Algorithms for D-modules, integration, and generalized functions

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Plan of the course

1st lecture	**Introduction:** Aim and an example **Chapter 1:** Basics of D-modules
2nd lecture	**Chapter 2:** Gröbner bases in the ring of differential operators **Chapter 3:** Distributions as generalized functions
3rd lecture	**Chapter 4:** D-module theoretic integration algorithm **Chapter 5:** Integration over the domain defined by polynomial inequalities

Introduction

A *D*-module is a system of linear (partial or ordinary) differential equations with polynomial (or analytic) coefficients.

There is a special class of D-modules which are called holonomic, the solution spaces of which are finite dimensional vector spaces.

A holonomic function is a differentiable or generalized function which is a solution of a holonomic system. For example, $\exp(f) = e^f$ is a holonomic function for any polynomial $f = f(x_1, \ldots, x_n)$.

Aim: To find a holonomic system which the integral of a holonomic function satisfies. The integration is to be performed over a domain defined by polynomial inequalities.

Example

Let us consider the integral

$$v(x) = \int_0^1 e^{xy}\,dy = \frac{e^x - 1}{x}$$

and assume that we did not know the answer. The integrand $u(x, y) := e^{xy}$ satisfies a holonomic system

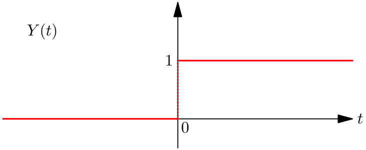
$$(\partial_x - y)u(x, y) = (\partial_y - x)u(x, y) = 0$$

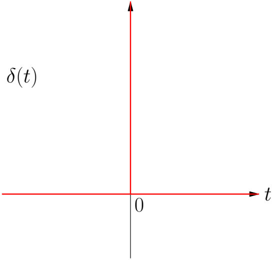
with $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$.

In order to apply integration algorithms for holonomic functions, we have to get rid of the boundary $y = 0, 1$.

The Heaviside function and the delta function

The Heaviside function $Y(t)$ is defined by $Y(t) = 1$ for $t > 0$ and $Y(t) = 0$ for $t < 0$. (Never mind the value at $t = 0$.)





$Y(t)$ is discontinuous at $t = 0$ and its derivative $Y'(t)$ as a generalized function coincides with Dirac's delta function $\delta(t)$.

$\delta(t)$ vanishes outside of $t = 0$ and $t\delta(t) = 0$ holds everywhere in $\mathbb{R}$.

Olvier Heaviside (1850–1925)

> *a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis.*

an excerpt from Wikipedia

Example (continued)

We rewrite the integral as

$$v(x) = \int_{-\infty}^{\infty} e^{xy} Y(y) Y(1-y)\, dy.$$

The new integrand $\tilde{u}(x,y) := e^{xy} Y(y) Y(1-y)$ satisfies a holonomic system

$$y(y-1)(\partial_y - x)\tilde{u}(x,y) = (\partial_x - y)\tilde{u}(x,y) = 0. \tag{1}$$

In fact one has

$$y(y-1)(\partial_y - x)(e^{xy} Y(y) Y(1-y)) = y(y-1)e^{xy}(\delta(y) - \delta(y-1)) = 0.$$

The integration algorithm applied to (1) outputs an answer

$$(x\partial_x^2 - (x-2)\partial_x - 1)v(x) = 0.$$

By rewriting the differential operator on the left-hand side as

$$x\partial_x^2 - (x-2)\partial_x - 1 = \partial_x(\partial_x - 1)x,$$

we know that $v(x) = x^{-1}(C_1 e^x + C_2)$ holds for $x \neq 0$ with some constants C_1, C_2. We get $C_1 = 1$ and $C_2 = -1$ from the fact that $v(x)$ is continuous at $x = 0$ with $v(0) = 1$. Thus we get

$$v(x) = \frac{e^x - 1}{x}.$$

General integrals

For a holonomic function $u(x_1, \dots, x_n)$, consider the integral

$$v(x_1, \dots, x_{n-d}) = \int_{D(x_1,\dots,x_{n-d})} u(x_1, \dots, x_n)\, dx_{n-d+1} \cdots dx_n,$$

$$D(x_1, \dots, x_{n-d}) := \{(x_{n-d+1}, \dots, x_n) \in \mathbb{R}^d \mid f_j(x_1, \dots, x_n) \geq 0 \quad (1 \leq j \leq m)\}$$

with polynomials $f_1, \dots, f_m$ with real coefficients. We rewrite it as

$$v(x_1, \dots, x_{n-d}) = \int_{\mathbb{R}^d} u(x_1, \dots, x_n) Y(f_1) \cdots Y(f_m)\, dx_{n-d+1} \cdots dx_n$$

and apply the D-module theoretic integration algorithm. Since generalized functions are involved, we cannot use differential operators with rational function coefficients.

For example, $x\delta(x) = 0$ DOES NOT imply

$$\delta(x) = \frac{1}{x}(x\delta(x)) = \frac{1}{x}0 = 0.$$

Hence we must work exclusively with differential operators with *polynomial coefficients*, that is, in the framework of (algebraic) D-modules.

We also need an algorithm to compute a holonomic system for the product $uY(f_1)\cdots Y(f_m)$ as a generalized function.

(The end of Introduction)

1. Basics of D-modules

References of Chapter 1

- J.-E. Björk: *Rings of Differential Operators*, North-Holland, 1979.
- M. Kashiwara: *Systems of Microdifferential Equations* (Progress in Math.), Birkhauser, 1983.
- M. Saito, B. Sturmfels, N. Takayama: *Gröbner Deformations of Hypergeometric Differential Equations*, Springer 2000.
- R. Hotta: *Introduction to Algebra — Groups and Modules* (in Japanese), Shokabo, 1987
- T. Tanisaki: *Non-commutative Algebra* (in Japanese), Iwanami, 2006

1.1 The ring of differential operators

Let K be a field of characteristic zero. In this course, we assume mostly that K is $\mathbb{C}$, the field of complex numbers.

Let $K[x] := K[x_1, \dots, x_n]$ be the ring of polynomials in indeterminates $x = (x_1, \dots, x_n)$ with coefficients in K.

A *derivation* $\theta : K[x] \to K[x]$ is a K-linear map that satisfies

$$\theta(fg) = \theta(f)g + f\theta(g) \qquad (\forall f, g \in K[x]).$$

The set $\mathrm{Der}_K K[x]$ of the derivations constitutes a $K[x]$-module.

For $i = 1, \dots, n$, define a derivation ∂_i by the partial derivative

$$\partial_i = \partial_{x_i} \ : \ K[x] \ni f \longmapsto \frac{\partial f}{\partial x_i} \in K[x].$$

Then $\partial_1, \dots, \partial_n$ are a $K[x]$-basis of $\mathrm{Der}_K K[x]$.

In fact, if $\theta \in \mathrm{Der}_K K[x]$, then it is easy to see that

$$\theta = \theta(x_1)\partial_1 + \cdots + \theta(x_n)\partial_n.$$

Let $\mathrm{End}_K K[x]$ be the K-algebra consisting of the K-linear endomorphisms of $K[x]$.

The ring D_n is defined as the K-subalgebra of $\mathrm{End}_K K[x]$ that is generated by $K[x]$ and $\mathrm{Der}_K K[x]$, or equivalently, by $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$.

An element $a = a(x)$ of $K[x]$ is regarded as an element of D_n as the multiplication $f \mapsto af$ for $f \in K[x]$. With this identification, D_n contains $K[x]$ as a subring.

D_n is called the *ring of differential operators* in the variables $x = (x_1, \dots, x_n)$ with polynomial coefficients, or, more simply, the n-th*Weyl algebra* over K.

D_n is a non-commutative K-algebra. In fact, for $a \in K[x]$ regarded as an element of D_n, the product in D_n satisfies

$$\partial_i a = a\partial_i + \partial_i(a) = a\partial_i + \frac{\partial a}{\partial x_i} \quad (i = 1, \dots, n).$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} = \{0, 1, 2, \dots\}$, we use the notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Then an element P of D_n is uniquely written in a finite sum

$$P = P(x, \partial) = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta \quad (a_{\alpha,\beta} \in K), \tag{2}$$

which is called the *normal form* of P.

Introducing commutative indeterminates $\xi = (\xi_1, \dots, \xi_n)$ correspoinding to ∂, we associate with this P a polynomial

$$P(x, \xi) := \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \xi^\beta \ \in K[x, \xi] = K[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$$

and call it the *total symbol* of P. Note that P must be in the normal form when ξ is substituted for ∂.

By this correspondence, D_n is isomorphic to $K[x, \xi]$ as a K-vector space but not as a ring, of course.

The product $R = PQ$ in D_n can be effectively computed by using the *Leibniz formula*

$$R(x, \xi) = \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} \left(\frac{\partial}{\partial \xi} \right)^{\nu} P(x, \xi) \cdot \left(\frac{\partial}{\partial x} \right)^{\nu} Q(x, \xi)$$

where $\nu! = \nu_1! \cdots \nu_n!$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$.

Example

Set $n = 1$ and write $x = x_1$ and $\partial = \partial_1$. Consider the product $R := \partial^m x^m$ with a non-negative integer m. Since the total symbol of ∂^m and x^m are ξ^m and x^m respectively, the Leibniz formula gives the total symbol $R(x, \xi)$ as

$$
\begin{aligned}
R(x,\xi) &= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left(\frac{\partial}{\partial \xi}\right)^{\nu} \xi^m \cdot \left(\frac{\partial}{\partial x}\right)^{\nu} x^m \\
&= \sum_{\nu=0}^{m} \frac{1}{\nu!} \{m(m-1)\cdots(m-\nu+1)\}^2 \xi^{m-\nu} x^{m-\nu}.
\end{aligned}
$$

This implies

$$
\partial^m x^m = \sum_{\nu=0}^{m} \frac{1}{\nu!} \{m(m-1)\cdots(m-\nu+1)\}^2 x^{m-\nu} \partial^{m-\nu}.
$$

Exercise 1.1. Prove the Leibniz formula.

Exercise 1.2. Set $n = 1$ and $x = x_1$, $\partial = \partial_1$. For a positive integer m, prove

$$x^m \partial^m = x\partial(x\partial - 1) \cdots (x\partial - m + 1).$$

The D-module formalism

Given $P_1, \dots, P_r \in D_n$, let us consider a system of linear (partial or ordinary) differential equations

$$P_1 u = \cdots = P_r u = 0 \tag{3}$$

for an unknown function u. Let $I := D_n P_1 + \cdots + D_n P_r$ be the left ideal of D_n generated by $P_1, \dots, P_r$. Then (3) is equivalent to

$$Pu = 0 \qquad (\forall P \in I).$$

Here we suppose that the unknown function u belongs to some 'function space' $\mathcal{F}$ which is a left D_n-module.

For $\mathcal{F}$ to be a left D_n-module, it is necessary that any function f belonging to $\mathcal{F}$ be infinitely differentiable and multiplication af by an aribrtrary polynomial $a \in K[x]$ make sense.

Here are examples of 'function spaces':

Example By the definition, $K[x]$ has a natural structure of left D_n-module since D_n is a subalgebra of $\mathrm{End}_K K[x]$. So $K[x]$ has two structures: a subring of D_n and a left D_n-module. Hence for $f \in K[x]$ and $P \in D_n$, Pf has two meanings:

- Pf as the product in D_n with f regarded as an element of the subring $K[x]$ of D_n.
- Pf as the action of P on the element f of the left D_n-module $K[x]$. In other words, we regard f as a function subject to derivations.

This might cause some confusion.

In "*Gröbner Deformations of Hypergeometric Differential Equations*" by Saito, Sturmfels, Takayama (Springer, 2000), the action of P on an element f of a left D_n-module is denoted conspicuously by $P \bullet f$ for distinction, which is new, I guess, both to analysts and algebraists.

I would like to write, only if needed, $Pf = P(f)$ to clarify the action of P on f, and $Pf = P \cdot f$ to emphasize that it be the product in D_n, following the traditional notation in D-module theory.

Example The field $K(x) = K(x_1, \dots, x_n)$ of rational functions has a natural structure of left D_n-module. For a point $p = (p_1, \dots, p_n)$ of the affine space K^n, the set of regular functions at p, i.e. the elements of $K(x)$ whose denominators do not vanish at p also has a natural structure of D_n-modules. More generally, for a multiplicative subset S of $K[x]$, the localization $K[x][S^{-1}]$ is also a left D_n-module.

Example Set $K = \mathbb{C}$. Let $C^\infty(U)$ be the set of the complex-valued C^∞ functions on an open set U of the n-dimensional real Euclidean space $\mathbb{R}^n$. Then each ∂_i acts on $C^\infty(U)$ as differentiation. This makes $C^\infty(U)$ a left D_n-module. Let $C_0^\infty(U)$ be the set of C^∞ functions on U with compact support. More precisely, $f \in C^\infty(U)$ belongs to $C_0^\infty(U)$ if and only if there is a compact subset K of U such that $f(x) = 0$ for any $x \in U \setminus K$. Then $C_0^\infty(U)$ is a left D_n-submodule of $C^\infty(U)$.

Other examples of such $\mathcal{F}$ are the set $\tilde{\mathcal{O}}(U)$ of possibly multi-valued analytic functions on an open subset U of $\mathbb{C}^n$, the set $\mathcal{D}'(U)$ of the Schwartz distributions on an open subset U of $\mathbb{R}^n$, and the set $S'(\mathbb{R}^n)$ of tempered distributions, which shall be introduced later, as well as the set $\mathcal{B}(U)$ of the hyperfunctions (of Mikio Sato) on an open subset U of $\mathbb{R}^n$.

Now for a left ideal I of D_n, consider the residue module $M := D_n/I$, which is a left D_n-module generated by the residue class $\overline{1}$ of $1 \in K[x] \subset D_n$. Fix a left D_n-module $\mathcal{F}$ as your favorite function space. A map $\varphi : M \to \mathcal{F}$ is D_n-linear, or a D_n-homomorphism, if

$$\varphi(u+v) = \varphi(u)+\varphi(v), \quad \varphi(Pu) = P\varphi(u) \quad (\forall u, v \in M,\ \forall P \in D_n).$$

Let $\mathrm{Hom}_{D_n}(M, \mathcal{F})$ be the set of the D_n-homomorphisms of M to $\mathcal{F}$, which is a K-vector space.

Since M is generated by $\overline{1}$ as left D_n-module, $\varphi \in \mathrm{Hom}_{D_n}(M, \mathcal{F})$ is uniquely determined by $\varphi(\overline{1})$.

On the other hand, for φ to be well-defined, it is necessary and sufficient that $\varphi(\overline{1})$ be annihilated by I, i.e., $P\varphi(\overline{1}) = 0$ for any $P \in I$.

In conclusion, we have an isomorphism as K-vector space

$$\mathrm{Hom}_{D_n}(M, \mathcal{F}) \ni \varphi \overset{\sim}{\longmapsto} \varphi(\overline{1}) \in \{f \in \mathcal{F} \mid Pf = 0 \quad (\forall P \in I)\}.$$

We started with a left ideal I of D_n generated by given $P_1, \ldots, P_n \in D_n$ and considered a left D_n-module $M = D_n/I$. Note that

$$I = \mathrm{Ann}_{D_n}\overline{1} = \{P \in D_n \mid P\overline{1} = 0 \in M\}.$$

We can argue in the reverse order:
Let M be a finitely generated left D_n-module and let $u_1, \dots, u_m \in M$ be generators of M, i.e. assume that for any $u \in M$, there exist $P_1, \dots, P_m \in D_n$ such that $u = P_1 u_1 + \cdots + P_m u_m$. Set

$$N := \{(P_1, \dots, P_m) \in (D_n)^m \mid P_1 u_1 + \cdots + P_m u_m = 0\},$$

which is a left D_n-submodule of M.

Since D_n is left (and right) Noetherian ring (this can be proved by using a Gröbner basis in D_n), N is also finitely generated over D_n.

Hence there exist

$$Q_i = (Q_{i1}, \dots, Q_{im}) \in (D_n)^m \qquad (i = 1, \dots, r)$$

which generate N as left D_n-module.

Then we have an exact sequence of left D_n-modules

$$(D_n)^r \xrightarrow{\psi} (D_n)^m \xrightarrow{\varphi} M \longrightarrow 0, \qquad (4)$$

which is called a *presentation* of M. Here φ and ψ are D_n-homomorphisms defined by, for $P_i \in D_n$,

$$\varphi((P_1, \ldots, P_m)) = P_1 u_1 + \cdots + P_m u_m,$$
$$\psi((P_1, \ldots, P_r)) = \begin{pmatrix} P_1 & \cdots & P_r \end{pmatrix} \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix}$$

and $N = \ker \varphi = \mathrm{im}\, \psi$ holds.

From (4) we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{D_n}(M,\mathcal{F}) \xrightarrow{\varphi^*} \mathrm{Hom}_{D_n}((D_n)^m,\mathcal{F}) \xrightarrow{\psi^*} \mathrm{Hom}_{D_n}((D_n)^r,\mathcal{F}).$$

Since $\mathrm{Hom}_{D_n}((D_n)^m,\mathcal{F})$ is isomorphic to $\mathcal{F}^m$, this yields

$$0 \longrightarrow \mathrm{Hom}_{D_n}(M,\mathcal{F}) \xrightarrow{\varphi^*} \mathcal{F}^m \xrightarrow{\psi^*} \mathcal{F}^r.$$

Regarding the elements of $\mathcal{F}^m$ as column vectors, we have, for $h \in \mathrm{Hom}_{D_n}(M,\mathcal{F})$ and $f_1,\dots,f_m \in \mathcal{F}$,

$$\varphi^*(h) = \begin{pmatrix} h(u_1) \\ \vdots \\ h(u_m) \end{pmatrix}, \quad \psi^*\left(\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}\right) = \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

Hence $\mathrm{Hom}_{D_n}(M, \mathcal{F})$ is isomorphic to the solution space of the system of linear differential equations

$$\sum_{j=1}^{m} Q_{ij} f_j = 0 \qquad (i = 1, \dots, r)$$

for unknown functions $f_1, \dots, f_m \in \mathcal{F}$. Note that the generators $u_1, \dots, u_m$ of M also satisfy the same equations in M.

Example Let us consider $K[x]$ as a left D_n-module. Since D_n contains $K[x]$ as a subring, $K[x]$ is generated by 1 as a left D_n-module. For $P \in D_n$, there exist $Q_1, \dots, Q_n \in D_n$ and $r(x) \in K[x]$ such that

$$P = Q_1\partial_1 + \cdots Q_n\partial_n + r(x).$$

Then $P(1) = r(x)$ vanishes if and only if $r(x) = 0$. This implies $K[x] \cong D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$ and a presentation of $K[x]$ is given by

$$(D_n)^n \xrightarrow{\cdot^t(\partial_1,\dots,\partial_n)} D_n \xrightarrow{\varphi} K[x] \longrightarrow 0$$

with $\varphi(P) = P(1)$. We have

$$\mathrm{Hom}_{D_n}(K[x], \mathcal{F}) \cong \{f \in \mathcal{F} \mid \partial_1 f = \cdots \partial_n f = 0\} = K$$

for $\mathcal{F} = K[x], K(x), K[x]_p$, or for $\mathcal{F} = C^\infty(U)$ if $K \subset \mathbb{C}$.
Exercise 1.3 Confirm the formulae above for $\varphi*$ and ψ^*.

1.3. Weight vector and filtration

A weight vector w for D_n is an integer vector

$$w = (w_1, \ldots, w_n; w_{n+1}, \cdots, w_{2n}) \in \mathbb{Z}^{2n}$$

with the conditions $w_i + w_{n+i} \geq 0$ for $i = 1, \ldots, n$, which are necessary in view of the commutation relation $\partial_i x_i = x_i \partial_i + 1$ in D_n. For a nonzero differential operator P of the form (2), we define its *w-order* to be

$$\mathrm{ord}_w(P) := \max\{\langle w, (\alpha, \beta)\rangle = w_1\alpha_1 + \cdots + w_n\alpha_n + w_{n+1}\beta_1 + \cdots + w_{2n}\beta_n \mid a_{\alpha,\beta} \neq 0\}.$$

We set $\mathrm{ord}_w(0) := -\infty$.

A weight vector w induces the *w-filtration*

$$F_k^w(D_n) := \{P \in D_n \mid \mathrm{ord}_w(P) \le k\} \quad (k \in \mathbb{Z})$$

on the ring D_n. This filtration satisfies the properties:

$$F_k^w(D_n) \subset F_{k+1}^w(D_n), \quad \bigcup_{k\in\mathbb{Z}} F_k^w(D_n) = D_n,$$

$$1 \in D_0^w(D_n), \quad F_k^w(D_n)F_l^w(D_n) \subset F_{k+l}^w(D_n), \quad \bigcap_{k\in\mathbb{Z}} F_k^w(D_n) = \{0\}.$$

The *w-graded ring* associated with this filtration is defined to be

$$\mathrm{gr}^w(D_n) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^w(D_n), \qquad \mathrm{gr}_k^w(D_n) := F_k^w(D_n)/F_{k-1}^w(D_n).$$

If $w_i + w_{n+i} > 0$ holds for all $i = 1, \ldots, n$, then $\mathrm{gr}^w(D_n)$ is isomorphic to the polynomial ring $K[x, \xi]$ as a K-algebra since in the right-hand side of the Leibniz formula, the terms with $\nu \geq 1$ are of lower w-order than the term with $\nu = 0$.

The *Rees algebra* $R^w(D_n)$ associated with the w-filtration is defined by

$$R^w(D_n) := \bigoplus_{k\in\mathbb{Z}} F_k^w(D_n)T^k \ \subset \ D_n[T]$$

with an indeterminate T. We have isomorphisms

$$R^w(D_n)/(T-1)R^w(D_n) \cong D_n, \qquad R^w(D_n)/TR^w(D_n) \cong \mathrm{gr}(D_n) \tag{5}$$

as K-algebra. Note that D_n, $\mathrm{gr}^w(D_n)$, and $R^w(D_n)$ are left (and right) Noetherian rings. (This can be proved by using Gröbner bases.)

Let M be a left D_n-module. A family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of K-subspaces $F_k(M)$ of M is called a *w-filtration* if it satisfies

1. $F_k(M) \subset F_{k+1}(M) \quad$ for all $k \in \mathbb{Z}$;
2. $\displaystyle\bigcup_{k\in\mathbb{Z}} F_k(M) = M$;
3. $F_j^w(D_n)F_k(M) \subset F_{j+k}(M) \quad$ for all $j, k \in \mathbb{Z}$;
4. Each $F_k(M)$ is finitely generated as a left $F_0^w(D_n)$-module.

A w-filtation $\{F_k(M)\}$ is called *good* if it also satisfies

5. there exists $k_1 \in \mathbb{Z}$ such that $F_j^w(D_n)F_{k_1}(M) = F_{j+k_1}(M)$ for any $j \leq 0$.
6. there exists $k_2 \in \mathbb{Z}$ such that $F_j^w(D_n)F_{k_2}(M) = F_{j+k_2}(M)$ for any $j \geq 0$.

A w-filtration $\{F_k(M)\}$ is good if and only if the associated Rees module $R(M) := \bigoplus_{k\in\mathbb{Z}} F_k(M)T^k$ is finitely generated as a left $R^w(D_n)$-module.

For a w-filtration $\{F_k(M)\}$, let

$$\mathrm{gr}(M) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k(M), \qquad \mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M)$$

be the associated graded module, which is a left $\mathrm{gr}^w(D_n)$-module. If $\{F_k(M)\}$ is good, then $\mathrm{gr}(M)$ is finitely generated over $\mathrm{gr}^w(D_n)$.

Exercise 1.4 Prove the K-algebra isomorphisms (5).

Exercise 1.5 Show that a w-filtration $\{F_k(M)\}$ of M is good if and only if the associated Rees module is finitely generated over $R^w(D_n)$.

1.4. Holonomic D-module and characteristic variety

Following J. Bernstein, let us define the notion of holonomic system by using the weight vector $(\mathbf{1},\mathbf{1}) = (1,\ldots,;1,\ldots,1) \in \mathbb{Z}^{2n}$. Note that

$$\langle(\mathbf{1},\mathbf{1}),(\alpha,\beta)\rangle = |\alpha| + |\beta| = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n.$$

Let M be a finitely generated left D_n-module and $\{F_k(M)\}$ be a good $(\mathbf{1},\mathbf{1})$-filtration. Then each $F_k(M)$ is a finite dimensional K-vector space since $F_0^{(\mathbf{1},\mathbf{1})}(D_n) = K$, and $F_k(M) = \{0\}$ for sufficiently small k.

By considering $\mathrm{gr}(M)$ as a graded $K[x,\xi]$-algebra, we see that there exists a (Hilbert) polynomial $p(k)$ in k such that

$$p(k) = \dim_K F_k(M) = \sum_{j\leq k} \dim_K F_j(M)/F_{j-1}(M) \qquad (k >> 0).$$

The degree of $p(k)$ does not depend on the choice of a good $(\mathbf{1},\mathbf{1})$-filtration $\{F_k(M)\}$ and is called the *dimension* of the module M, which we denote by $\dim M$.

Let $d = \dim M$. Then the leading coefficent c_d of $p(k)$ is a positive rational number such that $d!c_d$ is an integer. We call $\mathrm{mult}\, M := d!c_d$ the *multiplicity* of M.

Theorem (Bernstein's inequality (around 1970))

$\dim M \geq n$ holds if $M \neq 0$.

Definition

A finitely generated left D_n-module M is called a *holonomic system* if $\dim M = n$ or $M = 0$.

Example Since

$$\dim_K F_k^{(\mathbf{1},\mathbf{1})}(D_n) = \binom{2n+k}{2n} = \frac{1}{(2n)!}k^{2n} + (\text{lower order terms in } k),$$

$\dim D_n = 2n$.

Example $K[x]$ is holonomic as a left D_n-module. In fact, set

$$F_k(K[x]) = \{f \in K[x] \mid \deg f \le k\} \quad (k \in \mathbb{Z}).$$

It is easy to see that

$$F_k^{(\mathbf{1},\mathbf{1})}(D_n)F_l(K[x]) = F_{l+k}(K[x])$$

holds for any integers $j, k \ge 0$. Hence $\{F_k(K[x])\}$ is a good $(\mathbf{1},\mathbf{1})$-filtration. It follows that $\dim K[x] = n$ from

$$\dim_K F_k(K[x]) = \binom{n+k}{n} = \frac{1}{n!}k^n + (\text{lower order terms in } k).$$

Let us recall another characterization of a holonomic system by using the weight vector $w = (\mathbf{0}, \mathbf{1}) = (0, \dots, 0; 1, \dots, 1)$. Let M be a finitely generated left D_n-module and $\{F_k(M)\}$ be a good $(\mathbf{0}, \mathbf{1})$-filtation of M. Then $\mathrm{gr}(M)$ is a finitely generated $K[x, \xi]$-module. Its support is an algebraic set of K^{2n} defined by

$$\mathrm{Supp}\,\mathrm{gr}(M) := \\ \{(p, q) \in K^n \times K^n \mid \mathrm{gr}(M)_{(p,q)} := K[x, \xi]_{(p,q)} \otimes_{K[x,\xi]} \mathrm{gr}(M) = 0\},$$

where $K[x, \xi]_{(p,q)}$ denotes the localization of $K[x, \xi]$ at (p, q), i.e., at the maximal ideal corresponding to (p, q). It can be proved that $\mathrm{Supp}\,\mathrm{gr}(M)$ is independent of a good $(\mathbf{0}, \mathbf{1})$-filtration $\{F_k(M)\}$ of M. $\mathrm{Supp}\,\mathrm{gr}(M)$ is called the *characteristic variety* of M and denoted by $\mathrm{Char}(M)$.

Theorem (Sato-Kashiwara-Kawai (1973), Gabber (1981))

Let K be an algebraically closed field of characteristic 0 and M be a finitely generated left D_n-module. If $M \neq 0$, then the dimension of each irreducible component of $\mathrm{Char}(M)$ is $\geq n$. More precisely, $\mathrm{Char}(M)$ is an involutive subset of the symplectic manifold (the cotangent bundle) $T^*\mathbb{C}^n = \mathbb{C}^{2n}$.

Theorem (Björk(1979))

Under the same assumption as the theorem above, dim M coincides with the maximum dimension of the irreducible components of $\mathrm{Char}(M)$.

Especially, M is holonomic if and only if the dimension of the characteristic variety is n or else $M = 0$.

Let P be a nonzero differential operator written in the form

$$P = P(x, \partial) = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta \quad (a_{\alpha,\beta} \in K)$$

and set $m := \mathrm{ord}_{(\mathbf{0},\mathbf{1})}(P)$. Then the *principal symbol* of P is the polynomial defined by

$$\sigma(P)(x, \xi) = \sum_{|\beta|=m} \sum_{\alpha} a_{\alpha,\beta} x^\alpha \xi^\beta.$$

This can be identified with the residue class of P in $\mathrm{gr}^{(\mathbf{0},\mathbf{1})}(D_n) \cong K[x, \xi]$. Note that $\sigma(P)(x, \xi)$ is homogeneous with respect to ξ.

If $M := D_n/I$ with a left ideal I of D_n, then we have, by the definition,

$$\mathrm{Char}(M) = \{(x, \xi) \in K^{2n} \mid \sigma(P)(x, \xi) = 0 \text{ for any } P \in I \setminus \{0\}\}.$$

In particular, if I is generated by $P_1, \ldots, P_r$, then we have

$$\mathrm{Char}(M) \subset \{(x, \xi) \in K^{2n} \mid \sigma(P_i)(x, \xi) = 0 \ (\forall i = 1, \ldots, r)\}.$$

Let $\pi : K^{2n} \ni (x, \xi) \mapsto x \in K^n$ be the projection. Then the *singular locus* of M

$$\mathrm{Sing}(M) := \pi(\mathrm{Char}(M) \setminus K^n \times \{0\})$$

is an algebraic set of K^n since $\mathrm{gr}(M)$ is homogeneous with respect to ξ. In particular, if M is holonomic, then $\mathrm{Sing}(M)$ is an algebraic set of codimension ≥ 1, or an empty set, since $\mathrm{Char}(M)$ is homogeneous with respect to ξ.

Example Let $P \in D_1$ with $P \neq 0$; i.e., let P be a linear ordinary differential operator with coefficients in $K[x] = K[x_1]$. Then P is written in the form

$$P = a_m(x)\partial^m + \cdots + a_0(x) \qquad (a_i(x) \in K[x],\ a_m(x) \neq 0).$$

Set $M = D_1/DP_1$. Then the dimension of the algebraic set

$$\begin{aligned}\mathrm{Char}(M) &= \{(x,\xi) \in K^2 \mid \sigma(P)(x,\xi) = a_m(x)\xi^m = 0\} \\ &= \{(x,\xi) \in K^2 \mid a_m(x) = 0\} \cup \{(x,0) \mid x \in K\}\end{aligned}$$

is one. Hence M is holonomic. $\mathrm{Sing}(M) = \{x \in K \mid a_m(x) = 0\}$.

Example As a left D_n-module, $K[x] \cong D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$.
Since $\sigma(\partial_i) = \xi_i$, we have

$$\mathrm{Char}(K[x]) \subset \{(x, \xi) \in K^{2n} \mid \xi = 0\},$$

and it follows that $\mathrm{Sing}(K[x]) = \emptyset$.
Since the dimension of $\mathrm{Char}(K[x])$ is $\geq n$, we have

$$\mathrm{Char}(K[x]) = \{(x, \xi) \in K^{2n} \mid \xi = 0\}.$$

This can be also proved as follows:
If $P \in D_n\partial_1 + \cdots + D_n\partial_n$, then $\sigma(P)(x, \xi)$ belongs to the ideal of $K[x, \xi]$ generated by $\xi_1, \ldots, \xi_n$.

Exercise 1.5. Let $K = \mathbb{C}$ and let $f(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. Consider a C^∞ function $e^{f(x)}$ on $\mathbb{R}^n$. Set $f_i = \partial_i(f)$ and

$$I := D_n(\partial_1 - f_1) + \cdots + D_n(\partial_n - f_n).$$

1. Show that $I = \mathrm{Ann}_{D_n} e^{f(x)} := \{P \in D_n \mid Pe^f(x) = 0\}$.
2. Show that $\mathrm{Hom}_{D_n}(M, C^\infty(\mathbb{R}^n)) \cong Ke^{f(x)}$.
3. Show that $M := D_n/I$ is holonomic by using the two definitions respectively.