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Algorithms for the b-function and D-modules associated with a polynomial

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Abstract

Let f be an arbitrary polynomial of n variables defined over a field of characteristic zero. We present algorithms for computing the b-function (Bernstein–Sato polynomial) of f, the D-module (the system of linear partial differential equations) for f^s, and the algebraic local cohomology group associated with f by using Gröbner bases for differential operators. © 1997 Elsevier Science B.V.

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1. Introduction

Let K be an algebraically closed field of characteristic zero and $O = O_{K^n}$ the sheaf of rings of regular functions on K^n. We denote by $D = D_{K^n} := O\langle\partial_1, \ldots, \partial_n\rangle$ the sheaf of rings of (algebraic) differential operators on K^n with $\partial = (\partial_1, \ldots, \partial_n) = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, where $x = (x_1, \ldots, x_n)$ stands for the coordinate system of K^n (cf. [4, 5]).

Let $f = f(x) \in K[x]$ be an arbitrary polynomial of n variables. Put $L := O[f^{-1}, s]f^s$, which is by definition a free $O[f^{-1}, s]$-module of rank one generated by f^s with a parameter s. Then L has a natural structure of left $D[s]$-module. We shall be concerned with the left $D[s]$-module $N := D[s]f^s$, which is a subsheaf of L. Put $J_f := \{P(s) \in D[s] \mid P(s)f^s = 0\}$. Then we have $N = D[s]/J_f$. Let us denote by N_0 the stalk of N at the origin $0 \in K^n$. Our aim is to present algorithms for the following problems by using Gröbner basis computation in the Weyl algebra (the ring of differential operators with polynomial coefficients) initiated by Galligo [11] (cf. also [8, 28]):

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(i) to compute the b-function (the Bernstein–Sato polynomial) $b_f(s)$ of f, which is by definition the monic polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$P(s,x,\partial)f^{s+1} = b(s)f^s \quad \text{in } N_0,$$

with some $P(s,x,\partial) \in D[s]_0$;

(ii) to find a set of generators of the sheaf of left ideals J_f of $D[s]$;

(iii) to find an explicit representation of the algebraic local cohomology group $H^1_{[Y]}(O) = O[f^{-1}]/O$ as a left D-module with $Y := \{x \in K^n \mid f(x) = 0\}$ (cf. [14] for the definition).

We can also compute the characteristic varieties and the multiplicities of N and $H^1_{[Y]}(O)$ by using the algorithms for (ii) and (iii) (cf. [21]). If the b-function $b_f(s)$ has no negative integral roots other than -1, then $O[f^{-1}]$ is isomorphic to N with s replaced by -1 (cf. [13]). Hence we can compute the structure of $O[f^{-1}]$ under this assumption.

Our methods for these three problems utilize the homogenization technique [22, 23, 26] with respect to the filtration of Kashiwara–Malgrange [15, 19] and the viewpoint of Malgrange [18] for studying the structure of N. We present two algorithms for solving the problem (i): one is independent of the problem (ii) and has been presented in [26] in a more general context but without any reference to implementation or examples; the other is newly obtained as a direct application of the algorithm for solving (ii). Details of our algorithm for the problem (iii) will appear elsewhere [24] as an application of computation of induced systems of D-modules. Hence the most essential points of the present paper lie in the solution to the problem (ii) as well as reports on actual implementation of algorithms for (i)–(iii) by using Kan [29] and partly Risa/Asir [20] with emphasis on the case with parameters.

When K coincides with the field $\boldsymbol{C}$ of complex numbers, we can also work with the sheaf D^{an} of *analytic* differential operators on $\boldsymbol{C}^n$. Our algorithms are also valid in this case without any modification since D^{an} is faithfully flat over D. In the actual computation, however, instead of assuming K to be algebraically closed, we assume that K is generated by a finite number of (algebraic or transcendental) elements over the field $\boldsymbol{Q}$ of rational numbers and that the algebraic relations among these elements are specified. Thus we can treat the case where f has parameters and/or f is defined over an algebraic number field.

In the classical case $K = \boldsymbol{C}$, problems (i)–(iii) have deep connections with the singularity structure of the hypersurface $f = 0$ and have been extensively studied theoretically (see e.g. [3, 13, 14, 18, 19]). Moreover, several algorithms for (i) and (ii) have been known under some conditions on f: An algorithm of computing $b_f(s)$ was first given by Sato et al. [27] when $f(x)$ is a relative invariant of a prehomogeneous vector space. Briançon, Maisonobe et al. [6, 17] have given an algorithm of computing $b_f(s)$ for $f(x)$ with *isolated singularity* (see also [12] for the case with parameters). Besides, Yano [32, 31] worked out many interesting examples of b-functions systematically; Aleksandrov–Kistlerov [1] have computed the b-functions for some discriminants of versal deformations, which have non-isolated singularities, by using computers following an observation of Yano–Sekiguchi [33]. These authors have also solved the problem (ii) in the course of solving (i) under respective conditions. However, as far as the present author knows, no general algorithms for (i)–(iii) are known that can be applied to an arbitrary polynomial f.

2. D-modules for f^s d'après Malgrange

We use the same notation as in the introduction. We define a sheaf of rings A_1D on K^n as follows: For a Zariski open set U of K^n, the set of sections of A_1D over U consists of the differential operators represented by a finite sum

$$P = \sum_{\mu,\nu,\alpha} a_{\mu\nu\alpha}(x) t^\mu \partial_t^\nu \partial^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$, $\mu, \nu \in \boldsymbol{N}$ with $\boldsymbol{N} := \{0, 1, 2, 3, \ldots,\}$, $\partial_t = \partial/\partial t$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, and $a_{\mu\nu\alpha}(x)$ is a regular function on (i.e. a rational function whose denominator never vanishes on) U.

As was observed by Malgrange [18], $L = O[f^{-1}, s]f^s$ has also a structure of left A_1D-module defined by

$$t(g(x,s)f^s) = g(x, s+1)f^{s+1}, \qquad \partial_t(g(x,s)f^s) = -sg(s-1)f^{s-1}$$

for a section $g(x,s)$ of $O[f^{-1}, s]$. Put $M := (A_1D)f^s$ and $N := D[s]f^s$. Then we have inclusions $N \subset M \subset L$.

Lemma 1. *The sheaf of left ideals*

$$I := (A_1D)(t - f(x)) + \sum_{i=1}^{n} (A_1D)(\partial_i + f_i\partial_t)$$

of A_1D with $f_i := \partial f/\partial x_i$ is maximal, i.e., its stalk I_p is a maximal left ideal of $(A_1D)_p$ for any $p \in K^n$.

Proof. The coordinate transformation $t' = t - f(x)$, $x' = x$ induces a ring automorphism of A_1D. Hence we may assume $f(x) = 0$ and $p = 0$. Thus, we can apply the same argument as [18, Lemma 4.1]. □

Proposition 2 (Malgrange [18]). *M is isomorphic to $(A_1D)/I$.*

Let J_f be the sheaf of left ideals of D consisting of sections $P(s)$ of $D[s]$ which satisfy $P(s)f^s = 0$. The following fact is the key to our solution of problem (ii).

Proposition 3. *For a Zariski open set U of K^n, the set of sections of J_f over U is given by*

$$\Gamma(U, J_f) = \{P(-s-1) \mid P(t\partial_t) \in \Gamma(U, I \cap D[t\partial_t])\}.$$

Proof. This follows immediately from Proposition 2, the relation $-\partial_t t f^s = s f^s$, and the fact that N is a subsheaf of M. □

For each integer k, we define a subsheaf $F_k(A_1 D)$ of $A_1 D$ consisting of sections of $A_1 D$ of the form

$$\sum_{\nu-\mu \le k} \sum_{\alpha} a_{\mu\nu\alpha}(x) t^{\mu} \partial_t^{\nu} \partial^{\alpha}$$

with $a_{\mu\nu\alpha}$ being a section of O. Then $\{F_k(A_1 D)\}_{k\in Z}$ constitutes a special case of the filtration introduced by Kashiwara [15] and Malgrange [19] for the study of vanishing cycle sheaves (the V-filtration). We make an essential use of the following fact for one of our algorithms of solving the problem (i).

Proposition 4. *For $b(s) \in K[s]$, we have $P(s) f^{s+1} = b(s) f^s$ in N_0 with some $P(s) \in D[s]_0$ if and only if $b(-\partial_t t) - Q \in I_0$ with some $Q \in F_{-1}(A_1 D)_0$.*

Proof. First assume $P(s) f^{s+1} = b(s) f^s$ with $P(s) \in D[s]_0$. Then we have $(b(-\partial_t t) - P(-\partial_t t) t) f^s = 0$ and $P(-\partial_t t) t$ belongs to $F_{-1}(A_1 D)_0$.

Conversely, suppose $b(-\partial_t t) - Q \in I_0$ with $Q \in F_{-1}(A_1 D)_0$. Expanding Q in the form $Q = \sum_{j=1}^{\infty} Q_j(t\partial_t) t^j$ with $Q_j(t\partial_t) \in D[t\partial_t]_0$, which is, in fact, a finite sum, put

$$\rho(Q) := \sum_{j=1}^{\infty} Q_j(-s-1) f^{j-1} \in D[s]_0.$$

Then we get $(b(-s-1) - \rho(Q) f) f^s = 0$. □

3. Gröbner bases with parameters and homogenization with respect to the V-filtration

Let K be a field of characteristic zero. The Buchberger algorithm for computing Gröbner basis does not require field extension. Hence, we can work in a field K over which the inputs are defined instead of working in the algebraic closure of K. We denote by $A_n(K)$ the Weyl algebra in variables x with coefficients in K [4].

Put $a = (a_1, \dots, a_\ell)$. We assume that a set $G(a)$ of generators of an ideal $J(a)$ of the polynomial ring $\mathbf{Q}[a] = \mathbf{Q}[a_1, \dots, a_\ell]$ is given so that K is isomorphic to the quotient field of $\mathbf{Q}[a]/J(a)$. (Thus, $J(a)$ must be a prime ideal.)

Adding new commutative variables $y = (y_1, \dots, y_m)$ as well as $a = (a_1, \dots, a_\ell)$, we work in the rings $A_{n+1}(\mathbf{Q})[y, a]$ and $A_{n+1}(K)[y]$ of differential operators with parameters. Hence their centers are $\mathbf{Q}[y, a]$ and $K[y]$, respectively. More concretely, an element P of $A_{n+1}(\mathbf{Q})[y, a]$ is written in a finite sum

$$P = \sum_{\mu,\nu,\alpha,\beta,\eta,\gamma} c_{\mu\nu\alpha\beta\eta\gamma} a^{\gamma} y^{\eta} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} \qquad (1)$$

with $\mu, \nu \in \boldsymbol{N}$, $\alpha, \beta \in \boldsymbol{N}^n$, $\eta \in \boldsymbol{N}^m$, $\gamma \in \boldsymbol{N}^\ell$, and $c_{\mu\nu\alpha\beta\eta\gamma} \in \boldsymbol{Q}$, while an element P of $A_{n+1}(K)[y]$ is written in the form

$$\sum_{\mu,\nu,\alpha,\beta,\eta} c_{\mu\nu\alpha\beta\eta} y^\eta t^\mu x^\alpha \partial_t^\nu \partial^\beta \tag{2}$$

with $\mu, \nu \in \boldsymbol{N}$, $\alpha, \beta \in \boldsymbol{N}^n$, $\eta \in \boldsymbol{N}^m$, and $c_{\mu\nu\alpha\beta\eta} \in K$.

Let us put $L_0 := \boldsymbol{N}^{2+2n+m}$ whose element $(\mu, \nu, \alpha, \beta, \eta)$ corresponds to the monomial $y^\eta t^\mu x^\alpha \partial_t^\nu \partial^\beta$ of $A_{n+1}(K)[y]$; also put $L := L_0 \times \boldsymbol{N}^\ell$ whose element $(\mu, \nu, \alpha, \beta, \eta, \gamma)$ corresponds to the monomial $a^\gamma y^\eta t^\mu x^\alpha \partial_t^\nu \partial^\beta$ of $A_{n+1}(\boldsymbol{Q})[y, a]$.

In general, a total order $\prec$ on L is called a *monomial order* if it satisfies

(A1) $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for any $\alpha, \beta, \gamma \in L$;

(A2) $0 \prec \alpha$ for any $\alpha \in L \setminus \{0\}$.

Moreover, we call a monomial order $\prec$ on L *parametric* (with respect to parameters a) if it satisfies

(A3) $(0, \gamma) \prec (\alpha, \gamma')$ for any $\alpha \in L_0 \setminus \{0\}$ and $\gamma, \gamma' \in \boldsymbol{N}^\ell$.

In the sequel, we denote by $\prec$ a monomial order on L satisfying (A1)–(A3), and by $\prec_0$ the restriction of $\prec$ to $L_0 \simeq L_0 \times \{0\} \subset L$.

For an element P of $A_{n+1}(\boldsymbol{Q})[y, a]$ of the form (1) and P of $A_{n+1}(K)[y]$ of the form (2), we define their *leading exponents* $\operatorname{lexp}(P)$ and $\operatorname{lexp}_0(P)$ with respect to the orders $\prec$ and $\prec_0$ to be the maximum elements of the sets

$$\{(\mu, \nu, \alpha, \beta, \eta, \gamma) \in L \mid c_{\mu\nu\alpha\beta\eta\gamma} \neq 0\},$$
$$\{(\mu, \nu, \alpha, \beta, \eta) \in L_0 \mid c_{\mu\nu\alpha\beta\eta} \neq 0\}$$

in the orders $\prec$ and $\prec_0$ respectively. Moreover, for a subset S of $A_{n+1}(\boldsymbol{Q})[y, a]$ and S_0 of $A_{n+1}(K)[y]$, we put

$$E(S) := \{\operatorname{lexp}(P) \mid P \in S \setminus \{0\}\},$$
$$E_0(S_0) := \{\operatorname{lexp}_0(P) \mid P \in S_0 \setminus \{0\}\}.$$

Definition 5. A finite subset $\boldsymbol{G}$ of a left ideal I of $A_{n+1}(\boldsymbol{Q})[y, a]$ (or of $A_{n+1}(K)[y]$) is called a *Gröbner basis* of I with respect to the order $\prec$ (or $\prec_0$) if

$$E(I) = \bigcup_{P \in G} (\operatorname{lexp}(P) + L), \qquad \left(\text{or } E_0(I) = \bigcup_{P \in G} (\operatorname{lexp}_0(P) + L_0)\right). \tag{3}$$

Moreover, $\boldsymbol{G}$ is called a *minimal Gröbner basis* if (3) never holds with $\boldsymbol{G}$ being replaced by a proper subset of $\boldsymbol{G}$.

If a finite set of generators of a left ideal I of $A_{n+1}(\boldsymbol{Q})[y, a]$ (or of $A_{n+1}(K)[y]$) is given, the Buchberger algorithm [7] computes a Gröbner basis of I as in the polynomial case (cf. [11, 8, 28]).

Our first aim is to make clear the meaning of the Gröbner basis computation with parameters a. This will be needed, e.g., for the computation of the b-function of a polynomial with parameters.

Let $\pi : \mathbf{Q}[a] \to \mathbf{Q}[a]/J(a) \subset K$ be the natural ring homomorphism and let $\varpi : L \to L_0$ be the projection. Then π extends to a ring homomorphism

$$\pi : A_{n+1}(\mathbf{Q})[y,a] \to A_{n+1}(K)[y].$$

For $P \in A_{n+1}(\mathbf{Q})[y,a]$ of the form (1), put $\mathrm{lexp}(P) = (\mu_0, \nu_0, \alpha_0, \beta_0, \eta_0, \gamma_0)$ and

$$\mathrm{lcoef}_0(P) := \sum_{\gamma} c_{\mu_0\nu_0\alpha_0\beta_0\eta_0\gamma} a^{\gamma} \in \mathbf{Q}[a].$$

For a left ideal I of $A_{n+1}(\mathbf{Q})[y,a]$, let $\pi(I)$ be the left ideal of $A_{n+1}(K)[y]$ which is generated by $\{\pi(P) \mid P \in I\}$.

Proposition 6. *Let I be a left ideal of $A_{n+1}(\mathbf{Q})[y,a]$ containing $J(a)$. Let $\mathbf{G}$ be a Gröbner basis of I with respect to $\prec$. Then $\pi(\mathbf{G}) := \{\pi(P) \mid P \in \mathbf{G},\ \pi(P) \neq 0\}$ is a Gröbner basis of $\pi(I)$ with respect to $\prec_0$.*

Proof. It suffices to prove

$$E_0(\pi(I)) = \bigcup_{Q \in \pi(\mathbf{G})} (\mathrm{lexp}_0(Q) + L_0). \tag{4}$$

Since $\pi(P) \in \pi(I)$ for each $P \in \mathbf{G}$, the inclusion $\supset$ in (4) is obvious. Put $G(a) := \mathbf{G} \cap \mathbf{Q}[a]$ and let $\tilde{J}(a)$ be the ideal of $\mathbf{Q}[a]$ generated by $G(a)$. Then $G(a)$ is a Gröbner basis of $I \cap \mathbf{Q}[a]$ with respect to the restriction of $\prec$ to $\{0\} \times \mathbf{N}^{\ell}$ since the order $\prec$ is an order for eliminating the variables other than a. It follows $\tilde{J}(a)$ contains $J(a)$. First, let us assume $\tilde{J}(a) \neq J(a)$. Then $G(a)$ contains an element $g(a) \in \mathbf{Q}[a]$ such that $\pi(g(a)) \neq 0$. Hence we have $\pi(I) = A_{n+1}(K)[y]$ in this case and the assertion of the theorem is valid.

Now let us assume $\tilde{J}(a) = J(a)$. We may assume that $\mathbf{G}$ is a minimal Gröbner basis. Our aim is to prove the inclusion $\subset$ in (4). Suppose $Q \in \pi(I) \setminus \{0\}$. Then there exist $g(a) \in \mathbf{Q}[a]$ and $P \in I$ so that $\pi(g(a)) \neq 0$ and $Q = \pi(g(a))^{-1}\pi(P)$. Then we have $\mathrm{lexp}_0(Q) = \mathrm{lexp}_0(\pi(P))$. Let P above be in the form (1) and put

$$c_{\mu\nu\alpha\beta\eta}(a) := \sum_{\gamma} c_{\mu\nu\alpha\beta\eta\gamma} a^{\gamma} \in \mathbf{Q}[a].$$

Let P' be the sum of the terms $c_{\mu\nu\alpha\beta\eta}(a) y^{\eta} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta}$ such that $c_{\mu\nu\alpha\beta\eta}(a) \notin J(a)$. Then we have $\pi(P) = \pi(P')$ and $P' \in I$ since $J(a) \subset I$. Note that $\mathrm{lexp}_0(\pi(P')) = \varpi(\mathrm{lexp}(P'))$ holds since $\pi(\mathrm{lcoef}_0(P')) \neq 0$ in view of the definition of P' and the condition (A3).

Moreover, dividing $\mathrm{lcoef}_0(P')$ by $G(a)$, we may assume

$$\mathrm{lexp}(\mathrm{lcoef}_0(P')) \notin \bigcup_{g \in G(a)} (\mathrm{lexp}(g) + L).$$

There exists $P_0 \in \mathbf{G}$ such that $\mathrm{lexp}(P') \in \mathrm{lexp}(P_0) + L$ since $\mathbf{G}$ is a Gröbner basis of I. In view of the observation above, P_0 does not belong to $G(a)$. Then we have $\mathrm{lexp}_0(\pi(P_0)) = \varpi(\mathrm{lexp}(P_0))$ since $\mathrm{lcoef}_0(P_0) \notin J(a)$ by virtue of the minimality of $\mathbf{G}$.

Thus, we get

$$\mathrm{lexp}_0(Q) = \varpi(\mathrm{lexp}(P')) \in \bigcup_{R\in\pi(G)} (\mathrm{lexp}_0(R) + L_0). \qquad \square$$

Next, let us consider the specialization of the parameters a. Let $\tilde{J}(a)$ be another prime ideal of $\boldsymbol{Q}[a]$ which contains $J(a)$. Then the natural ring homomorphism $\boldsymbol{Q}[a]/J(a) \to \boldsymbol{Q}[a]/\tilde{J}(a)$ can be regarded as a specialization of the parameters a. Let us denote by $\tilde{\pi}: \boldsymbol{Q}[a] \to \boldsymbol{Q}[a]/\tilde{J}(a) \subset \tilde{K}$ the canonical ring homomorphism with $\tilde{K}$ being the quotient field of $\boldsymbol{Q}[a]/\tilde{J}(a)$. Then $\tilde{\pi}$ extends to a ring homomorphism $\tilde{\pi}: A_{n+1}(\boldsymbol{Q})[y,a] \to A_{n+1}(\tilde{K})[y]$.

Proposition 7. *Let I be a left ideal of $A_{n+1}(\boldsymbol{Q})[y,a]$ containing $J(a)$. Let $\boldsymbol{G}$ be a Gröbner basis of I with respect to $\prec$. Assume* $\mathrm{lcoef}_0(P) \notin \tilde{J}(a)$ *for any $P \in \boldsymbol{G}$ such that $\pi(P) \neq 0$. Then $\tilde{\pi}(\boldsymbol{G}) := \{\tilde{\pi}(P) \mid P \in \boldsymbol{G},\ \tilde{\pi}(P) \neq 0\}$ is a Gröbner basis (with respect to $\prec_0$) of the left ideal $\tilde{\pi}(I)$ of $A_{n+1}(\tilde{K})[y]$ generated by $\{\tilde{\pi}(P) \mid P \in I\}$.*

Proof. It is easy to see that $\tilde{\pi}(\boldsymbol{G})$ generates $\tilde{\pi}(I)$. Set $\boldsymbol{G} = \{P_1,\dots,P_d\}$. We may assume that $\boldsymbol{G}$ is a minimal Gröbner basis. Applying Proposition 6 to the case $J(a) = \{0\}$, we know that $\boldsymbol{G}$ also constitutes a Gröbner basis in $A_{n+1}(\boldsymbol{Q}(a))[y]$, where $\boldsymbol{Q}(a)$ denotes the field of rational functions of a. For $1 \le i < j \le d$, let $\mathrm{lcoef}_0(P_j)S_{ji}P_i - \mathrm{lcoef}_0(P_i)S_{ij}P_j$ be the S-polynomial of P_i and P_j in $A_{n+1}(\boldsymbol{Q}(a))[y]$, where S_{ji} and S_{ij} are minimum monomials in $A_{n+1}(\boldsymbol{Q})[y]$ such that $\mathrm{lexp}_0(S_{ji}P_i) = \mathrm{lexp}_0(S_{ij}P_j)$ holds (here lexp_0 denotes the leading exponent of an element of $A_{n+1}(\boldsymbol{Q}(a))[y]$ with respect to $\prec_0$). Then there exist $Q_{ijk} \in A_{n+1}(\boldsymbol{Q}(a))[y]$ so that

$$\mathrm{lcoef}_0(P_j)S_{ji}P_i - \mathrm{lcoef}_0(P_i)S_{ij}P_j = \sum_{k=1}^{d} Q_{ijk}P_k \tag{5}$$

and that $\mathrm{lexp}_0(Q_{ijk}P_k) \prec \mathrm{lexp}_0(S_{ji}P_i)$ or else $Q_{ijk} = 0$. In view of the division algorithm to obtain (5), we can take Q_{ijk} so that its denominator is a power of $\mathrm{lcoef}_0(P_k)$.

Now assume $\pi(P_k) \neq 0$ for $k = 1,\dots,d'$, and $\pi(P_k) = 0$ for $k = d'+1,\dots,d$. There exists $g \in \boldsymbol{Q}[a] \setminus \tilde{J}(a)$ such that $gQ_{ijk} \in A_{n+1}(\boldsymbol{Q})[y,a]$ for $k = 1,\dots,d'$ since $\mathrm{lcoef}_0(P_k) \notin \tilde{J}(a)$ and $\tilde{J}(a)$ is prime. Then by (5) we have

$$\begin{aligned}&\tilde{\pi}(\mathrm{lcoef}_0(P_j))\tilde{\pi}(S_{ji})\tilde{\pi}(P_i) - \tilde{\pi}(\mathrm{lcoef}_0(P_i))\tilde{\pi}(S_{ij})\tilde{\pi}(P_j)\\ &\quad = \frac{1}{\tilde{\pi}(g)}\sum_{k=1}^{d'} \tilde{\pi}(gQ_{ijk})\tilde{\pi}(P_k),\end{aligned}$$

and $\mathrm{lexp}_0(\tilde{\pi}(gQ_{ijk}P_k)) \prec \mathrm{lexp}_0(\tilde{\pi}(S_{ji}P_i))$ or else $\tilde{\pi}(Q_{ijk}) = 0$ for $1 \le i < j \le d'$. This implies that $\tilde{\pi}(\boldsymbol{G})$ is a Gröbner basis with respect to $\prec_0$. $\square$

Next, let us introduce the notion of homogeneity and homogenization with respect to the V-filtration. Now that the relation between Gröbner bases of $A_{n+1}(\boldsymbol{Q})[y,a]$ and of $A_{n+1}(K)[y]$ is established, we have only to work with $A_{n+1}(K)[y]$.

Definition 8. Let P be an element of $A_{n+1}(K)$ of the form

$$P = \sum_{\mu,\nu,\alpha,\beta} c_{\mu\nu\alpha\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \tag{6}$$

with $c_{\mu\nu\alpha\beta} \in K$. Then the *F-order* $\mathrm{ord}_F(P)$ of P is defined by

$$\mathrm{ord}_F(P) := \max\{\nu - \mu \mid c_{\mu\nu\alpha\beta} \neq 0 \text{ for some } \alpha, \beta \in \boldsymbol{N}^n\}.$$

If $k = \mathrm{ord}_F(P)$, the *formal symbol* $\hat{\sigma}(P)$ of P is defined by

$$\hat{\sigma}(P) = \hat{\sigma}_k(P) := \sum_{\nu-\mu=k} \sum_{\alpha,\beta} c_{\mu\nu\alpha\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in A_{n+1}(K).$$

Definition 9. Let s be a new commutative variable and let P be a non-zero element of $A_{n+1}(K)$ of F-order m. Then we define $\psi(P) = \psi(P)(s) \in A_n(K)[s]$ by

$$\psi(P)(t\partial_t) = \begin{cases} \hat{\sigma}_0(t^m P) & \text{if } m \geq 0, \\ \hat{\sigma}_0(\partial_t^{-m} P) & \text{if } m < 0. \end{cases}$$

In order to define the homogeneity for elements of $A_{n+1}(K)[y]$, we fix a weight vector $\delta = (\delta_1, \ldots, \delta_m) \in \boldsymbol{Z}^m$ and write $\langle \delta, \eta \rangle = \delta_1 \eta_1 + \cdots + \delta_m \eta_m$ for $\eta = (\eta_1, \ldots, \eta_m) \in \boldsymbol{N}^m$ We shall assume $\delta_1 = -1$ throughout the present paper.

Definition 10 (*F-homogeneity*). We call an element P of $A_{n+1}(K)[y]$ *F-homogeneous* (of order k) if it is written in the form (2) and there exists an integer k so that $c_{\mu\nu\alpha\beta\eta} \neq 0$ if $\nu - \mu + \langle \delta, \eta \rangle \neq k$. Moreover, a left ideal of $A_{n+1}(K)[y]$ is called F-homogeneous if it is generated by F-homogeneous elements.

Lemma 11. *If two elements P, Q of $A_{n+1}(K)[y]$ are both F-homogeneous, then so is PQ. In particular, the Buchberger algorithm for computing Gröbner bases preserves the F-homogeneity.*

Definition 12 (*F-homogenization*). For an element P of $A_{n+1}(K)$ of the form (6), put $k := \min\{\nu - \mu \mid c_{\mu\nu\alpha\beta} \neq 0 \text{ for some } \alpha, \beta \in \boldsymbol{N}^n\}$. Then the *F-homogenization* $P^h \in A_{n+1}(K)[y_1]$ of P is defined by

$$P^h = P^h(y_1) := \sum_{\mu,\nu,\alpha,\beta} c_{\mu\nu\alpha\beta} y_1^{\nu-\mu-k} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in A_{n+1}(K)[y_1].$$

P^h is F-homogeneous of order k.

Lemma 13. *For $P, Q \in A_{n+1}(K)$, we have $(PQ)^h = P^h Q^h$.*

Lemma 14. *For $P_1, \ldots, P_d \in A_{n+1}(K)$, there exist $\eta, \eta_1, \ldots, \eta_d \in \boldsymbol{N}$ so that*

$$y_1^\eta (P_1 + \cdots + P_d)^h = y_1^{\eta_1}(P_1)^h + \cdots + y_1^{\eta_d}(P_d)^h.$$

Lemma 15. *Let I be an F-homogeneous left ideal of $A_{n+1}(K)[y_1]$ and put $I(1):=\{P(1)\,|\,P(y_1)\in I\}$. Then, for an element P of $A_{n+1}(K)$, we have $P\in I(1)$ if and only if there exists $\eta\in \boldsymbol{N}$ so that $y_1^{\eta}P^h\in I$.*

Proof. Assume $P\in I(1)$. Then there exist F-homogeneous $Q_1(y_1),\dots,Q_d(y_1)\in I$ such that $P=Q_1(1)+\cdots+Q_d(1)$. Then by the preceding lemma, there exist $\eta,\eta_1,\dots,\eta_d\in \boldsymbol{N}$ so that

$$y_1^{\eta}P^h = y_1^{\eta_1}Q_1(1)^h + \cdots + y_1^{\eta_d}Q_d(1)^h.$$

It is easy to see by the definition that there exist $\eta_j'\in \boldsymbol{N}$ so that $Q_j(y_1)=y_1^{\eta_j'}Q_j(1)^h$. This implies $y_1^{\eta+\eta'}P^h\in I$ with $\eta' := \max\{\eta_j'\,|\,j=1,\dots,d\}$. The converse implication is obvious. □

Now, we consider two special orders $\prec_1$ and $\prec_2$ which behave nicely with respect to the V-filtration and the F-homogenization. We will make essential use of these orders in the algorithms for the problems (i)–(iii) stated in the introduction. A prototype of our argument has been presented in [22, 23, 26].

First, putting $y=y_1$, $\eta=\eta_1\in \boldsymbol{N}$ with $m=1$, let us consider an order $\prec_1$ on $L:=\boldsymbol{N}^{2+2n+1}\ni(\mu,\nu,\alpha,\beta,\eta)$ which satisfies (A1), (A2) and

(A4) $\eta<\eta'$ implies $(\mu,\nu,\alpha,\beta,\eta)\prec_1(\mu',\nu',\alpha',\beta',\eta')$ for any $\mu,\nu,\mu',\nu',\eta,\eta'\in \boldsymbol{N}$, $\alpha,\beta,\alpha',\beta'\in \boldsymbol{N}^n$.

Let us denote by $\mathrm{lexp}(P(y_1))\in L$ the leading exponent of $P(y_1)\in A_{n+1}(K)[y_1]$ with respect to $\prec_1$. The weight vector for $y=y_1$ is $\delta=\delta_1=-1$ in this case.

Lemma 16. *Let $P(y_1)$, $Q(y_1)$ be nonzero elements of $A_{n+1}(K)[y_1]$ which are F-homogeneous of the same order. Then* $\mathrm{lexp}(P(y_1))\preceq_1\mathrm{lexp}(Q(y_1))$ *implies* $\mathrm{ord}_F(P(1))\le \mathrm{ord}_F(Q(1))$.

Proof. Put

$$\mathrm{lexp}(P(y_1))=(\mu,\nu,\alpha,\beta,\eta),\qquad \mathrm{lexp}(Q(y_1))=(\mu',\nu',\alpha',\beta',\eta').$$

We have $\nu-\mu-\eta=\nu'-\mu'-\eta'$ by the assumption. Hence $\eta\le\eta'$ implies $\mathrm{ord}_F(P(1))\le\mathrm{ord}_F(Q(1))$. □

We denote by $A_n(K)[t\partial_t]$ the subring of $A_{n+1}(K)$ generated by x, ∂ and $t\partial_t$, which is isomorphic to $A_n(K)[s]$.

Theorem 17 (Oaku [26]). *Let I be an F-homogeneous left ideal of $A_{n+1}(K)[y_1]$. Suppose that $\boldsymbol{G}$ is a Gröbner basis of I with respect to $\prec_1$ consisting of F-homogeneous operators. Put $I(1):=\{P(1)\,|\,P(y_1)\in I\}$. Let $\psi(I(1))$ be the left ideal of $A_n(K)[s]$ generated by the set $\{\psi(P)(s)\,|\,P\in I(1)\setminus\{0\},\mathrm{ord}_F(P)=0\}$. Then $\psi(I(1))$ is generated by the set $\psi(\boldsymbol{G}(1)):=\{\psi(P(1))\,|\,P(y_1)\in\boldsymbol{G}\}$.*

Proof. Put $G = \{P_1(y_1), \ldots, P_d(y_1)\}$. Suppose $P \in I(1)$ and $\mathrm{ord}_F(P) = 0$. Then by Lemma 15 there exist $\eta \in N$ and F-homogeneous $Q_1(y_1), \ldots, Q_d(y_1) \in A_{n+1}(K)[y_1]$ so that

$$y_1^\eta P^h(y_1) = Q_1(y_1)P_1(y_1) + \cdots + Q_d(y_1)P_d(y_1) \tag{7}$$

and that $\mathrm{lexp}(Q_j(y_1)P_j(y_1)) \preceq_1 \mathrm{lexp}(y_1^\eta P^h(y_1))$ or $Q_j(y_1) = 0$ for each j. Since the both sides of (7) are F-homogeneous of the same order, we have $\mathrm{ord}_F(Q_j(1)P_j(1)) \le \mathrm{ord}_F(P)$ by Lemma 16, and

$$P = Q_1(1)P_1(1) + \cdots + Q_d(1)P_d(1).$$

Let $P_j(1)$ be of F-order m_j. Then the F-order of $Q_j(1)$ is not greater than $-m_j$. Hence we can take $Q_j' \in A_n(K)[t\partial_t]$ so that $\hat\sigma_{-m_j}(Q_j(1)) = Q_j' S_j$, where $S_j := t^{m_j}$ if $m_j \ge 0$ and $S_j := \partial_t^{-m_j}$ if $m_j < 0$. Then we have

$$\psi(P)(t\partial_t) = \sum_{j=1}^d Q_j' S_j \hat\sigma_{m_j}(P_j(1)) = \sum_{j=1}^d Q_j' \psi(P_j(1))(t\partial_t). \qquad \square$$

Next, putting $y = (y_1, y_2)$ and $\eta = (\eta_1, \eta_2)$ with $m = 2$, let us consider an order $\prec_2$ on $L := N^{2+2n+2} \ni (\mu, \nu, \alpha, \beta, \eta)$ which satisfies (A1), (A2) and

(A5) If $\eta \neq 0$, we have $(\mu, \nu, \alpha, \beta, 0) \prec_2 (\mu', \nu', \alpha', \beta', \eta)$ for any $\mu, \nu, \mu', \nu' \in N$, $\alpha, \beta, \alpha', \beta' \in N^n$.

We put $\delta = (-1, 1)$ in order to define the F-homogeneity of an element of $A_{n+1}(K)[y]$.

Theorem 18. *Let I be an F-homogeneous left ideal of $A_{n+1}(K)[y_1]$. Denote by $\tilde I$ the left ideal of $A_{n+1}(K)[y]$ generated by I and $1 - y_1y_2$. Let G be a Gröbner basis of $\tilde I$ with respect to the order $\prec_2$ consisting of F-homogeneous operators. Put $G_0 := G \cap A_{n+1}(K)$. Then the left ideal $I_0 := I(1) \cap A_n(K)[t\partial_t]$ of $A_n(K)[t\partial_t]$ is generated by $\psi(G_0) := \{\psi(P)(t\partial_t) \mid P \in G_0\}$.*

Proof. Suppose $P \in G_0$. Let $\{U_1(y_1), \ldots, U_k(y_1)\}$ be a set of generators of I. Since P belongs to $\tilde I$, there exist $V_0(y), V_1(y), \ldots, V_d(y) \in A_{n+1}(K)[y]$ so that

$$P = V_0(y)(1 - y_1y_2) + V_1(y)U_1(y_1) + \cdots + V_d(y)U_d(y_1).$$

Putting $y = (1, 1)$, we get $P \in I(1)$. Since P is F-homogeneous and free of y, there exists some monomial S in $A_{n+1}(K)$ so that $\psi(P)(t\partial_t) = SP \in I_0$. Hence we have $G_0 \subset I_0$.

To prove that G_0 generates I_0, suppose $P \in I_0$. Then by Lemma 15 there exists $\eta_1 \in N$ so that $y_1^{\eta_1} P \in I$ since $P^h = P$. Hence, we have

$$P = (1 - y_1^{\eta_1} y_2^{\eta_1})P + y_2^{\eta_1} y_1^{\eta_1} P \in \tilde I.$$

Set $G_0 = \{P_1, \ldots, P_d\}$. Since P is free of y and since $\prec_2$ eliminates y, there exist $Q_1, \ldots, Q_d \in A_{n+1}(K)$ so that $P = Q_1P_1 + \cdots + Q_dP_d$. Since P and $P_1, \ldots, P_d$ are F-homogeneous and free of y, we may assume so are $Q_1, \ldots, Q_d$. Put $m_j := \mathrm{ord}_F(P_j)$ and let S_j be as in the proof of the preceding theorem. Then there exist $Q'_j \in A_n(K)[t\partial_t]$ so that $Q_j = Q'_j S_j$. Thus, we obtain

$$P = Q'_1 S_1 P_1 + \cdots + Q'_d S_d P_d = Q'_1 \psi(P_1)(t\partial_t) + \cdots + Q'_d \psi(P_d)(t\partial_t). \qquad \square$$

4. The D-module for f^s

Let $f(x) \in K[x]$ be an arbitrary polynomial and retain the same notation as in the preceding sections. In particular, we assume K to be algebraically closed theoretically while, in the computation of Gröbner bases, we may assume that K is generated by a finite set of elements over $\boldsymbol{Q}$.

In the sequel, we work in the ring $A_{n+1}(K)[y]$ with $y = (y_1, y_2)$ with the weight vector $\delta = (-1, 1)$ for y. The following theorem gives an algorithm to compute the $D[s]$-module $N = D[s]f^s = D[s]/J_f$.

Theorem 19. *Let $\tilde{I}$ be the left ideal of $A_{n+1}(K)[y]$ generated by*

$$1 - y_1 y_2, \quad t - y_1 f(x), \quad \partial_i + y_1 f_i(x)\partial_t \quad (i = 1, \ldots, n)$$

with $f_i(x) := \partial f/\partial x_i$. Let $\boldsymbol{G}$ be a Gröbner basis of $\tilde{I}$ with respect to $\prec_2$ consisting of F-homogeneous operators. Put

$$\boldsymbol{G}_0 := \boldsymbol{G} \cap A_{n+1}(K), \qquad \psi(\boldsymbol{G}_0) := \{\psi(P)(-s-1) \mid P \in \boldsymbol{G}_0\}.$$

Then the sheaf J_f of left ideals of $D[s]$ on K^n is generated by $\psi(\boldsymbol{G}_0)$ whose elements are regarded as sections of $D[s]$ over K^n.

Proof. It suffices to prove that the stalk $(J_f)_0$ of J_f at 0 is generated by $\psi(\boldsymbol{G}_0)$. Let I be a left ideal of $A_{n+1}(K)[y_1]$ generated by $t - y_1 f(x)$ and $\partial_i + y_1 f_i(x)\partial_t$ $(i = 1, \ldots, n)$ and put

$$I(1) := \{P(1) \mid P(y_1) \in I\}, \qquad I_0 := I(1) \cap A_n(K)[t\partial_t].$$

We denote by $\psi(I_0)$ the left ideal of $A_n[s]$ generated by $\{\psi(P)(-s-1) \mid P \in I_0\}$. First, let us show $(J_f)_0 = D[s]_0 \psi(I_0)$. Assume $P(s) \in \psi(I_0)$. Then we have $P(-\partial_t t) \in I_0$. Hence there exist $Q_0, Q_1, \ldots, Q_n \in A_{n+1}(K)$ so that

$$P(-\partial_t t) = Q_0 \cdot (t - f(x)) + Q_1 \cdot (\partial_1 + f_1(x)\partial_t) + \cdots + Q_n \cdot (\partial_n + f_n(x)\partial_t). \qquad (8)$$

This implies $P(s) \in (J_f)_0$ in view of Proposition 3.

Conversely, suppose $P(s) \in (J_f)_0$. Then there exist $Q_0, Q_1, \ldots, Q_n \in (A_1 D)_0$ which satisfy (8). By the definition of $(A_1 D)_0$, there exists $c(x) \in K[x]$ so that $c(0) \neq 0$ and that $c(x)P(-\partial_t t)$, $c(x)Q_0, c(x)Q_1, \ldots, c(x)Q_n$ all belong to $A_{n+1}(K)$. Hence, we have $c(x)P(s) \in \psi(I_0)$ by definition. This implies $P(s) \in D[s]_0 \psi(I_0)$. Since $\psi(I_0)$ is generated by $\psi(\boldsymbol{G}_0)$ by virtue of Theorem 18, we are done. $\square$

Next, let us consider the specialization of the parameter s. Let s_0 be an element of K and put

$$N|_{s=s_0} := N/(s-s_0)N = D[s]/(J_f + D[s](s-s_0)) \simeq D/(J_f|_{s=s_0}),$$

where $J_f|_{s=s_0} := \{P(s_0) \mid P(s) \in J_f\}$. It is known that $N|_{s=s_0}$ is a holonomic system and also known is an estimate of its characteristic variety [13].

Put $Df^{s_0} := D/J_f(s_0)$ with $J_f(s_0) := \{P \in D \mid Pf^{s_0} = 0\}$. Then there is a natural surjective D-homomorphism $\rho : N|_{s=s_0} \to Df^{s_0}$. Let $\tilde{b}_f(s)$ be the global b-function of $f(x)$ (see Section 5 for the definition). Assume $\tilde{b}_f(s_0 - j) \neq 0$ for any $j = 1, 2, 3, \ldots$. Then ρ is an isomorphism on K^n [13, Proposition 6.2].

Our aim is to give an algorithm to compute $N|_{s=s_0}$ for a given s_0. We assume here that $f(x)$ is defined over a subfield K_0 of K and let $g(s) \in K_0[s]$ be the minimal polynomial of s_0 over K_0. (If s_0 is transcendental over K_0, then we put $g(s) := 0$.) Let $\pi : K_0[s] \to K_0[s]/g(s)K_0[s] \subset K$ be the canonical ring homomorphism. Then π extends to a ring homomorphism $\pi : A_n(K_0)[s] \to A_n(K)$.

Let $\psi(\boldsymbol{G}_0)$ be as in Theorem 19 with K replaced by K_0. Let $\prec$ be a monomial order on $\boldsymbol{N}^{2n+1} \ni (\alpha, \beta, \mu)$ which is parametric with respect to s; i.e., $\prec$ satisfies (A1)–(A3) with $L_0 := \boldsymbol{N}^{2n}$ and $L := \boldsymbol{N}^{2n} \times \boldsymbol{N}$. Moreover, we assume that $\prec$ eliminates ∂; i.e., it satisfies

(A6) If $\beta, \beta' \in \boldsymbol{N}^n$ satisfy $|\beta| > |\beta'|$, then we have $(\alpha, \beta, \mu) \succ (\alpha', \beta', \mu')$ for any $\alpha, \alpha' \in \boldsymbol{N}^n$ and $\mu, \mu' \in \boldsymbol{N}$.

Let $\prec_0$ be the restriction of $\prec$ to $\boldsymbol{N}^{2n} \times \{0\}$. For an element

$$P = \sum_{\alpha, \beta, \mu} c_{\alpha\beta\mu} s^\mu x^\alpha \partial^\beta$$

of $A_n(K_0)[s]$, let $(\alpha_0, \beta_0, \mu_0)$ be the leading exponent of P with respect to $\prec$. Then we set

$$\mathrm{lcoef}_0(P) := \sum_{\mu \geq 0} c_{\alpha_0\beta_0\mu} s^\mu \in K_0[s].$$

Moreover, we define the *order* of P by

$$\mathrm{ord}(P) := \max\{|\beta| \mid c_{\alpha\beta\mu} \neq 0 \text{ for some } \alpha \in \boldsymbol{N}^n \text{ and } \mu \in \boldsymbol{N}\},$$

and if $\mathrm{ord}(P) = k$, we define the *principal symbol* of P by

$$\sigma(P) = \sigma_k(P) := \sum_{|\beta| = k} \sum_{\mu, \alpha} c_{\alpha\beta\mu} s^\mu x^\alpha \xi^\beta \in K[x, \xi, s]$$

with a commutative variable $\xi = (\xi_1, \ldots, \xi_n)$.

Proposition 20. *Let $\psi(\boldsymbol{G}_0)$ be as in Theorem 19 with K replaced by K_0. Let $\boldsymbol{G}'$ be a Gröbner basis (with respect to the order $\prec$ above) of the left ideal of $A_n(K_0)[s]$ generated by $\psi(\boldsymbol{G}_0)$ and $g(s)$. Assume that $\pi(\mathrm{lcoef}_0(P)) \neq 0$ for any $P \in \boldsymbol{G}'$ such that $\pi(P) \neq 0$. Then $\pi(\boldsymbol{G}') := \{\pi(P(s)) \mid P(s) \in \boldsymbol{G}'\}$ constitutes a set of* *involutory generators of* $J_f|_{s=s_0}$; *i.e.*, $\pi(\mathbf{G}')$ *generates* $J_f|_{s=s_0}$ *over* D, *and* $\sigma(\pi(\mathbf{G}')) := \{\sigma(\pi(Q)) \mid Q \in \mathbf{G}'\}$ *generates the sheaf of ideals* $\sigma(J_f|_{s=s_0})$ *of* $O_{K^{2n}}$ *which is generated by* $\{\sigma(P) \mid P \in J_f|_{s=s_0}\}$. *In particular, the characteristic variety of* $N|_{s=s_0}$ *is given by*

$$\mathrm{Char}(N|_{s=s_0}) = \{(x,\xi) \in K^{2n} \mid \sigma(\pi(P))(x,\xi) = 0 \textit{ for any } P(s) \in \mathbf{G}'\}.$$

Proof. By applying Proposition 7 with a replaced by s and $J(a)$ by $(g(s))$, we know that $\pi(\mathbf{G}')$ is a Gröbner basis (with respect to $\prec_0$) of $J_f|_{s=s_0}$. The involutivity follows from the condition (A6) (cf. [21,25]). □

5. The Bernstein–Sato polynomial

In this section, we present two algorithms for computing the b-function of an arbitrary polynomial. Let us begin with some definitions and remarks. Let K be an algebraically closed field of characteristic zero and let $f(x) \in K[x]$ be an arbitrary polynomial of n variables. Let $N := D[s]f^s = D/J_f$ be as in introduction. Then the *local b-function* (at the origin) $b_f(s)$ of $f(x)$ is the monic polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$P(s)f^{s+1} = b(s)f^s \quad \text{in } N_0 \tag{9}$$

with some $P(s) \in D[s]_0$; the *global b-function* $\tilde{b}_f(s)$ of $f(x)$ is the monic polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$P(s)f^{s+1} = b(s)f^s \quad \text{in } \Gamma(K^n, N) \tag{10}$$

with some $P(s) \in A_n(K)[s]$.

The existence of $\tilde{b}_f(s)$ was proved by Bernstein [3]. Note that $b_f(s)$ is a divisor of $\tilde{b}_f(s)$. If, e.g., $f(x)$ is quasi-homogeneous, or $f(x)$ has 0 as its only singularity, then the local and the global b-functions coincide. It is also to be noted that if $f(x)$ is defined over a subfield K_0 of K, then the above definitions with K replaced by K_0 yield the same b-function. Hence, in the actual computation, we do not have to assume that K is algebraically closed. Kashiwara [13] proved that the roots of $b_f(s)$ are negative rational numbers. In particular, we have $b_f(s) \in \mathbf{Q}[s]$ in fact.

For the first algorithm, we use the order $\prec_1$ introduced in Section 3.

Theorem 21 (Oaku [26]). (i) *Let* I *be a left ideal of* $A_{n+1}(K)[y_1]$ *generated by*

$$t - y_1 f(x), \qquad \partial_i + y_1 f_i(x)\partial_t \quad (i = 1, \dots, n)$$

with $f_i(x) := \partial f/\partial x_i$. *Let* $\mathbf{G}$ *be a Gröbner basis of* I *with respect to the order* $\prec_1$ *consisting of F-homogeneous operators. Put* $\psi(\mathbf{G}) := \{\psi(P(1)) \mid P(y_1) \in \mathbf{G}\}$.

(ii) *Let* $\prec$ *be an order on* $\mathbf{N}^{2n+1}$ *satisfying* (A1), (A2), (A6), *and let* $\mathbf{G}_1$ *be a Gröbner basis of the left ideal of* $A_n(K)[s]$ *generated by* $\psi(\mathbf{G})$ *with respect to* $\prec$. *Let* J *be the ideal of* $K[x,s]$ *generated by* $\mathbf{G}_1 \cap K[x,s]$.

Under the assumptions (i) *and* (ii), $\tilde{b}_f(-s-1)$ *is the monic generator of the ideal* $J \cap K[s]$ *of* $K[s]$, *while* $b_f(-s-1)$ *is the monic generator of the ideal* $O_0 J \cap K[s]$ *of* $K[s]$.

Proof. Let $I(1)$ be the left ideal of $A_n(K)[s]$ generated by $t - f(x)$ and $\partial_i + f_i(x)\partial_t$ $(i = 1, \ldots, n)$. Applying Theorem 17, we know that $\psi(\boldsymbol{G})$ generates the left ideal $\psi(I(1))$ of $A_n[s]$ generated by $\{\psi(P) \mid P \in I(1) \setminus \{0\}, \operatorname{ord}_F(P) = 0\}$. Since the order $\prec$ eliminates ∂, $\boldsymbol{G}_1 \cap K[x,s]$ generates the ideal $\psi(I(1)) \cap K[x,s]$. We also know that $\boldsymbol{G}_1 \cap K[x,s]$ generates the ideal of $O_0[s]$ which is generated by $\{\psi(P) \mid P \in (A_1 D)_0 I(1)\} \cap O_0[s]$ by a localization argument similar to the one used in the proof of Theorem 19. Combining the above arguments with Proposition 4, we know that $b_f(-s-1)$ is the monic generator of $O_0[s]J \cap K[s]$. On the other hand, $\tilde{b}_f(-s-1)$ is the monic generator of $J \cap K[s]$ since $\Gamma(K^n, O[s]) = K[x,s]$ and $\Gamma(K^n, D[s]) = A_n(K)[s]$. $\square$

Now that we have a set of generators of J, we can compute $\tilde{b}_f(-s-1)$ immediately by a Gröbner basis computation in $K[x,s]$ with respect to an order eliminating x. The monic generator of $O_0[s] \cap J$ can be computed by the following algorithm where we regard K as being generated by a finite set of generators over $\boldsymbol{Q}$ instead of assuming that K is algebraically closed. The following algorithm is a slight modification of [26, Algorithm 4.5].

Algorithm 1. *Input:* generators $f_1(x,s), \ldots, f_k(x,s)$ of an ideal J of $K[x,s]$:

(i) Compute the monic generator $f_0(s)$ of the ideal $J(0)$ of $K[s]$ that is generated by $f_1(0,s), \ldots, f_k(0,s)$ by Gröbner basis or GCD computation; if $f_0(s) = 1$, then put $b(s) := 1$ and quit;

(ii) Compute the irreducible decomposition $f_0(s) = g_1(s)^{\mu_1} \cdots g_d(s)^{\mu_d}$ in $K[s]$;

(iii) For $i := 1$ to d do {

by computing the ideal quotient $J : g_i(s)^\ell$ for $\ell = \mu_i, \mu_i + 1, \ldots$ repeatedly, determine the least $\ell \geq \mu_i$ so that $J : g_i(s)^\ell$ contains an element $a_i(x,s) \in K[x,s]$ such that $a_i(0,s)$ is not a multiple of $g_i(s)$. (This process can be done by Gröbner basis computation in $K[x,s]$ and division in $K[s]$.) Denote this ℓ by ℓ_i;

}

(iv) Put $b(s) := g_1(s)^{\ell_1} \cdots g_d(s)^{\ell_d}$;

Output: $b(s)$ is the monic generator of $O_0[s]J \cap K[s]$.

Proposition 22. *Assume* $O_0[s]J \cap K[s] \neq \{0\}$. *Then* $b(s)$ *is the monic generator of* $O_0[s]J \cap K[s]$ *in the above algorithm.*

Proof. Let $h(s)$ be the monic generator of $O_0[s]J \cap K[s]$. First, $f_0(s)$ divides $h(s)$ since $J(0) \cap K[s] \supset O_0[s]J \cap K[s]$. Then it also follows that $b(s)$ divides $h(s)$ in view of the definition of ℓ_i and the fact that there exists $c(x) \in K[x]$ so that $c(0) \neq 0$ and $c(x)h(s) \in J$. This also assures the existence of $b(s)$ i.e. that the algorithm does not fail to stop.

It remains to prove $b(s) \in O_0[s]J \cap K[s]$. Put $Q := J : b(s)$. Let $\overline{K}$ be the algebraic closure of K and set

$$V(Q) := \{(x,s) \in \overline{K}^{n+1} \mid g(x,s) = 0 \text{ for any } g \in Q\}.$$

Then we have

$$\begin{aligned} V(Q) \cap (\{0\} \times \overline{K}) &\subset V(J) \cap (\{0\} \times \overline{K}) \\ &= \{(0,s) \mid s \in \overline{K},\ f_1(0,s) = \cdots = f_k(0,s) = 0\} \\ &= \bigcup_{i=1}^{d} \{(0,s) \mid g_i(s) = 0\}. \end{aligned}$$

Since Q contains $a_i(x,s)$ and $a_i(0,s)$ is not a multiple of $g_i(s)$ for each $i = 1,\dots,d$, we have $V(Q) \cap (\{0\} \times \overline{K}) = \emptyset$. Moreover there exists $c(x) \in K[x]$ so that $c(x)h(s) \in J \subset Q$ and $c(0) \neq 0$. Hence it follows that there exists $q(x) \in Q \cap K[x]$ so that $q(0) \neq 0$ (see e.g. [9, p. 162]). This implies $q(x)b(s) \in J$, and hence $b(s) \in O_0[s]J \cap K[s]$. □

Combining Theorem 21 and this algorithm, we have obtained an algorithm of computing $b_f(s)$. Now let us describe another algorithm for $b_f(s)$ which is based on Theorem 19.

Theorem 23. *In the same notation as in Theorem* 19, *let us denote by* I_f *the left ideal of* $A_n(K)[s]$ *generated by* $\psi(G_0)$ *and* f. *Let* G_2 *be a Gröbner basis of* I_f *with respect to the order* $\prec$ *satisfying* (A1), (A2), (A6). *Let* J *be the ideal of* $K[x,s]$ *generated by* $G_2 \cap K[x,s]$. *Then* $\tilde{b}_f(s)$ *is the monic generator of* $J \cap K[s]$, *while* $b_f(s)$ *is the monic generator of* $O_0[s]J \cap K[s]$.

Proof. In view of (9) and (10), $b_f(s)$ and $\tilde{b}_f(s)$ are the monic generators of the ideals $(J_f + D[s]f)_0 \cap K[s]$ and $\Gamma(K^n, J_f + D[s]f) \cap K[s]$, respectively (cf. [11, 30]). Hence for the proof of the theorem, we can use the same argument as in Theorem 21. □

In the actual computation corresponding to Theorems 21, 23 and Algorithm 1, we may assume that K is the quotient field of $\mathbf{Q}[a]/J(a)$ as in Section 2 and can apply Propositions 6 and 7 in the computation of Gröbner bases. In particular, we can treat the case where f has parameters, and can obtain a sufficient condition on the special values of the parameters for the result to be valid after the specialization.

6. The algebraic local cohomology group

Let K be an algebraically closed field of characteristic zero and let $f(x) \in K[x]$ be an arbitrary polynomial. Put $Y := \{x \in K^n \mid f(x) = 0\}$. Then the algebraic local cohomology group $H^k_{[Y]}(O)$ has a structure of left D-module and vanishes if $k \neq 1$ (cf. [14]). Moreover, $H^1_{[Y]}(O)$ is isomorphic to $O[f^{-1}]/O$ although its structure as left D-module is not necessarily obvious. Our purpose is to give an algorithm of computing the left D-module $H^1_{[Y]}(O)$ as an application of the computation of the b-function.

Let P be an element of $A_{n+1}(K)$ of F-order at most k. Then we can write P in the form

$$P = \sum_{j=0}^{k} P_j(t\partial_t, x, \partial)\partial_t^j + R$$

uniquely with $P_j \in A_n(K)[t\partial_t]$ and $R \in A_{n+1}(K)$ with $\mathrm{ord}_F(R) \le -1$. Then we put

$$\varphi(P,k) := (P_0(0,x,\partial), \ldots, (k-1)!P_{k-1}(0,x,\partial), k!P_k(0,x,\partial)) \in D^{k+1}.$$

The proof of the following theorem is based on an algorithm to compute the induced system (or the restriction) of a D-module, details of which will appear elsewhere [24].

Theorem 24. *Let I and $\boldsymbol{G}$ be as in* (i) *of Theorem* 21 *and let $\tilde{b}_f(s)$ be the global b-function of $f(x)$. Put $k := \max\{j \in \boldsymbol{Z} \mid \tilde{b}_f(-j-1) = 0\}$. (Note that $k \ge 0$ since $\tilde{b}_f(-1) = 0$.) Then $H^1_{[Y]}(O) = O[f^{-1}]/O$ is generated by the residue classes $[f^{-j-1}]$ of f^{-j-1} with $j = 0, 1, \ldots, k$ as left D-module. Moreover, $H^1_{[Y]}(O)$ is isomorphic to D^{k+1}/L, where L is the left D-module generated by*

$$\{\varphi(\partial_t^\nu P(1), k) \mid P \in \boldsymbol{G},\ \nu \in \boldsymbol{N},\ \nu + \mathrm{ord}_F(P(1)) \le k\}.$$

The algebraic local cohomology group $H^1_{[Y]}(O)$ is closely related to the D-module $O[f^{-1}]$ as is seen by the exact sequence

$$0 \to O \to O[f^{-1}] \to H^1_{[Y]}(O) \to 0. \tag{11}$$

In particular, we get an algorithm of computing the characteristic variety and multiplicities of the D-module $O[f^{-1}]$ by virtue of the preceding theorem.

Proposition 25. *Under the same assumptions as in the preceding theorem, $O[f^{-1}]$ is generated by $f^{-1}, \ldots, f^{-k-1}$ as a left D-module. If $k = 0$, then we have an isomorphism $O[f^{-1}] \simeq N_f|_{s=-1}$ as left D-modules. Hence we have an algorithm of computing the structure of the D-module $O[f^{-1}]$ if $\tilde{b}_f(\nu) \ne 0$ for $\nu = -2, -3, \ldots$.*

Proof. The first assertion follows from Theorem 24 and the exact sequence (11). If $k = 0$, then substituting $-2, -3, \ldots$ for s in (10), we know that $O[f^{-1}] = Df^{-1}$. We also have $Df^{-1} = N_f|_{s=-1}$ by [13, Proposition 6.2]. □

7. Remarks on the analytic case

In this section, let us assume that K is the field $\boldsymbol{C}$ of complex numbers and use the usual topology of $\boldsymbol{C}^n$ instead of the Zariski topology. Then we can use the sheaf O^{an} of analytic functions on $\boldsymbol{C}^n$, and the sheaf D^{an} of analytic differential operators on

$\mathbf{C}^n$ instead of O and D, respectively. Let us call such objects *analytic* as are obtained by replacing O and D by O^{an} and D^{an}, respectively. Our aim is to show that our algorithms presented so far yield solutions also for analytic objects.

Let $A_1 D^{an}$ be the sheaf on $\mathbf{C}^n$ defined as follows: For an open set U of $\mathbf{C}^n$, the set of sections of $A_1 D^{an}$ over U consists of the differential operators represented by a finite sum

$$P = \sum_{\mu,\nu,\alpha} a_{\mu\nu\alpha}(x) t^{\mu} \partial_t^{\nu} \partial^{\alpha}$$

with $a_{\mu\nu\alpha}(x) \in \Gamma(U, O^{an})$. Put also

$$I^{an} := (A_1 D^{an})(t - f(x)) + \sum_{j=1}^{n} (A_1 D^{an})(\partial_i + f_i \partial_t),$$

$$J_f^{an} := \{P(s) \in D^{an}[s] \mid P(s) f^s = 0\}$$

with $f_i := \partial f / \partial x_i$. Then the arguments in Section 2 also hold for these analytic objects (cf. [18]). Hence the following lemma assures that $\psi(G_0)$ of Theorem 19 also generates J_f^{an}.

Lemma 26. *We have* $I^{an} \cap D^{an}[t\partial_t] = D^{an} \otimes_D (I \cap D[t\partial_t])$.

Proof. This is an immediate consequence of the faithful flatness of D^{an} over D. □

For the validity of Theorem 23 and Algorithm 1 in the analytic case, we need the following two lemmas, which follow from the faithful flatness of O^{an} over O.

Lemma 27. *We have*

$$(J_f^{an} + D^{an}[s] f) \cap O^{an}[s] = O^{an}[s] \otimes_{O[s]} ((J_f + D[s] f) \cap O[s]).$$

Lemma 28. *For an ideal* J *of* $\mathbf{C}[x, s]$, *we have* $(O^{an})_0[s] J \cap \mathbf{C}[s] = O_0[s] J \cap \mathbf{C}[s]$.

Thus we have proved that the local b-function in the algebraic sense and the one in the analytic sense coincide. This also guarantees the correctness of Theorem 21 in the analytic case. Finally, Theorem 24 is also valid in the analytic case since we have

$$D^{an} \otimes_D H^1_{[Y]}(O) = O^{an} \otimes_O (\mathbf{C}[f^{-1}]/O) = O^{an}[f^{-1}]/O^{an} = H^1_{[Y]}(O^{an}).$$

8. Implementation and examples of computation

We have implemented our algorithms presented so far in a computer algebra system Kan of Takayama [29] and partly in Risa/Asir [20]. Kan is a system designed especially for Gröbner basis computation in rings of polynomials, differential operators, and (q-) difference operators. Hence we use Kan for Gröbner basis computations in

Table 1
b-functions for f with isolated singularity

f	b_f	A1	A2
x^5+y^5	$(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})$ $(s+1)^2(s+\frac{6}{5})(s+\frac{7}{5})(s+\frac{8}{5})$	3.1s	3.25s
$x^5+y^3x^3+y^5$	$(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})$ $(s+1)^2(s+\frac{6}{5})(s+\frac{7}{5})$	105s	518s
$x^3+y^3+z^3$	$(s+1)^2(s+\frac{4}{3})(s+\frac{5}{3})(s+2)$	1.3s	1.7s
$x^3+z^2y^2x^2+zyx$ $+y^3+z^3$	$(s+1)^2(s+\frac{4}{3})(s+\frac{5}{3})(s+2)$	238s	548s
$x^6+y^4+z^3$	$(s+\frac{3}{4})(s+\frac{11}{12})(s+1)^2(s+\frac{13}{12})$ $(s+\frac{7}{6})(s+\frac{5}{4})(s+\frac{4}{3})(s+\frac{17}{12})$ $(s+\frac{3}{2})(s+\frac{19}{12})(s+\frac{5}{3})(s+\frac{7}{4})$ $(s+\frac{11}{6})(s+\frac{23}{12})(s+2)(s+\frac{25}{12})$ $(s+\frac{9}{4})$	104s	116s
$x^4+zyx+y^4+z^3$	$(s+1)^3(s+\frac{5}{4})(s+\frac{4}{3})(s+\frac{3}{2})$ $(s+\frac{5}{3})(s+\frac{7}{4})$	219s	266s

the Weyl algebra while we use a general-purpose computer algebra system Risa/Asir for factorization, Gröbner basis computation, and prime (and primary) decomposition in the polynomial ring.

Let us begin with examples of computation of b-functions. In Tables 1 and 2, A1 refers to the algorithm based on Theorem 21 and Algorithm 1 while A2 refers to the one based on Theorem 23 and Algorithm 1. The Gröbner basis computations corresponding to Theorems 21 and 23 are executed by Kan; Algorithm 1 is performed by Risa/Asir. The computation time indicates the sum of the computation time of Kan and Risa/Asir on Sun 4/20 (256 Mbyte memory). The time of handing on the output of Kan to Risa/Asir, which is done by writing to and reading from a file, is not included.

Most of the examples in Table 1 are included in [32, 31] (see also [6]). See [32, pp. 198–200] for some of the examples in Table 2.

As an example with a parameter, put $f := x^4 + y^4 + z^2 + axyz$. Assuming a to be transcendental over $\boldsymbol{Q}$, we obtain the b-function of f over the field $\boldsymbol{Q}(a)$ as

$$b_f(s) = (s+1)^2(s+\tfrac{5}{4})(s+\tfrac{3}{2})(s+\tfrac{7}{4})(s+2).$$

Table 2
b-functions for f with non-isolated singularities

f	b_f	A1	A2
$x^3 + z^2y^2$	$\left(s+\frac{5}{6}\right)^2 (s+1) \left(s+\frac{7}{6}\right)^2 \left(s+\frac{4}{3}\right)$ $\left(s+\frac{5}{3}\right)$	2.2s	3.1s
$x^4 + x^3 + z^3y^3 + z^2y^2$	the same as above	14s	99s
$(x^3 - z^2y^2)^2$	$\left(s+\frac{5}{12}\right)^2 \left(s+\frac{1}{2}\right) \left(s+\frac{7}{12}\right)^2 \left(s+\frac{2}{3}\right)$ $\left(s+\frac{5}{6}\right) \left(s+\frac{11}{12}\right)^2 (s+1) \left(s+\frac{13}{12}\right)^2$ $\left(s+\frac{7}{6}\right) \left(s+\frac{4}{3}\right)$	268s	286s
$x^5 - z^2y^2$	$\left(s+\frac{7}{10}\right)^2 \left(s+\frac{9}{10}\right)^2 (s+1) \left(s+\frac{11}{10}\right)^2$ $\left(s+\frac{6}{5}\right) \left(s+\frac{13}{10}\right)^2 \left(s+\frac{7}{5}\right) \left(s+\frac{8}{5}\right) \left(s+\frac{9}{5}\right)$	12.7s	11.5s
$x^5 - z^3y^2$	$\left(s+\frac{8}{15}\right) \left(s+\frac{7}{10}\right) \left(s+\frac{11}{15}\right) \left(s+\frac{13}{15}\right)$ $\left(s+\frac{9}{10}\right) \left(s+\frac{14}{15}\right) (s+1) \left(s+\frac{16}{15}\right)$ $\left(s+\frac{11}{10}\right) \left(s+\frac{17}{15}\right) \left(s+\frac{6}{5}\right) \left(s+\frac{19}{15}\right)$ $\left(s+\frac{13}{10}\right) \left(s+\frac{7}{5}\right) \left(s+\frac{22}{15}\right) \left(s+\frac{8}{5}\right) \left(s+\frac{9}{5}\right)$	32s	32s
$x^3 - 3zyx + y^3$	$(s+1)^3 \left(s+\frac{4}{3}\right) \left(s+\frac{5}{3}\right)$	5.5s	5.3s
$x^3 + y^3 + z^3 - 3xyz$	$(s+1)^3$	0.9s	0.8s
$y(x^5 - z^2y^2)$	$\left(s+\frac{7}{15}\right) \left(s+\frac{3}{5}\right) \left(s+\frac{7}{10}\right) \left(s+\frac{11}{15}\right)$ $\left(s+\frac{4}{5}\right) \left(s+\frac{13}{15}\right) \left(s+\frac{9}{10}\right) \left(s+\frac{14}{15}\right)$ $(s+1)^2 \left(s+\frac{16}{15}\right) \left(s+\frac{11}{10}\right) \left(s+\frac{17}{15}\right)$ $\left(s+\frac{6}{5}\right) \left(s+\frac{19}{15}\right) \left(s+\frac{13}{10}\right) \left(s+\frac{7}{5}\right)$ $\left(s+\frac{23}{15}\right)$	337s	376s
$y(x^3 - z^2y^2)$	$\left(s+\frac{5}{9}\right) \left(s+\frac{7}{9}\right) \left(s+\frac{5}{6}\right) \left(s+\frac{8}{9}\right)$ $(s+1)^2 \left(s+\frac{10}{9}\right) \left(s+\frac{7}{6}\right) \left(s+\frac{11}{9}\right)$ $\left(s+\frac{13}{9}\right)$	17s	19s
$y((y+1)x^3 - z^2y^2)$	the same as above	356s	185s

(The computation time is 114s by A1.) By using Proposition 7, we know that this also gives the b-function of f with an arbitrary a which is not necessarily transcendental over $\boldsymbol{Q}$ under the condition

$$a(a^2-8)(a^2+8)(3a^4-128)(a^4-192)\neq 0.$$

If a does not satisfy this condition, we can use Proposition 6 with $J(a)$ being the ideal generated by each irreducible component of the left hand side of the above condition. When $a=0$ or $3a^4-128=0$ or $a^4-192=0$, we can verify that the b-function of f is the same as above (11s, 110s, 76s, respectively, by A1) while the b-function of f is

$$b_f(s)=(s+1)^3\left(s+\tfrac{3}{2}\right)$$

if $a^2-8=0$ or $a^2+8=0$ (18s each by A1). Note that f has non-isolated singularities if and only if $(a^2-8)(a^2+8)=0$.

Now, let us show some examples of computation of $N:=D[s]f^s=D[s]/J_f$ and $H^1_{[Y]}(O)$ with $Y:=\{(x,y,z)\in K^3 \mid f(x,y,z)=0\}$. First, let us consider $f:=y(x^5-y^2z^2)$. By Theorem 19, we get as an involutory basis of the ideal J_f the following 7 operators (21s):

- $-2x\partial_x+10y\partial_y-15z\partial_z$,
- $y\partial_y-z\partial_z-s$,
- $-2y^2z\partial_x-5x^4\partial_z$,
- $-4y^2z\partial_x^2+25x^3z\partial_z^2+5x^3(-10s-3)\partial_z$,
- $-8y^2z\partial_x^3-125x^2z^2\partial_z^3+25x^2z(20s-1)\partial_z^2$
  $-5x^2(10s+3)(10s+1)\partial_z$,
- $16y^2z\partial_x^4-625xz^3\partial_z^4+750xz^2(5s-2)\partial_z^3-75xz(100s^2-30s+3)\partial_z^2$
  $+5x(10s+1)(10s+3)(10s-1)\partial_z$,
- $-32y^2z\partial_x^5-3125z^4\partial_z^5+6250z^3(4s-3)\partial_z^4-1875z^2(40s^2-40s+11)\partial_z^3$
  $+625z(40s^2-20s+3)(4s-1)\partial_z^2$
  $-5(10s-1)(10s+1)(10s-3)(10s+3)\partial_z$

with $\partial_x=\partial/\partial x$, $\partial_y=\partial/\partial y$, $\partial_z=\partial/\partial z$. Since the principal symbols of these operators do not involve s, we know that $J_f(s_0)$ is also generated by these operators with s replaced by a special value s_0 provided that $b_f(s_0-v)\neq 0$ for $v=1,2,\ldots$. In particular, these operators with s replaced by -1 constitute a set of involutory generators of the annihilator ideal for f^{-1} in $O[f^{-1}]$. (The global b-function coincides with $b_f(s)$ in this case.) By applying Theorem 24, we have $H^1_{[Y]}(O)=D/J$ with the following operators as a set of involutory generators of the sheaf of left ideals J (14s):

- $-2x\partial_x+10y\partial_y-15z\partial_z$,
- $y\partial_y-z\partial_z+1$,
- $-2y^2z\partial_x-5x^4\partial_z$,
- $y(-x^5+z^2y^2)$,
- $-4y^2z\partial_x^2+25x^3z\partial_z^2+35x^3\partial_z$,
- $8y^2z\partial_x^3+125x^2z^2\partial_z^3+525x^2z\partial_z^2+315x^2\partial_z$,

- $16y^2z\partial_x^4 - 625xz^3\partial_z^4 - 5250xz^2\partial_z^3 - 9975xz\partial_z^2 - 3465x\partial_z$,
- $-32y^2z\partial_x^5 - 3125z^4\partial_z^5 - 43750z^3\partial_z^4 - 170625z^2\partial_z^3 - 196875z\partial_z^2$ $-45045\partial_z$.

Put the cotangent bundle of K^3 as

$$T^*K^3 := \{(x, y, z, \xi\,\mathrm{d}x + \eta\,\mathrm{d}y + \zeta\,\mathrm{d}z) \mid (x, y, z) \in K^3,\ (\xi, \eta, \zeta) \in K^3\}.$$

In general, let V be a non-singular variety in K^3 defined by

$$V = \{(x, y, z) \in U \mid g_1(x, y, z) = \cdots = g_\ell(x, y, z) = 0\}$$

with a Zariski open set U of K^3 and $g_1, \ldots, g_\ell \in K[x, y, z]$ so that $\mathrm{d}g_1, \ldots, \mathrm{d}g_\ell$ are linearly independent on V. Then the conormal bundle of V is a subset of T^*K^3 defined by

$$T_V^*K^3 := \{(x, y, z, c_1\mathrm{d}g_1 + \cdots + c_d\,\mathrm{d}g_\ell) \mid (x, y, z) \in V,\ c_1, \ldots, c_\ell \in K\}.$$

For any $s_0 \in K$, the characteristic variety of $N|_{s=s_0}$ is given by

$$\mathrm{Char}(N|_{s=s_0}) = T_{V_0}^*K^3 \cup T_{V_1}^*K^3 \cup T_{V_2}^*K^3 \cup T_{V_3}^*K^3 \cup T_{V_4}^*K^3 \cup T_{V_5}^*K^3$$

with

$$V_0 := \{(x, y, z) \in K^3 \mid x = y = z = 0\},$$

$$V_1 := \{(x, y, z) \in K^3 \mid x = y = 0\} \setminus V_0,$$

$$V_2 := \{(x, y, z) \in K^3 \mid x = z = 0\} \setminus V_0,$$

$$V_3 := \{(x, y, z) \in K^3 \mid y = 0\} \setminus V_1,$$

$$V_4 := \{(x, y, z) \in K^3 \mid x^5 - y^2z^2 = 0\} \setminus (V_1 \cup V_2),$$

$$V_5 := \{(x, y, z) \in K^3 \mid f(x, y, z) \neq 0\}.$$

For this irreducible decomposition of the characteristic variety, we use the prime decomposition program of Risa/Asir. The multiplicities of $T_{V_j}^*K^3$ are 3, 2, 1, 1, 1, 1, respectively. We get the multiplicities by computing the (local) Hilbert polynomials of the ideal generated by the principal symbols of the generators of J_f listed above through the homogenization and Gröbner basis computation in the polynomial ring (cf. [16]). The characteristic variety of $H^1_{[Y]}(O)$ is given by

$$\mathrm{Char}(H^1_{[Y]}(O)) = T_{V_0}^*K^3 \cup T_{V_1}^*K^3 \cup T_{V_2}^*K^3 \cup T_{V_3}^*K^3 \cup T_{V_4}^*K^3$$

and the multiplicity of each component is the same as above.

Finally, put $f := x^4 + y^4 + z^2 + axyz$ and assume that the parameter a satisfies $a^2 + 8 = 0$. Then we get as involutory generators of the ideal J_f the following 12 operators (14s):

- $x\partial_x + y\partial_y + 2z\partial_z - 4s$,
- $-ya\partial_x + xa\partial_y + 4(x + y)(x - y)\partial_z$,

- $(ayx + 2z)\partial_y + (-azx - 4y^3)\partial_z$,
- $2z\partial_x + x^2a\partial_y + y(-4yx - az)\partial_z$,
- $(-azx - 4y^3)\partial_x + (4x^3 + azy)\partial_y$,
- $ya\partial_x^2 + 4y^2\partial_x\partial_z + ya\partial_y^2 + 2(2yx + az)\partial_y\partial_z + 8xz\partial_z^2 - 4as\partial_y - 16xs\partial_z$,
- $(x^4 + azyx + y^4 + z^2)\partial_y + s(-azx - 4y^3)$,
- $-y^2a\partial_x^2 - 4z\partial_x\partial_y - x^2a\partial_y^2 + 2z(-4yx - az)\partial_z^2$
  $+4((4s + 2)yx + saz)\partial_z$,
- $2z\partial_x^2 - yza\partial_x\partial_z + 2z\partial_y^2 - 3xza\partial_y\partial_z + 8y^2z\partial_z^2 + 2xa(2s + 1)\partial_y$
  $+8y^2(-2s - 1)\partial_z$,
- $2z\partial_x^3 + y^2a\partial_x^2\partial_y - 3yza\partial_x^2\partial_z + 6z\partial_x\partial_y^2 + x^2a\partial_y^3 + yza\partial_y^2\partial_z$
  $+4z^2a\partial_y\partial_z^2 - 16xz^2\partial_z^3 + 4ya(s + 1)\partial_x^2$
  $+2za(-4s + 1)\partial_y\partial_z + 8xz(8s + 1)\partial_z^2 - 8x(2s + 1)(4s + 1)\partial_z$,
- $y^2a\partial_x^3 + 6z\partial_x^2\partial_y - yza\partial_x\partial_y\partial_z + 2z^2a\partial_x\partial_z^2 + (-ayx + 2z)\partial_y^3$
  $-5xza\partial_y^2\partial_z - 16yz^2\partial_z^3 + za(-4s - 1)\partial_x\partial_z + 2xa(4s + 1)\partial_y^2$
  $+8yz(8s + 1)\partial_z^2 - 8y(2s + 1)(4s + 1)\partial_z$,
- $2z\partial_x^4 + 2y^2a\partial_x^3\partial_y - 3yza\partial_x^3\partial_z + 12z\partial_x^2\partial_y^2 + 6z^2a\partial_x\partial_y\partial_z^2$
  $+2(-ayx + z)\partial_y^4 - 7xza\partial_y^3\partial_z + 32z^3\partial_z^4 + 2ya(2s + 3)\partial_x^3$
  $-12zas\partial_x\partial_y\partial_z + 12xas\partial_y^3 + 48z^2(-4s + 1)\partial_z^3$
  $+384zs^2\partial_z^2 - 32(2s + 1)(4s + 1)s\partial_z$.

For any $s_0 \in K$, the characteristic variety of $N|_{s=s_0}$ is given by

$$\mathrm{Char}(N|_{s=s_0}) = T^*_{W_0}K^3 \cup T^*_{W_1}K^3 \cup T^*_{W_2}K^3 \cup T^*_{W_3}K^3$$

with

$$W_0 := \{(x, y, z) \in K^3 \mid x = y = z = 0\},$$

$$W_1 := \{(x, y, z) \in K^3 \mid x^2 + y^2 = 4xy - az = 0\} \setminus W_0,$$

$$W_2 := \{(x, y, z) \in K^3 \mid f(x, y, z, a) = 0\} \setminus W_1,$$

$$W_3 := \{(x, y, z) \in K^3 \mid f(x, y, z, a) \neq 0\}.$$

Moreover, the multiplicity of each $T^*_{W_j}K^3$ is one. (Note that $T^*_{W_1}K^3$ decomposes into two components each of which is of multiplicity one.) The characteristic variety of $H^1_{[Y]}(O)$ is given by

$$\mathrm{Char}(H^1_{[Y]}(O)) = T^*_{W_0}K^3 \cup T^*_{W_1}K^3 \cup T^*_{W_2}K^3$$

and the multiplicity of each component is one also.

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