Some D-module theoretic aspects of the local cohomology of a polynomial ring

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Local cohomology of the polynomial ring

Let K be a field of characteristic 0 and $R = K[x] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K.

For an ideal I of R and an integer j, the j-th local cohomology group $H^j_I(R)$ of R with support in I is defined as the j-th right derived functor of the functor Γ_I taking support in I. (It depends only on the radical $\sqrt{I}$ of I.)

For example, if $I = (f)$ with $f \in R \setminus \{0\}$, then $H^j_I(R) = 0$ for $j \neq 1$ and $H^1_I(R) = R[f^{-1}]/R$.

It is an R-module but is not finitely generated over R in general.

D-module structure of the local cohomology

Let $D_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ with $\partial_i = \partial/\partial x_i$ be the n-th Weyl algebra, or the ring of differential operators with polynomial coefficients in the variables $x_1, \dots, x_n$.

Then each $H^j_I(R)$ has a natural structure of left D_n-module. Moreover, it is finitely generated over D_n and is holonomic, i.e., its D-module theoretic dimension equals n if it is not zero.

So what might be of interest is the multiplicity of $H^1_{(f)}(R)$ as D-module in the sense of Bernstein.

(The dimension and the mulitiplicity will be explained later.)

An algorithm for the local cohomology

Let I be generated by $f_1, \ldots, f_d \in R$. Introducing new variables $t_1, \ldots, t_d$, set $\tilde{R} = K[t_1, \ldots, t_d, x_1, \ldots, x_n]$ and let J be the ideal of $\tilde{R}$ generated by $t_i - f_i$ $(i = 1, \ldots, d)$. Then

$$H_J^d(\tilde{R}) = D_{n+d}/N$$

with the left ideal N of D_{n+d} generated by

$$t_j - f_j \quad (j = 1, \ldots, d), \quad \partial_{x_i} + \sum_{k=1}^{d} \frac{\partial f_k}{\partial x_i} \partial_{t_k} \quad (i = 1, \ldots, n).$$

$H_I^j(R)$ equals the j-th cohomology of the D-module theoretic restriction of $H_J^d(\tilde{R})$ to the subspace $t_1 = \cdots = t_d = 0$ of K^{d+n}. This yields an algorithm to compute $H_I^j(R)$, combined with the restriction algorithm. (There is another algorithm due to U. Walther.)

Generators of the d-th local cohomology

Let $b(s)$ be the b-function, or the indicial polynomial, of $H^d_J(\tilde{R})$ with respect to the subspace $t_1 = \cdots = t_d = 0$ and let $-m$ be the smallest integer root of $b(s)$.

Then, in terms of the Cech cohomology, $H^d_I(R)$ is generated by the residue classes of $f_1^{-1-m_1} \cdots f_d^{-1-m_d} \in R_{f_1 \cdots f_d}$ with $m_1 + \cdots + m_d \leq m$ as a left D_n-module. Here $R_{f_1 \cdots f_d} = R[(f_1 \cdots f_d)^{-1}]$ denotes the localization of R by the multiplicative set $\{(f_1 \cdots f_d)^i \mid i \geq 0\}$.

$b(-s-d)$ coincides with the Bernstein-Sato polynomial $b_{(f_1,\ldots,f_d)}(s)$ of the variety defined by I in the sense of Budur-Mustata-Saito, which coincides with the classical Bernstein-Sato polynomial if $d = 1$.

They proved that $b_{(f_1,\ldots,f_d)}(s)$ is inedependent of the choice of the generators $f_1, \ldots, f_d$ of $\sqrt{I}$ as long as d is fixed.

Dimension and multiplicity of a D-module

For each integer k, set

$$F_k(D_n) = \{ \sum_{|\alpha|+|\beta| \le k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in K \}.$$

In particular, $F_k(D_n) = 0$ for $k < 0$ and $F_0(D_n) = K$. The filtration $\{F_k(D_n)\}_{k\in\mathbb{Z}}$ is called the Bernstein filtration on D_n.
Let M be a finitely generated left D_n-module. A family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of K-subspaces of M is called a Bernstein filtration of M if it satisfies

1. $F_k(M) \subset F_{k+1}(M) \quad (\forall k \in \mathbb{Z}), \qquad \bigcup_{k\in\mathbb{Z}} F_k(M) = M$
2. $F_j(D_n)F_k(M) \subset F_{j+k}(M) \quad (\forall j, k \in \mathbb{Z})$
3. $F_k(M) = 0$ for $k \ll 0$

Moreover, $\{F_k(M)\}$ is called a good Bernstein filtration if

1. $F_k(M)$ is finite dimensional over K for any $k \in \mathbb{Z}$.
2. $F_j(D_n)F_k(M) = F_{j+k}(M)$ $(\forall j \geq 0)$ holds for $k \gg 0$.

Then there exists a (Hilbert) polynomial
$h(T) = h_d T^d + h_{d-1} T^{d-1} + \cdots + h_0 \in \mathbb{Q}[T]$ such that

$$\dim_K F_k(M) = h(k) \quad (k \gg 0)$$

and $d!h_d$ is a positive integer.
The leading term of $h(T)$ does not depend on the choice of a good Bernstein filtration $\{F_k(M)\}$. So J. Bernstein defined

- $\dim M := d = \deg h(T)$ (the dimension of M).
- $\mathrm{mult}\, M := d!h_d$ (the multiplicity of M).

Basic examples

- D_n: Since

$$\dim_K F_k^{(\mathbf{1},\mathbf{1})}(D_n) = \binom{2n+k}{2n} = \frac{1}{(2n)!}k^{2n} + (\text{lower order terms in } k)$$

we have $\dim D_n = 2n$ and $\mathrm{mult}\, D_n = 1$.

- $R = K[x]$: Set

$$F_k(R) = \{f \in R \mid \deg f \leq k\} \quad (k \in \mathbb{Z}).$$

Then $\{F_k(R)\}$ is a good Bernstein filtration of $K[x]$. Since

$$\dim_K F_k(R) = \binom{n+k}{n} = \frac{1}{n!}k^n + (\text{lower order terms in } k),$$

we have $\dim R = n$ and $\mathrm{mult}\, R = 1$. In particular, R is a holonomic D_n-module.

Basic facts on dimension and multiplicity

Let M be a finitely generated left D_n-module.

- If $M \neq 0$, then $n \leq \dim M \leq 2n$ (Bernstein's inequality, 1970). M is said to be holonomic if $M = 0$ or $\dim M = n$. R is a holonomic D_n-module.
- If M is holonomic, then $\mathrm{length}\, M \leq \mathrm{mult}\, M$, where $\mathrm{length}\, M$ is the length of M as a left D_n-module.
- $\dim M$ and $\mathrm{length}\, M$ are invariants of M as left D_n-module.
- $\mathrm{mult}\, M$ is invariant under affine (i.e., linear transformations + shifting) coordinate transformations of K^n.
- If M is holonomic, then $H^j_I(M)$ is a holonomic D_n-module for any ideal I of R and for any integer j.

The D-module for f^s and the b-function

Let $f \in R$ be a nonzero polynomial and s be an indeterminate. Set

$$N := D_n[s]f^s = D_n[s]/\mathrm{Ann}_{D_n[s]} f^s \subset R[f^{-1}, s]f^s,$$

where f^s is regarded as a free geneartor of $R[f^{-1}, s]f^s$. Then N has a natural structure of left $D_n[s]$-module induced by the differentiation

$$\partial_i(f^s) = s\frac{\partial f}{\partial x_i} f^{-1} f^s \quad (i = 1, \dots, n).$$

(However, N is not holonomic as left D_n-module.)
The b-function, or the Bernstein-Sato polynomial $b_f(s)$ of f is the monic polynomial in s of the least degree such that

$$P(s)f^{s+1} = b_f(s)f^s \quad (\exists P(s) \in D_n[s]).$$

There are algorithms for computing $\mathrm{Ann}_{D_n} f^s$ and $b_f(s)$ by using Gröbner bases in the ring of differential operators.

The D-module for f^λ with $\lambda \in K$

For $\lambda \in K$, set

$$N_\lambda := N/(s-\lambda)N = D_n f^\lambda \qquad (f^\lambda := f^s \mod (s-\lambda)N).$$

N_λ is a holonomic D_n-module.

Proposition (Kashiwara)

If a nongenative integer m satisfies $b_f(-m-\nu) \neq 0$ for any $\nu = 1, 2, 3, \ldots$, then $N_{-m} \cong R[f^{-1}]$ as left D_n-module.

By using this isomorphism, we can compute the structure of $R[f^{-1}]$ as a left D_n-module, starting from that of $N = D_n[s]/\mathrm{Ann}_{D_n[s]} f^s$.

Generators of $H^1_{(f)}(R)$

In terms of the the Cech cohomology, we have

$$H^1_{(f)}(R) = R[f^{-1}]/R.$$

Both $H^1_{(f)}(R)$ and $R[f^{-1}]$ are holonomic D_n-modules and

$$\mathrm{mult}\, H^1_{(f)}(R) = \mathrm{mult}\, R[f^{-1}] - 1, \quad \mathrm{length}\, H^1_{(f)}(R) = \mathrm{length}\, R[f^{-1}] - 1.$$

If $b_f(-m-\nu) \neq 0$, then $H^1_{(f)}(R)$ is generated by $[f^{-m}]$.

Proposition (essentially by Kashiwara)

$H^1_{(f)}(R)$ is generated by the residue class $[f^{-1}]$ over D_n.
$\Leftrightarrow$ $R[f^{-1}]$ is generated by f^{-1} over D_n.
$\Leftrightarrow$ $b_f(\nu) \neq 0$ for any integer $\nu \leq -2$.

Length and multiplicity of N_λ

Theorem (Kashiwara)

If $b_f(\lambda+\nu) \neq 0$ for any $\nu \in \mathbb{Z}$, then $\mathrm{length}\, N_\lambda = 1$, i.e., N_λ is an irreducible D_n-module. On the other hand, $\mathrm{length}\, N_j \geq 2$ for any $j \in \mathbb{Z}$.

Proposition

For any $\lambda \in K$ and for any $j \in \mathbb{Z}$,

$$\mathrm{length}\, N_{\lambda+j} = \mathrm{length}\, N_\lambda, \quad \mathrm{mult}\, N_{\lambda+j} = \mathrm{mult}\, N_\lambda.$$

As the simplest example, set $f = x = x_1$ with $n = 1$. Then

$$N_\lambda = D_1/D_1(x\partial_x - \lambda), \quad \mathrm{mult}\, N_\lambda = 2 \text{ for any } \lambda \in K,$$
$$\mathrm{length}\, N_\lambda = 1 \text{ for any } \lambda \notin \mathbb{Z}, \quad \mathrm{length}\, N_j = 2 \text{ for any } j \in \mathbb{Z}.$$

In fact, $\cdots \cong N_{-2} \cong N_{-1} \not\cong N_0 \cong N_1 \cong \cdots$.

Sketch of the proof

Let

$$t : N \ni a(x,s)f^s \longmapsto a(x,s+1)ff^s \in N$$

be the shift operator with respect to s. Bernstein and Kashiwara proved that N/tN is a holonomic D_n-module and $b_f(s)$ is the minimal polynomial of s acting on N/tN.

Kashiwara's argument on the irreducibility of N_λ is based on the following commuting diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & K_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{t} & N & \longrightarrow & N/tN & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s-\lambda-1} & & \downarrow{\scriptstyle s-\lambda} & & \downarrow{\scriptstyle s-\lambda} & & \\
0 & \longrightarrow & N & \xrightarrow{t} & N & \longrightarrow & N/tN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{\lambda+1} & \xrightarrow{t} & N_\lambda & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

b-function-free algorithm for $\operatorname{mult} H^1_{(f)}(R)$

Since $\operatorname{mult} H^1_{(f)}(R) = \operatorname{mult} R[f^{-1}] - 1$, we have only to compute $\operatorname{mult} R[f^{-1}]$.

Step 1: Compute a finite set G of generators of the left ideal $\operatorname{Ann}_{D_n[s]} f^s$ by using the aglorithm by O, or by Briançon-Maisonobe, which are based on Gröbner basis computation in the ring of differential operators, or in the ring of difference-differential operators.

Step 2: Choose an arbitrary integer k, e.g., $k = 0$, and specialze s to k:

$$G|_{s=k} := \{P(k) \mid P(s) \in G\}.$$

Step 3: Compute a Gröbner basis G_0 of the left ideal of D_n generated by $G|_{s=k}$ with respect to a term order $\prec$ compatible with the total degree, e.g., total degree (reverse) lexicographic order.

Step 4: Let $\langle \mathrm{in}_\prec(G_0)\rangle$ be the monomial ideal in the polynomial ring $K[x,\xi]$ generated by the initial monomials of the elements of G_0. Compute the (Hilbert) polynomial $h(T)$ such that

$$h(k) = \sum_{j=0}^{k} \dim_K (K[x,\xi]/\langle \mathrm{in}(G_0)\rangle)_j \quad (k \gg 0),$$

where the rightmost subscript denotes the j-th homogeneous part.

Output: The leading coefficient of $h(T)$ multiplied by $n!$ gives $\mathrm{mult}\, R[f^{-1}] = \mathrm{mult}\, H^1_{(f)}(R) + 1$.

Proof of the correctness

Let $-m$ be the minimum integer root of $b_f(s)$. Then $R[f^{-2}] \cong N_{-m}$. Hence, for any $k \in \mathbb{Z}$, we have

$$\mathrm{mult}\, R[f^{-1}] = \mathrm{mult}\, N_{-m} = \mathrm{mult}\, N_k.$$

An upper bound of $\mathrm{mult}\, H^1_{(f)}(R)$

Proposition

The multiplicity of $M := H^1_{(f)}(R)$ is at most $(\deg f + 1)^n - 1$.

Proof: Set $d := \deg f$. Then

$$F_k(M) := \left\{ \left[\frac{a}{f^{k+1}} \right] \mid a \in K[x_1, \dots, x_n], \deg a \le (d+1)k \right\} \quad (k \in \mathbb{Z})$$

is a (not necessarily good) Bernstein filtration of M with

$$\dim_K F_k(M) = \binom{n + (d+1)k}{n} - \binom{n + (d+1)k - d(k+1)}{n}$$
$$= \frac{\{(d+1)k\}^n}{n!} - \frac{k^n}{n!} + (\text{lower order terms w.r.t. } k)$$

This implies $m(M) \le (d+1)^n - 1$.

One variable case ($n = 1$)

Proposition

If $f \in R = K[x]$ (the ring of polynomials in one indeterminate x) is nonzero and square-free, then $\mathrm{mult}\, H^1_{(f)}(R) = \deg f$.

Proof: $M := H^1_{(f)}(R) = R[f^{-1}]/R \cong D/Df$.
Set $F_k(M) := F_k(D_n)[f^{-1}] \cong F_k(D_n)/F_{k-d}(D_n)f$ with $d := \deg f$.
Then

$$\dim F_k(M) = \dim F_k(D_n) - \dim F_{k-d}(D_n) = \binom{k+2}{2} - \binom{k-d+2}{2} = dk + \text{const.}$$

Example in two variables, 1

Proposition

Set $f = x^m + y^n \in R = K[x, y]$ with $1 \le m \le n$. Then the multiplicity of $M := H^1_{(f)}(R)$ is $2n - 1$.

Proof: Since the b-function $b_f(s)$ of f does not have any negative integer ≤ -2 as a root, we have $M := H^1_{(f)}(R) = D[f^{-1}]$. The annihilator $\mathrm{Ann}_D[f^{-1}]$ is generated by

$$f, \quad E := nx\partial_x + my\partial_y + mn, \quad P := ny^{n-1}\partial_x - mx^{m-1}\partial_y$$

with $\partial_x = \partial/\partial x,\ \partial_y = \partial/\partial y$. $G = \{f, E, P\}$ is a Gröbner basis of $\mathrm{Ann}_D[f^{-1}]$ w.r.t. the total-degree reverse lexicographic order $\prec$ such that $x \succ y \succ \partial_x \succ \partial_y$.

In case $m < n$: We have

$$\mathrm{sp}(f, E) = nx\partial_x f - y^n E = x^m E - my\partial_y f,$$
$$\mathrm{sp}(f, P) = n\partial_x f - yP = x^{m-1} E,$$
$$\mathrm{sp}(E, P) = y^{n-1} E - xP = m\partial_y f.$$

The initial monomials of the Gröbner basis G are

$$\mathrm{in}_<(f) = y^n, \quad \mathrm{in}_<(E) = x\xi, \quad \mathrm{in}_<(P) = y^{n-1}\xi,$$

where ξ and η are the commutative variables corresponding to ∂_x and ∂_y repectively.

Hence for $N \geq n$,

$$
\begin{aligned}
&\dim_K F_N(D_n)/(\mathrm{Ann}_D[f^{-1}] \cap F_N(D_n)) \\
&= \sharp(\{x^i y^j \xi^k \eta^l \mid i+j+k+l \leq N\} \setminus \langle y^n, x\xi, y^{n-1}\xi \rangle) \\
&= \sharp\{x^i y^j \eta^l \mid i+j+l \leq N,\, 0 \leq j \leq n-1\} \\
&\quad + \sharp\{y^j \xi^k \eta^l \mid j+k+l \leq N,\, 0 \leq j \leq n-2,\, k \geq 1\} \\
&= \sum_{j=0}^{n-1} \binom{2+N-j}{2} + \sum_{j=0}^{n-2} \binom{2+N-j-1}{2} \\
&= \frac{2n-1}{2} N^2 + \cdots
\end{aligned}
$$

In case $m = n$: We have

$$\mathrm{sp}(f, E) = nx\partial_x f - y^n E = yP,$$

$$\mathrm{sp}(f, P) = n\partial_x f - yP = y^n P + nx^{n-1}\partial_y f$$

$$\mathrm{sp}(E, P) = y^{n-1}E - xP = m\partial_y f.$$

$$\mathrm{in}_<(f) = x^n, \quad \mathrm{in}_<(E) = x\xi, \quad \mathrm{in}_<(P) = y^{n-1}\xi.$$

Hence for $N \geq n$,

$$\begin{aligned}
&\dim_K F_N(D_n)/(\mathrm{Ann}_D[f^{-1}] \cap F_N(D_n)) \\
&= \sharp(\{x^i y^j \xi^k \eta^l \mid i+j+k+l \leq N\} \setminus \langle x^n, x\xi, y^{n-1}\xi\rangle) \\
&= \sharp\{x^i y^j \eta^l \mid i+j+l \leq N,\ 0 \leq i \leq n-1\} \\
&\quad + \sharp\{y^j \xi^k \eta^l \mid j+k+l \leq N,\ 0 \leq j \leq n-2,\ k \geq 1\} \\
&= \sum_{j=0}^{n-1} \binom{2+N-j}{2} + \sum_{j=0}^{n-2} \binom{2+N-j-1}{2} \\
&= \frac{2n-1}{2} N^2 + \cdots
\end{aligned}$$

Example in two variables, 2

Proposition

Set $f = x^m + y^n + 1 \in R = K[x, y]$ with $1 \leq m \leq n$. Then the multiplicity of $M := H^1_{(f)}(R)$ is $nm + n - m$

Proof: Since the curve $f = 0$ is non-singular, the b-function is $b_f(s) = s + 1$. Hence $M := H^1_{(f)}(R) = D[f^{-1}]$. The annihilator $\mathrm{Ann}_D[f^{-1}]$ is generated by

$$f, \qquad P := ny^{n-1}\partial_x - mx^{m-1}\partial_y$$

since $f = 0$ is non-singular.

In case $n = m$:
$G = \{f, P\}$ is a Gröbner basis of $\mathrm{Ann}_D[f^{-1}]$ w.r.t. the total-degree reverse lexicographic order $\prec$ such that $x \succ y \succ \partial_x \succ \partial_y$. In fact

$$\mathrm{sp}(f, P) = ny^{n-1}\partial_x f - x^n P = ny^{n-1}\partial_x f + x^n P$$

Since $\mathrm{in}_<(f) = x^n$ and $\mathrm{in}_<(P) = y^{n-1}\xi$, we have

$$\begin{aligned}
&\dim_K F_N(D_n)/(\mathrm{Ann}_{D_n}[f^{-1}] \cap F_N(D_n)) \\
&= \sharp(\{x^i y^j \xi^k \eta^l \mid i+j+k+l \le N\} \setminus \langle x^n, y^{n-1}\xi\rangle) \\
&= \sharp\{x^i y^j \eta^l \mid i+j+l \le N,\ 0 \le i \le n-1\} \\
&+ \sharp\{x^i y^j \xi^k \eta^l \mid i+j+k+l \le N,\ 0 \le i \le n-1,\ 0 \le j \le n-2,\ k \ge 1\} \\
&= \sum_{i=0}^{n-1} \binom{2+N-i}{2} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} \binom{2+N-i-j-1}{2} \\
&= \frac{n^2}{2}N^2 + \cdots
\end{aligned}$$

In case $m < n$:
The Gröbner basis of $\mathrm{Ann}_D[f^{-1}]$ w.r.t. the same order is $G = \{f, P, Q\}$ with

$$Q := n(x^m + 1)\partial_x + mx^{m-1}y\partial_y + mnx^{m-1}.$$

In fact

$$\begin{aligned}
\mathrm{sp}(f, P) &= mn^2\partial_x f - yP = Q, \\
\mathrm{sp}(f, Q) &= mnx^m\partial_x f - my^n Q \\
&= -m^2x^{m-1}y\partial_y f + mx^m Q - mn\partial_x f + Q, \\
\mathrm{sp}(P, Q) &= x^m P - y^{n-1}Q = -mx^{m-1}\partial_y f + P
\end{aligned}$$

Since $\mathrm{in}_<(f) = y^n$, $\mathrm{in}_<(P) = ny^{n-1}\xi$, $\mathrm{in}_<(Q) = nx^m\xi$, we have

$$
\begin{aligned}
&\dim_K F_N(D_n)/(\mathrm{Ann}_D[f^{-1}] \cap F_N(D_n)) \\
&= \sharp(\{x^i y^j \xi^k \eta^l \mid i+j+k+l \le N\} \setminus \langle y^n, y^{n-1}\xi, x^m\xi\rangle) \\
&= \sharp\{x^i y^j \eta^l \mid i+j+l \le N,\ 0 \le i \le n-1\} \\
&+ \sharp\{x^i y^j \xi^k \eta^l \mid i+j+k+l \le N,\ 0 \le i \le m-1, \\
&\quad 0 \le j \le n-2,\ k \ge 1\} \\
&= \sum_{i=0}^{n-1} \binom{2+N-i}{2} + \sum_{i=0}^{m-1}\sum_{j=0}^{n-2} \binom{2+N-i-j-1}{2} \\
&= \frac{n+m(n-1)}{2} N^2 + \cdots
\end{aligned}
$$

Hyperplane arrangements

Let $f_1, \dots, f_m \in K[x] = K[x_1, \dots, x_n]$ be linear (i.e., of first degree) polynomials and set $F = f_1 \cdots f_m$. We assume that $f_1, \dots, f_m$ are pairwise distinct up to nonzero constant and set

$$H_i = \{x \in K^n \mid f_i(x) = 0\}.$$

Then $\mathcal{A} := \{H_i\}$ defines an arrangement of hyperplanes in K^n.

- The only integer root of $b_F(s)$ is -1 (A. Leykin).
  $\Rightarrow H^1_{(F)}(R)$ is generated by $[1/F]$.

Proposition 1

$\mathrm{mult}\, H^1_{(F)}(R) = \mathrm{length}\, H^1_{(F)}(R)$.

Explicit formulae in special cases

Set $\mathrm{mult}\,\mathcal{A} := \mathrm{mult}\,H^1_{(F)}(R)$ and $\mathrm{length}\,\mathcal{A} := \mathrm{length}\,H^1_{(F)}(R)$.

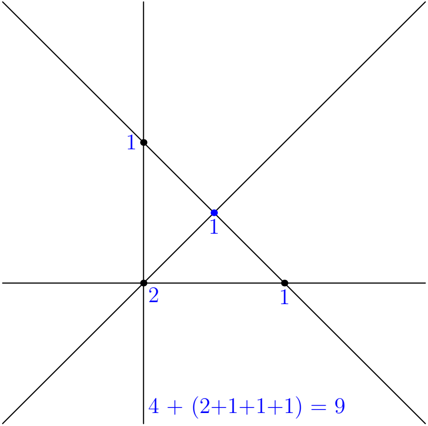
Let $L(\mathcal{A})$ be the set of the distinct intersections, other than the empty set, of some elements of $\mathcal{A}$. For an element Z of $L(\mathcal{A})$, let us define its multiplicity by

$$\mathrm{mult}_{\mathcal{A}} Z := \sharp\{i \in \{1, \ldots, m\} \mid Z \subset H_i\} - \mathrm{codim}\, Z + 1.$$

Proposition 2

If $n = 2$, then

$$\mathrm{mult}\,\mathcal{A} = \mathrm{length}\,\mathcal{A} \;=\; \#\mathcal{A} \;+ \sum_{Z \in L(\mathcal{A}),\, \mathrm{codim}\, Z = 2} \mathrm{mult}\,_{\mathcal{A}} Z.$$



Proposition 3

If $n = 3$ and $\mathcal{A}$ is central, then

$$\mathrm{mult}\,\mathcal{A} = \mathrm{length}\,\mathcal{A} \;=\; 2 \sum_{Z \in L(\mathcal{A}),\, \mathrm{codim}\, Z = 2} \mathrm{mult}_{\mathcal{A}} Z \;+ 1.$$

Proof of Propositions 1,2,3

By induction on $m = \#\mathcal{A}$. Propositions 1,2,3 hold trivially for $m = 1$. Assume they hold for $m - 1$ and set $\mathcal{A}_{m-1} = \{H_1, \ldots, H_{m-1}\}$. We regard $\mathcal{A}' := \{H_i \cap H_m \mid 1 \leq i \leq m-1\}$ as a hyperplane arrangement in H_m. Set $F_{m-1} = f_1 \cdots f_{m-1}$. We have a Mayer-Vietoris sequence

$$0 \to H^1_{(F_{m-1})}(R) \oplus H^1_{(f_m)}(R) \to H^1_{(F)}(R) \to H^2_{(F_{m-1})+(f_m)}(R) \to 0.$$

$$0 \to H^1_{(F_{m-1})}(R) \oplus H^1_{(f_m)}(R) \to H^1_{(F)}(R) \to H^2_{(F_{m-1})+(f_m)}(R) \to 0.$$

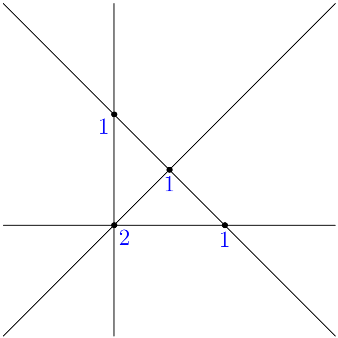
Since $H^i_{(f_m)}(R) = 0$ for $i \neq 1$, we have

$$\mathrm{mult}\, H^2_{(F_{m-1})+(f_m)}(R) = \mathrm{mult}\, H^1_{(F_{m-1})}(H^1_{(f_m)}(R)) = \mathrm{mult}\, \mathcal{A}'.$$

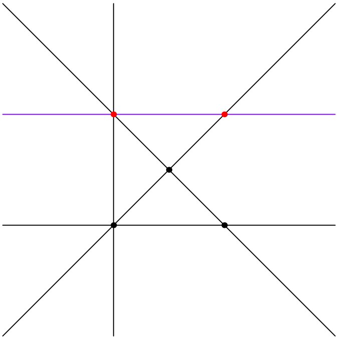
This also holds for length instead of mult. Hence we get

$$\mathrm{mult}\, \mathcal{A} = \mathrm{mult}\, \mathcal{A}_{m-1} + \mathrm{mult}\, \mathcal{A}' + 1,$$
$$\mathrm{length}\, \mathcal{A} = \mathrm{length}\, \mathcal{A}_{m-1} + \mathrm{length}\, \mathcal{A}' + 1.$$

Propositions 1,2,3 can be proved by using these recursive formulae.



$4 + (2{+}1{+}1{+}1) = 9$



$9 + (1{+}1) + 1 = 12$