Plan of the course

1st lecture **Introduction:** Aim and an example
Chapter 1: Basics of D-modules

2nd lecture **Chapter 2:** Gröbner bases in the ring of differential operators
Chapter 3: Distributions as generalized functions

3rd lecture **Chapter 4:** D-module theoretic integration algorithm
Chapter 5: Integration over the domain defined by polynomial inequalities

2. Gröbner bases in the ring of differential operators

References of Chapter 2

- M. Saito, B. Sturmfels, N. Takayama: *Gröbner Deformations of Hypergeometric Differential Equations*, Springer 2000.

2.1. Definitions and basic properties

Recall that $\xi = (\xi_1, \dots, \xi_n)$ are the commutative variables corresponding to derivations $\partial_1 = \partial_{x_1}, \dots, \partial_n = \partial_{x_n}$. Let

$$M(x, \xi) = \{x^\alpha \xi^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$$

be the set of the monomials in $K[x, \xi]$. A total order $\prec$ on $M(x, \xi)$ is called a *monomial order* if it satisfies for $u, v, w \in M(x, \xi)$

1. $u \prec v \quad \Rightarrow \quad uw \prec vw$;
2. $1 \prec x_i \xi_i$ for any $i = 1, \dots, n$.

A monomial order is called a *term order* if

3. $1 \prec x^\alpha \xi^\beta$ for any $(\alpha, \beta) \in \mathbb{N}^{2n} \setminus \{(\mathbf{0}, \mathbf{0})\}$.

This is equivalent to the condition that the monomial order $\prec$ be a well-ordering.

Now fix a monomial order $\prec$. For a nonzero element $P = \sum_{\alpha,\beta} a_{\alpha\beta} x^{\alpha} \partial^{\beta}$ of D_n, its initial monomial $\mathrm{in}_{\prec}(P)$ is defined to be the maximum nonzero monomial

$$\mathrm{in}_{\prec}(P) = \max{}_{\prec}\{x^{\alpha}\xi^{\beta} \mid a_{\alpha\beta} \neq 0\}$$

of $P(x, \xi)$ with respect to $\prec$.

Note that $\mathrm{in}_{\prec}(P)$ belongs to $K[x, \xi]$ instead of D_n so that monomial ideals make sense.

By using the Leibniz formula and the conditions (1) and (2), we can verify that $\mathrm{in}_\prec(PQ) = \mathrm{in}_\prec(P)\mathrm{in}_\prec(Q) = \mathrm{in}_\prec(QP)$ holds in $K[x, \xi]$ for nonzero $P, Q \in D_n$.

Definition (Gröbner basis)

Let I be a left ideal of D_n. A finite subset G of I is called a Gröbner basis of I with respect to a monomial order $\prec$ if

1. G generates I as a left ideal;
2. $\mathrm{in}_\prec(G) := \{\mathrm{in}_\prec(P) \mid P \in G\}$ generates the monomial ideal $\mathrm{in}_\prec(I)$ in $K[x, \xi]$ which is generated by the set $\{\mathrm{in}_\prec(P) \mid P \in I,\ P \neq 0\}$.

Proposition

For any left ideal I of D_n, and any monomial order $\prec$, there exists a Gröbner basis G of I with respect to $\prec$.

Proof: Let G be a finite generating set of I. Since $\mathrm{in}_\prec(I)$ is a monomial ideal of $K[x, \xi]$, there exists a finite set G' of I such that $\{\mathrm{in}_\prec(P) \mid P \in G'\}$ generates $\mathrm{in}_\prec(I)$. Then $G \cup G'$ is a Gröbner basis of I with respect to $\prec$.

For a term order, we can compute a Gröbner basis of I by using division and Buchberger's criterion applied to D_n.

Now let $w \in \mathbb{Z}^{2n}$ be a weight vector. A monomial order $\prec$ on $M(x, \xi)$ is *adapted* to w if

$$x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'} \ \Rightarrow \ \langle w, (\alpha, \beta) \rangle \leq \langle w, (\alpha', \beta') \rangle.$$

There exists a term order that is adapted to w if and only if $w_i \geq 0$ for any $i = 1, \ldots, n$.

For an arbitrary monomial order $\prec$, define another monomial order $\prec_w$ by

$$x^\alpha \xi^\beta \prec_w x^{\alpha'} \xi^{\beta'} \ \Leftrightarrow \ \langle w, (\alpha, \beta) \rangle < \langle w, (\alpha', \beta') \rangle$$
$$\text{or } (\langle w, (\alpha, \beta) \rangle = \langle w, (\alpha', \beta') \rangle \text{ and } x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'}).$$

Then $\prec_w$ is adapted to w.

Recall: For a weight vector $w \in \mathbb{Z}^{2n}$, the w-filtration of D_n is

$$F_k^w(D_n) = \{P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} = 0 \text{ if } \langle w, (\alpha, \beta) \rangle > k\}.$$

The associated graded ring is

$$\mathrm{gr}^w(D_n) := \bigoplus_{k \geq 0} \mathrm{gr}_k^F(D_n), \qquad \mathrm{gr}_k^F(D_n) := F_k^w(D_n)/F_{k-1}^w(D_n).$$

If $P \in F_k(D_n) \setminus F_{k-1}(D_n)$, let $\overline{P}$ be the residue class in $\mathrm{gr}_k^w(D_n)$.

Proposition

Let I be a left ideal of D_n and G be a Gröbner basis of I with respect to a monomial order $\prec$ which is adapted to a weight vector w. Then $\mathrm{gr}(G) := \{\overline{P} \mid P \in G\}$ generates the w-graded left ideal

$$\mathrm{gr}(I) := \bigoplus_{k \in \mathbb{Z}} (I \cap F_k^w(D_n))/(I \cap F_{k-1}^w(D_n))$$

of $\mathrm{gr}^w(D_n)$. Such G is called a *w-involutive basis* of I.

Computing $\mathrm{Char}(M)$

Corollary

If $P_1, \dots, P_r$ are a Gröbner basis of a left ideal I of D_n with respect to a term order which is adapted to $(\mathbf{0}, \mathbf{1})$, then

$$\mathrm{Char}(D_n/I) = \{(x, \xi) \in K^{2n} \mid \sigma(P_i)(x, \xi) = 0 \ (1 \leq \forall i \leq r)\}.$$

Example As a left D_n-module, $K[x] \cong D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$. Since $\partial_1, \dots, \partial_n$ are a Gröbner basis with respect to any term order which is adapted to $(\mathbf{0}, \mathbf{1})$, and $\sigma(\partial_i) = \xi_i$, we have

$$\mathrm{Char}(K[x]) = \{(x, \xi) \in K^{2n} \mid \xi = 0\},$$

and it follows that $\mathrm{Sing}(K[x]) = \emptyset$.

2.2. Homogenization trick

For a monomial order $\prec$ in which 1 is not the smallest element, the division algorithm cannot be performed directly. To bypass this difficulty, we introduce the $(\mathbf{1},\mathbf{1})$-homogenized ring. First, recall the Rees algebra

$$R^{(\mathbf{1},\mathbf{1})}(D_n) = \bigoplus_{k\in\mathbb{Z}} F_k^{(\mathbf{1},\mathbf{1})}(D)T^n$$

of D_n with respect to the $(\mathbf{1},\mathbf{1})$-filtration.

Let $D_n^{(h)}$ be the K-vector space with the basis $\{x^\alpha\partial^\beta h^k \mid \alpha,\beta\in\mathbb{N}^n,\ k\in\mathbb{N}\}$, where h is a new indeterminate.

Define a K-isomorphism $\Psi : R^{(\mathbf{1},\mathbf{1})}(D_n) \to D_n^{(h)}$ by

$$\Psi(x^\alpha \partial^\beta T^k) = x^\alpha \partial^\beta h^{k-|\alpha|-|\beta|}.$$

Note that $x^\alpha \partial^\beta T^k \in R^{(\mathbf{1},\mathbf{1})}(D_n)$ means $|\alpha| + |\beta| \leq k$.

We can make $D_n^{(h)}$ a graded K-algebra by using the graded K-algebra structure of $R^{(\mathbf{1},\mathbf{1})}(D_n)$ via Ψ.

Let us call this $D^{(h)}$ the *homogenized Weyl algebra*, which was introduced, in connection with Gröbner bases, by Takayama and Assi-Castro-Granger independently. In fact, $D^{(h)}$ was implemented by Takayama in his computer algebra system Kan as early as 1994.

The image of $F_k^{(\mathbf{1},\mathbf{1})}(D_n)$ by Ψ consists of the elements of $D_n^{(h)}$ which are homogeneous of degree k in x, ∂, h. For an element P of D_n, we set

$$P^{(h)} := \Psi(PT^k) \quad \text{with } k := \mathrm{ord}_{(\mathbf{1},\mathbf{1})} P,$$

which is called the (($\mathbf{1}, \mathbf{1}$)-) homogenization of P. For example, since $\partial_i x_j T^2 = (x_i \partial_j + \delta_{ij}) T^2$ holds in $R^{(\mathbf{1},\mathbf{1})}(D_n)$, we have

$$\partial_i x_j = \Psi(\partial_i x_j T^2) = \Psi(x_i \partial_j T^2) + \delta_{ij} \Psi(T^2) = x_i \partial_j + \delta_{ij} h^2.$$

More generally, for elements P, Q of $D_n^{(h)}$, let $P(x, \xi, h)$ and $Q(x, \xi, h)$ be their total symbols defined in a similar manner as in D_n. Then the total symbol of $R := PQ$ is given by

$$R(x, \xi, h) = \sum_{\nu \in \mathbb{N}^n} \frac{h^{2\nu}}{\nu!} \left(\frac{\partial}{\partial \xi}\right)^{\nu} P(x, \xi, h) \cdot \left(\frac{\partial}{\partial x}\right)^{\nu} Q(x, \xi, h).$$

Now let $\prec$ be an arbitrary monomial order on $M(x, \xi)$. We define a monomial order $\prec_h$ on $M(x, \xi, h)$ by

$$x^{\alpha}\xi^{\beta}h^{j} \prec_h x^{\alpha'}\xi^{\beta'}h^{k} \iff |\alpha| + |\beta| + j < |\alpha'| + |\beta'| + k$$
$$\text{or } (|\alpha| + |\beta| + j = |\alpha'| + |\beta'| + k \text{ and } x^{\alpha}\xi^{\beta} \prec x^{\alpha'}\xi^{\beta'}).$$

Then $\prec_h$ is clearly a term order. Hence the division and the Buchberger algeorithm works with $\prec_h$ in $D_n^{(h)}$.

Theorem (Takayama, Assi-Castro-Granger)

Let I be the left ideal of D_n generated by nonzero $P_1, \ldots, P_r$. and $\prec$ an arbitrary monomial order on $M(x, \xi)$. Let J be a left ideal of $D_n^{(h)}$ generated by $P_1^{(h)}, \ldots, P_r^{(h)}$. Let $\{Q_1', \ldots, Q_l'\}$ be a Gröbner basis of J with respect to $\prec_h$, which can be computed by Buchberger's algorithm.

Set $Q_i := Q_i'|_{h=1}$ for $i = 1, \ldots, r$. Then $\{Q_1, \ldots, Q_r\}$ is a Gröbner basis of I with respect to $\prec$. Moreover, for any nonzero element P of I, there exist $U_1, \ldots, U_l \in D_n$ such that

$$P = U_1 Q_1 + \cdots + U_l Q_l, \qquad \mathrm{in}_\prec(U_i Q_i) \preceq \mathrm{in}_\prec P \text{ if } U_i Q_i \neq 0.$$

In particular, if $\prec$ is adapted to w, then for any $k \in \mathbb{Z}$, we have

$$I \cap F_k^w(D_n) = F_{k-m_1}^w(D_n)\overline{P}_1 + \cdots + F_{k-m_l}^w(D_n)\overline{P}_l \text{ with } m_i := \mathrm{ord}_w P_i.$$