\mathbf{m}]((D_\pi)^r) = \bigoplus_{i=1}^{r} \bigoplus_{|\gamma| \le k_1 - m_i} t^\gamma D_n$$

as left D_n-modules. Hence ψ is a homomorphisms of free left D_n-modules of finite rank, $\operatorname{coker} \psi$ can be explicitly computed by linear algebra over D_n. This gives the relations among the generators $\{t^\gamma[u] \mid |\gamma| \le k_1\}$ of $\pi_* M$. By elimination, we can obtain $\operatorname{Ann}_{D_n}[u]$ so that $D_n[u] \cong D_n/\operatorname{Ann}_{D_n}[u]$ is a left D_n-submodule of $\pi_* M$.

Example (again)

Set $n = d = 1$ and write $x = x_1$, $t = t_1$. Consider

$$M := D_2/(D_2P_1 + D_2P_2) \quad \text{with} \quad P_1 = t - \partial_x, \quad P_2 = t(t-1)(\partial_t - x).$$

Set $u := \overline{1} \in M$ and $F_k(M) := F_k^w(D_n)u$.

Let $\overline{u}$ be the residue class of u in $\mathrm{gr}_0(M)$. Since $0 = \partial_t P_1 \overline{u} = \partial_t t \overline{u}$, the b-function of M w.r.t. w is $b(s) = s$. So $k_1 = 0$.

A Gröbner basis of $I := D_2P_1 + D_2P_2$ with respect to a monomial order adapted to $(1, 0, -1, 0)$ is $\{P_1, P_2, P_3\}$ with

$$P_3 = x\partial_x^2 - x\partial_x + 2\partial_x - 1 - \partial_t\partial_x^2 + \partial_t\partial_x.$$

The w-order of P_1, P_2, P_3 are $1, 2, 0$ respectively.

Hence we have an exact sequence

$$F_{-2}(D_\pi) \oplus F_{-1}(D_\pi) \oplus F_0(D_\pi) \xrightarrow{\overline{\psi}} F_0(D_\pi) \longrightarrow \pi_* M \longrightarrow 0,$$

where $\overline{\psi}$ is induced from the column vector ${}^t(P_1\ P_2\ P_3)$. Since $D_\pi \cong D_1[t]$, we have $F_k(D_\pi) = 0$ for $k < 0$ and $F_0(D_\pi) = D_1$. Since operators of the form $\partial_t P$ with $P \in D_2$ vanishes in D_π,

$$\overline{\psi}((0\ 0\ Q)) = QP_3 = Q(x, \partial_x)(x\partial_x^2 - x\partial_x + 2\partial_x - 1)$$

holds for any $Q = Q(x, \partial_x) \in D_1$. This implies

$$\pi_* M = D_1 / D_1(x\partial_x^2 - x\partial_x + 2\partial_x - 1)$$