Plan of the course

1st lecture **Introduction:** Aim and an example
Chapter 1: Basics of D-modules

2nd lecture **Chapter 2:** Gröbner bases in the ring of differential operators
Chapter 3: Distributions as generalized functions

3rd lecture **Chapter 4:** D-module theoretic integration algorithm
Chapter 5: Integration over the domain defined by polynomial inequalities

3. Distributions as generalized functions

3.1. Definitions and basic properties

Definition

Let $C_0^\infty(U)$ be the set of the C^∞ functions on an open set U of $\mathbb{R}^n$ with compact support. A distribution u on U is a linear mapping

$$u : C_0^\infty(U) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

such that $\lim_{j\to\infty}\langle u, \varphi_j \rangle = 0$ holds for a sequence $\{\varphi_j\}$ of $C_0^\infty(U)$ if there is a compact set $K \subset U$ such that $\varphi_j = 0$ on $U \setminus K$ and

$$\lim_{j\to\infty} \sup_{x\in U} |\partial^\alpha \varphi_j(x)| = 0 \quad \text{for any } \alpha \in \mathbb{N}^n,$$

where $x = (x_1, \dots, x_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$. The set of the distributions on U is denoted by $\mathcal{D}'(U)$.

Remark

In distribution theory, $C_0^\infty(U)$ is also denoted by $\mathcal{D}(U)$ equipped with a natural topology. $\mathcal{D}'(U)$ stands for the dual space of $\mathcal{D}(U)$, i.e., the set of continuous linear maps of $\mathcal{D}(U)$ to $\mathbb{C}$.

A Lebesgue measurable function $u(x)$ defined on an open set U of $\mathbb{R}^n$ is called locally integrable on U if it is integrable on any compact subset of U.

We can regard a locally integrable function $u(x)$ on U as a distribution on U through the pairing

$$\langle u, \varphi \rangle = \int_U u(x)\varphi(x)\,dx \qquad (\forall \varphi \in C_0^\infty(U)).$$

Identifying two locally integrable functions which are equal to each other almost everywhere in U (i.e. outside a set of measure 0), we can regard the set of the locally integrable functions on U as a subspace of $\mathcal{D}'(U)$.

Let u be a distribution on U. The derivative $\partial_k u$ of u with respect to x_k is defined by

$$\langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

For a C^∞ function a on U, the product au is defined by

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

In particular, by these actions of the derivations and the polynomial multiplications, $\mathcal{D}'(U)$ has a natural structure of left D_n-module.

Example Set $n = 1$. The Heaviside function $Y(x)$ is the measurable function on $\mathbb{R}$ such that $Y(x) = 1$ for $x > 0$ and $Y(x) = 0$ for $x < 0$. The Dirac delta function $\delta(x)$ is a distribution on $\mathbb{R}$ defined by

$$\langle \delta(x), \varphi \rangle = \varphi(0) \quad (\varphi \in \mathbb{C}_0^\infty(\mathbb{R})).$$

The derivative of $Y(x)$ as a distribution coincides with $\delta(x)$ since

$$-\langle Y(x), \varphi'(x) \rangle = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta(x), \varphi \rangle \quad (\varphi \in C_0^\infty(\mathbb{R})).$$

The derivative $\delta'(x)$ of $\delta(x)$ is defined by

$$\langle \delta'(x), \varphi(x) \rangle = -\langle \delta(x), \varphi'(x) \rangle = -\varphi'(0).$$

Restriction and support

Let $u \in \mathcal{D}'(U)$ with an open set U of $\mathbb{R}^n$. Let V be an open subset of V. Then there exists a natural inclusion $C_0^\infty(V) \subset C_0^\infty(U)$. The restiction $v := u|_V$ of u to V is defined by

$$\langle v, \varphi \rangle = \langle u, \varphi \rangle \qquad (\forall \varphi \in C_0^\infty(V)).$$

Then $U \longmapsto \mathcal{D}'(U)$, where U are open sets of $\mathbb{R}^n$, constitues a sheaf on $\mathbb{R}^n$. For $u \in \mathcal{D}'(U)$, the *support* $\operatorname{supp} u$ is defined to be the smallest closed set Z in U such that $u|_{U \setminus Z} = 0$, i.e., $\langle u, \varphi \rangle = 0$ for any $\varphi \in C_0^\infty(U \setminus Z)$.

For example, with $x = x_1$ we have $\operatorname{supp} \delta(x) = \{0\}$ and $\operatorname{supp} Y(x) = \{x \in \mathbb{R} \mid x \geq 0\}$.

The set of the distributions on U whose supports are compact sets of U is denoted by $\mathcal{E}'(U)$. ($\mathcal{E}'(U)$ means the dual space of $\mathcal{E}(U) = C^\infty(U)$.

Let $u \in \mathcal{E}'(U)$ and $K := \operatorname{supp} u$. Let $1(x)$ be the constant function with value 1. Then the paring

$$\langle u, 1(x) \rangle = \langle u, \chi(x) \rangle$$

is well-defined with an arbitrary $\chi \in C_0^\infty(U)$ such that $\chi(x) = 1$ on an open set $V \subset U$ such that $K \subset V$. In fact, assume $\tilde{\chi} \in C_0^\infty(U)$ satisfies the same condition. Then since

$$\operatorname{supp}(\chi - \tilde{\chi}) \cap \operatorname{supp} u = \emptyset,$$

$\langle u, \chi \rangle = \langle u, \tilde{\chi} \rangle$ holds.

Now let M be a finitely generated D_n-module with $K = \mathbb{C}$. Recall that $\mathrm{Hom}_{D_n}(M, \mathcal{D}'(U))$ represents the solution space in $\mathcal{D}'(U)$ of the system M of linear differential equations.

Theorem (Kashiwara)

Let M be a holonomic D_n-module and U an open set of $\mathbb{R}^n$. Then

1. $\mathrm{Hom}_{D_n}(M, \mathcal{D}'(U))$ is a finite dimensional vector space over $\mathbb{C}$.
2. Each element of $\mathrm{Hom}_{D_n}(M, \mathcal{D}'(U))$ is real analytic on $U' := U \setminus \mathrm{Sing}(M)$; i.e., the natural $\mathbb{C}$-homomorphism $\mathrm{Hom}_{D_n}(M, \mathcal{A}(U')) \to \mathrm{Hom}_{D_n}(M, \mathcal{D}'(U'))$ is an isomorphism, where $\mathcal{A}(U')$ denotes the set of complex-valued real analytic functions on U'.

Example $\mathrm{Hom}_{D_n}(\mathbb{C}[x], \mathcal{D}'(\mathbb{R}^n)) \cong \mathbb{C}$. In fact, $\mathbb{C}[x] = D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$ and we can prove that if $u \in \mathcal{D}(\mathbb{R}^n)$ satisfies $\partial_1 u = \cdots \partial_n u = 0$, then u is a constant function. Since $\mathrm{Sing}\mathbb{C}[x] = \emptyset$, u is real analytic on $\mathbb{R}^n$.

Example Set $M := D_n/(D_n x_1 + \cdots + D_n x_n)$. Then $\mathrm{Hom}_{D_n}(M, \mathcal{D}'(\mathbb{R}^n))$ is one dimensional and spanned by $\delta(x)$, the n-dimensional delta function defined by

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0, \ldots, 0) \qquad (\forall \varphi \in C_0^\infty(\mathbb{R}^n)).$$

Since $\mathrm{Sing}M = \{0\}$, u is real analytic on $\mathbb{R}^n \setminus \{0\}$.

Example Set $n = 1$ and $M := D_1/D_1 x\partial$. Then $\mathrm{Hom}_{D_1}(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ is one dimensional and spanned by $Y(x)$. Since $\mathrm{Sing}M = \{0\}$, $Y(x)$ is real analytic on $\mathbb{R} \setminus \{0\}$.

3.2. Product of distributions

The product of two distributions cannot be defined in general. There are some cases where the product is well-defined: Let U be an open set of $\mathbb{R}^n$.

1. For $u_1 \in C^\infty(U)$ and $u_2 \in \mathcal{D}'(U)$, the product $u = u_1 u_2$ is well-defined as an element of $\mathcal{D}'(U)$ and the Leibniz rule $\partial_i(u_1 u_2) = (\partial_i u_1)u_2 + u_1(\partial_1 u_2)$ holds for $i = 1, \dots, n$.
2. Let u_1 and u_2 be measurable functions on U. If both u_1 and u_2 are locally square-integrable (i.e., $|u_1|^2$ and $|u_2|^2$ are locally integrable) or else if u_1 is bounded and u_2 is locally integrable, then the product $u = u_1 u_2$ is well-defined as a locally integrable function. But the Leibniz rule does not make sense; in fact, the product $(\partial_1 u_1)u_2$ cannot be defined in general.

For example, in one variable $x = x_1$, the product $\delta(x)^2$ or $Y(x)\delta(x)$ cannot be defined as distributions.

If $u(x)$ is locally integrable, then $Y(x)u(x)$ is also a locally integrable function. But $\delta(x)u(x)$ cannot be defined in general. In particular, the Leibniz rule

$$\partial_x(Y(x)u(x)) = Y(x)u'(x) + \delta(x)u(x)$$

does not make sense in general unless u is C^∞ while the lefthand side is well-defined as distribution.

Integration of a distribution

Let us consider distributions in variables (x, t) with $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_d)$. We regard t as the integration variables and x as parameters. Let $\pi : \mathbb{R}^{n+d} \ni (x, t) \mapsto x \in \mathbb{R}^n$ be the projection. Let U be an open set of $\mathbb{R}^n$ and let u be a distribution defined on $\pi^{-1}(U) = U \times \mathbb{R}^d$.

We would like to define the integral $\int_{\mathbb{R}^d} u(x, t)\, dt$ as a distribution on U. For this, we need some 'tameness' of u with respect to t. There are two special cases where the integration is well-defined.

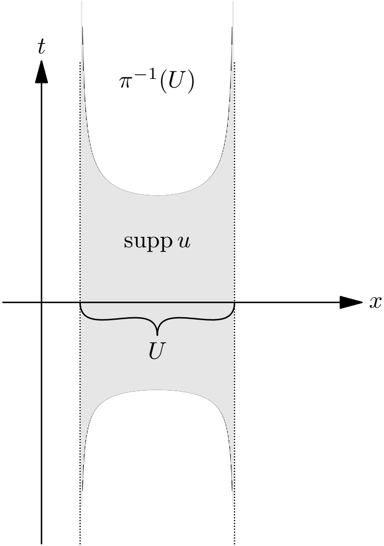
• $u(x,t)$ is a C^∞ function on $\pi^{-1}(U)$ and is rapidly decreasing with respect to t, i.e., $Pu(x,t)$ is bounded on $\pi^{-1}(K)$ for any compact subset K of U, and for any diffenrential operator $P \in D_{n+d}$. Here D_{n+d} denotes the ring of differntial operators in the variables (x,t). Let us denote by $\mathcal{ES}(U)$ the set of such distributions.

The integral of $u \in \mathcal{ES}(U)$ in t is naturally defined by

$$\int_{\mathbb{R}^d} u(x,t)\,dt,$$

which is a C^∞ function on U.

• u is a distribution on $\pi^{-1}(U)$ such that $\pi : \operatorname{supp} u \to \mathbb{R}^n$ is proper, i.e., for any compact set K of U, $\pi^{-1}(K) \cap \operatorname{supp} u$ is compact.



Let us denote by $\mathcal{D}'\mathcal{E}'(U)$ the set of such distributions. Then for $u \in \mathcal{D}'\mathcal{E}'(U)$, its integral with respect to t is defined by

$$\left\langle \int_{\mathbb{R}^d} u(x,t)\,dt,\ \varphi \right\rangle = \langle u(x,t), \varphi(x)1(t) \rangle \qquad (\forall \varphi \in C^\infty(U)),$$

where $1(t)$ denotes the constant function with value 1. This integral belongs to $\mathcal{D}'(U)$.

More precisely, the pairing above is defined as follows:

Choose $\chi(x,t) \in C^\infty(\pi^{-1}(U))$ such that $\chi(x,t) = 1$ on an open set W of $\pi^{-1}(U)$ containing $\operatorname{supp} u(x,t)$ and that $\pi : \operatorname{supp} \chi(x,t) \longrightarrow U$ is proper. Then we define

$$\langle u(x,t),\, \varphi(x)1(t)\rangle := \langle u(x,t),\, \varphi(x)\chi(x,t)\rangle.$$

The righthand side does not depend on such $\chi(x,t)$ since $\operatorname{supp}(1-\chi) \cap \operatorname{supp} u = \emptyset$.

Example Let $f(x,t)$ be a C^∞ function on $\mathbb{R}^2$. Since $\mathrm{supp}\, f(x,t)\delta(t) \subset \{(x,t) \mid t = 0\}$, $f(x,t)\delta(t)$ belongs to $\mathcal{D}'\mathcal{E}'(\mathbb{R})$. By the definition,

$$\left\langle \int_{\mathbb{R}} f(x,t)\delta(t)\,dt,\ \varphi(x) \right\rangle = \langle f(x,t)\delta(t),\ \varphi(x)1(t)\rangle$$
$$= \langle 1(x)\delta(t),\ \varphi(x)1(t)f(x,t)\rangle = \int_{\mathbb{R}} f(x,0)\varphi(x)\,dx$$

holds for any $\varphi \in C_0^\infty(\mathbb{R})$. Hence $\displaystyle\int_{\mathbb{R}} f(x,t)\delta(t)\,dt = f(x,0)$, which belongs to $C^\infty(\mathbb{R})$.

Each case is too restrictive but the sum of two cases suffices mostly for our purposes.

Lemma

Let $u \in \mathcal{D}'(\pi^{-1}(U))$ *belongs to* $\mathcal{ES}(U) + \mathcal{D}'\mathcal{E}'(U)$ *and choose* $u_1 \in \mathcal{ES}(U)$ *and* $u_2 \in \mathcal{D}'\mathcal{E}'(U)$ *such that* $u = u_1 + u_2$*. Then the integral of* u *is defined by*

$$\int_{\mathbb{R}^d} u(x,t)\,dt = \int_{\mathbb{R}^d} u_1(x,t)\,dt + \int_{\mathbb{R}^d} u_2(x,t)\,dt$$

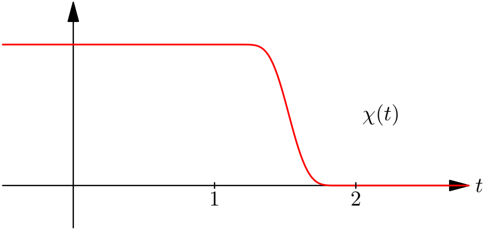
and is independent of the choice of u_1 *and* u_2*.*

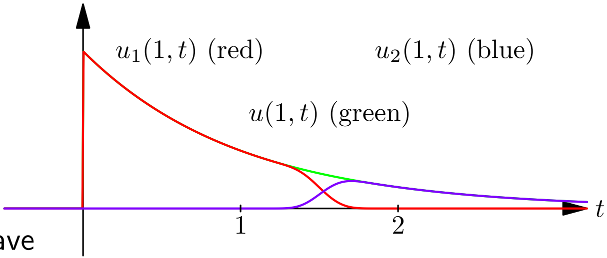
Proof: Let $u_1' \in \mathcal{E}\mathcal{S}(U)$ and $u_2' \in \mathcal{D}'\mathcal{E}'(U)$ also satisfy $u = u_1' + u_2'$. Then $w := u_1 - u_1' = u_2' - u_2$ belongs to $\mathcal{E}\mathcal{S}(U) \cap \mathcal{D}'\mathcal{E}'(U)$. This means that w is a C^∞ function with proper support with respect to π, that is, the map $\pi : \mathrm{supp}\, w \to \mathbb{R}^n$ is proper. Hence the integral $\int_{\mathbb{R}^d} w(x,t)\,dt$ of w as an element of
$\mathcal{E}\mathcal{S}(U)$ coincides with the one as an element of $\mathcal{D}'\mathcal{E}'(U)$.
This proves the well-definedness of the integral of u.

Example Let us consider the integral

$$v(x) = \int_0^\infty e^{-xt}\,dt = \int_{-\infty}^\infty e^{-xt} Y(t)\,dt$$

for $x \in U := \{x \in \mathbb{R} \mid x > 0\}$. Let $\chi(t)$ be a C^∞ function on $\mathbb{R}$ such that $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$.





Then we have

$$e^{-xt}Y(t) = e^{-xt}\chi(t)Y(t) + e^{-xt}(1-\chi(t))Y(t)$$

with $u_1 := e^{-xt}\chi(t)Y(t)$ belonging to $\mathcal{D}'\mathcal{E}'(U)$. In fact it is a measurable function with support in $\mathbb{R} \times [0,2]$).
$u_2 := e^{-xt}(1-\chi(t))Y(t)$ belongs to $\mathcal{ES}(U)$ since $u_2(x,t) = e^{-xt}$ for $t \geq 2$ and $u_2(x,t) = 0$ for $t \leq 1$.

Since u_1 and u_2 are both mesurable, we have, for any $\varphi \in C^\infty(\mathbb{R})$,

$$\begin{aligned}\langle u, \varphi \rangle &= \int_{-\infty}^{\infty} u_1(x,t)\varphi(t)\,dt + \int_{-\infty}^{\infty} u_2(x,t)\varphi(t)\,dt \\ &= \int_0^{\infty} e^{-xt}\chi(t)\varphi(t)\,dt + \int_0^{\infty} e^{-xt}(1-\chi(t))\varphi(t)\,dt \\ &= \int_0^{\infty} e^{-xt}\varphi(t)\,dt.\end{aligned}$$

Differentiation under the integral sign

Let U be an open set of $\mathbb{R}^n$ and $u = u(x,t)$ be an element of $\mathcal{ES}(U) + \mathcal{D}'\mathcal{E}'(U)$. Then for any $P = P(x, \partial_x) \in D_n$, we have

$$P(x,\partial_x) \int_{\mathbb{R}^d} u(x,t)\,dt = \int_{\mathbb{R}^d} P(x,\partial_x) u(x,t)\,dt.$$

Proof: The case $u \in \mathcal{ES}(U)$ is the classical differentiation under the integral sign. So, assume $u(x,t)$ belongs to $\mathcal{D}'\mathcal{E}'(U)$. We have only to prove the equality for $P = x_i$ and $P = \partial_{x_i}$.

Let $\varphi(x) \in C^\infty(U)$. Then

$$\left\langle \partial_{x_i} \int_{\mathbb{R}^d} u(x,t)\,dt,\ \varphi(x) \right\rangle = -\left\langle \int_{\mathbb{R}^d} u(x,t)\,dt,\ \partial_{x_i}\varphi(x) \right\rangle$$
$$= -\langle u(x,t), (\partial_{x_i}\varphi(x))1(t)\rangle = -\langle u(x,t), \partial_{x_i}(\varphi(x)1(t))\rangle$$
$$= \langle \partial_{x_i} u(x,t), \varphi(x)1(t)\rangle = \left\langle \int_{\mathbb{R}^d} \partial_{x_i} u(x,t)\,dt,\ \varphi(x) \right\rangle$$

and

$$\left\langle x_i \int_{\mathbb{R}^d} u(x,t)\,dt,\ \varphi(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} u(x,t)\,dt,\ x_i\varphi(x) \right\rangle$$
$$= \langle u(x,t), x_i\varphi(x)1(t)\rangle = \left\langle \int_{\mathbb{R}^d} x_i u(x,t)\,dt,\ \varphi(x) \right\rangle.$$