

Computation of holonomic systems for Feynman amplitudes associated with some simple diagrams

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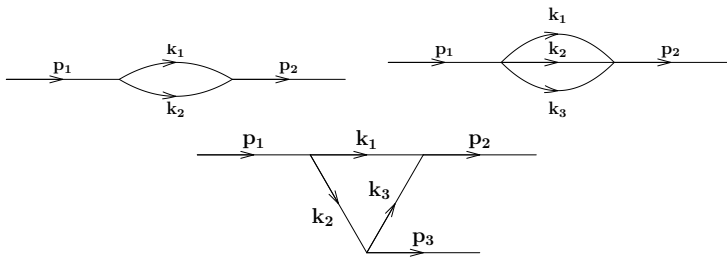
Holonomic systems are a class of systems of linear (partial or ordinary) differential equations. One of the most fundamental properties of a holonomic system is that its solution space is finite-dimensional.

A Feynman amplitude is the integral of a rational function, or more generally, the product of complex powers of polynomials. Hence we can, in principle, apply the following two facts in (computational) D -module theory:

- 1 For (multi-variate) polynomials $f_1, \dots, f_d$ and complex numbers $\lambda_1, \dots, \lambda_d$, the multi-valued analytic function $f_1^{\lambda_1} \cdots f_d^{\lambda_d}$ satisfies a holonomic system, which can be computed algorithmically. Moreover, if we regard λ 's as parameters, then it satisfies a difference-differential system which corresponds to a holonomic system by Mellin transform.
- 2 If a function satisfies a holonomic system, its integral with respect to some of its variables also satisfies a holonomic system, which can be computed algorithmically.

In the integration, it would also be natural to regard the integrand as a local cohomology class associated with $f_1, \dots, f_d$, which corresponds to the Cutkosky-type phase space integral in case of Feynman amplitude.

However, actual computation is hard in general because of the complexity. I have only succeeded in the computation for Feynman amplitudes associated with simple diagrams as below mostly in the two-dimensional space-time.

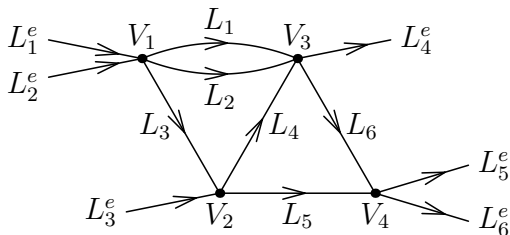


Feynman diagrams and Feynman integrals

Let G be a connected Feynman graph (diagram), i.e., G consists of

- vertices $V_1, \dots, V_{n'}$,
- oriented line segments $L_1, \dots, L_N$ called internal lines,
- oriented half-lines $L_1^e, \dots, L_n^e$ called external lines.

The end-points of each internal line L_i are two distinct vertices, and each external line has only one end-point, which coincides with one of the vertices.

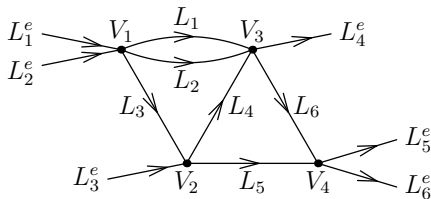


- We associate ν -dimensional vector $\mathbf{p}_r$ to each external line L_r^e ($1 \leq r \leq n'$),
and ν -dimensional vector $\mathbf{k}_l$ and a real number (mass) $m_l \geq 0$ to each internal line L_l ($1 \leq l \leq N$).
- For a vertex V_j and an internal or external line L_l , the incidence number $[j : l]$ is defined as follows:

$[j : l] = 1$ if L_l ends at V_j ,

$[j : l] = -1$ if L_l starts from V_j ,

$[j : l] = 0$ otherwise.



The Feynman integral associated with G is defined to be

$$F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n'} \delta\left(\sum_{r=1}^n [j:r] \mathbf{p}_r + \sum_{l=1}^N [j:l] \mathbf{k}_l\right)}{\prod_{l=1}^N (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1} 0)} \prod_{l=1}^N d^\nu \mathbf{k}_l.$$

Here δ denotes the ν -dimensional delta function,

$$\mathbf{k}_l^2 := k_{l0}^2 - k_{l1}^2 - \dots - k_{l\nu}^2,$$

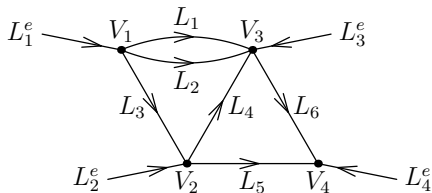
$d^\nu \mathbf{k}_l$ is the ν -dimensional volume element,

and $(\dots + \sqrt{-1} 0)$ means the limit $(\dots + \sqrt{-1} \varepsilon)$ as $\varepsilon \rightarrow +0$.

The integrand is well-defined as a generalized function at least if all $m_l > 0$.

In what follows, we assume that G is external, i.e., for each vertex V_j , there exists a unique external line, which we may assume to be L_j^e , that ends at V_j and that no external line starts from V_j . Then $n = n'$ holds and the Feynman integral is

$$F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^n \delta(\mathbf{p}_j + \sum_{l=1}^N [j : l] \mathbf{k}_l)}{\prod_{l=1}^N (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1} 0)} \prod_{l=1}^N d^\nu \mathbf{k}_l$$



Rewriting the Feynman integral

The delta factors of the integrand of the Feynman integral correspond to the linear equations (momentum preservation)

$$p_j + \sum_{l=1}^N [j : l] k_l = 0 \quad (1 \leq j \leq n)$$

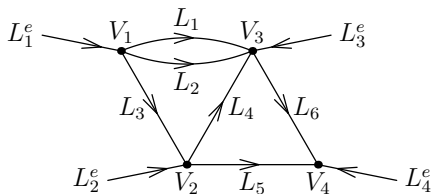
for indeterminates p_j and k_l which correspond to the vectors $\mathbf{p}_j$ and $\mathbf{k}_l$. These equations define an N -dimensional linear subspace of $\mathbb{R}^{n+N}$, which is contained in the hyperplane $p_1 + \cdots + p_n = 0$ since $\sum_{j=1}^n [j : l] = 0$.

Lemma

Let A be the $n \times N$ matrix whose (j, l) -element is $[j : l]$. Then the rank of A is $n - 1$.

For the example below, the matrix A is given by

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



In view of the lemma above, we can choose a set of indices

$$J = \{l_1, \dots, l_{N-n+1}\} \subset \{1, \dots, N\}$$

and integers a_{l_r} and b_{l_j} so that the system

$$p_j + \sum_{l=1}^N [j : l] k_l = 0 \quad (1 \leq j \leq n)$$

of linear equations is equivalent to

$$\sum_{j=1}^n p_j = 0, \quad k_l - \psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}}) = 0 \quad (l \in J^c).$$

Then the Feynman integral is written in the form

$$\begin{aligned}
 F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) &= \int_{\mathbb{R}^{N\nu}} \delta(\mathbf{p}_1 + \dots + \mathbf{p}_n) \\
 &\quad \times \prod_{l \in J^c} \delta(\mathbf{k}_l - \psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})) \\
 &\quad \times \prod_{l=1}^N (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l=1}^N d\mathbf{k}_l \\
 &= \delta(\mathbf{p}_1 + \dots + \mathbf{p}_n) \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})
 \end{aligned}$$

with the amplitude function

$$\begin{aligned}
 \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) &= \int_{\mathbb{R}^{(N-n+1)\nu}} \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \\
 &\quad \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l \in J} d\mathbf{k}_l.
 \end{aligned}$$

D -modules

In general, let D_n be the ring of differential operators in the variables $x = (x_1, \dots, x_n)$ with polynomial coefficients (the n -th Weyl algebra). D_n is the $\mathbb{C}$ -algebra generated by $x_1, \dots, x_n$ and $\partial_{x_1}, \dots, \partial_{x_n}$ with commutation relations

$$\partial_{x_i} x_j - x_j \partial_{x_i} = \delta_{ij}, \quad x_i x_j = x_j x_i, \quad \partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}.$$

A system

$$\sum_{j=1}^l P_{ij} u_j = 0 \quad (i = 1, \dots, m)$$

of linear differential equations with $P_{ij} \in D_n$ for unknown functions $u_1, \dots, u_l$ corresponds to the left D_n -module

$$M = D_n^l / (D_n \vec{P}_1 + \dots + D_n \vec{P}_m), \quad \vec{P}_i = (P_{i1}, \dots, P_{il}).$$

In fact, the solution space of this system in a function space $\mathcal{F}$ is given by the left D_n -module homomorphisms

$$\mathrm{Hom}_{D_n}(M, \mathcal{F}) \simeq \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_l \end{pmatrix} \in \mathcal{F}^l \left| \sum_{j=1}^l P_{ij} u_j = 0 \quad (i = 1, \dots, m) \right. \right\}.$$

This follows from the exact sequence

$$D_n^m \xrightarrow{\psi} D_n^l \xrightarrow{\varphi} M \longrightarrow 0$$

of left D_n -modules. Here φ and ψ are D_n -homomorphisms defined by

$$\varphi((P_1, \dots, P_l)) = [(P_1, \dots, P_l)] \quad (\text{the residue class in } M),$$

$$\psi((Q_1, \dots, Q_m)) = (Q_1 \quad \cdots \quad Q_m) \begin{pmatrix} P_{11} & \cdots & P_{1l} \\ \vdots & & \vdots \\ P_{m1} & \cdots & P_{ml} \end{pmatrix}.$$

Characteristic variety and holonomic systems

For the sake of simplicity, let us consider a system

$$\sum_{j=1}^m P_j u_j = 0 \quad (P_j \in D_n)$$

for a single unknown function u , i.e., the D_n -module

$$M = D_n/I, \quad I := D_n P_1 + \cdots + D_n P_m.$$

In general, a nonzero $P \in D_n$ is written in the form

$$P = \sum_{|\alpha| \leq l} a_\alpha(x) \partial_x^\alpha, \quad a_\alpha(x) \in \mathbb{C}[x]$$

with multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \dots\}$. The principal symbol of P is defined to be

$$\sigma(P) = \sigma(P)(x, \xi) = \sum_{|\alpha|=l} a_\alpha(x) \xi^\alpha \in \mathbb{C}[x, \xi]$$

with commutative indeterminates $\xi = (\xi_1, \dots, \xi_n)$.

Definition

The characteristic variety of $M = D_n/I$ is the algebraic set

$$\text{Char}(M) = \{(x, \xi) \in \mathbb{C}^{2n} \mid \sigma(P)(x, \xi) = 0 \quad (\forall P \in I \setminus \{0\})\}$$

of the cotangent bundle $T^*\mathbb{C}^n = \mathbb{C}^{2n}$.

Theorem (Sato-Kawai-Kashiwara, J. Bernstein)

The dimension of $\text{Char}(M)$ is not less than n if $M \neq 0$.

Definition

The left D_n -module $M = D_n/I$ is holonomic if $\text{Char}(M)$ is n -dimensional or else $M = 0$.

Theorem (Kashiwara)

The solution space of a holonomic system is finite dimensional.

Local cohomology

In general, let $f_1, \dots, f_d$ be polynomials in the variables $x = (x_1, \dots, x_n)$ with complex coefficients such that the variety

$$Y = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_d(x) = 0\}$$

is d -codimensional, i.e., $f_1, \dots, f_d$ are of complete intersection. Then the (algebraic) d -th local cohomology group associated with $f_1, \dots, f_d$ is defined to be the quotient space

$$H_Y^d(\mathbb{C}[x]) := \mathbb{C}[x, f^{-1}] / \sum_{k=1}^n \mathbb{C}[x, (f/f_k)^{-1}]$$

with $f = f_1 \cdots f_d$. It consists of the cohomology classes $[g/f^\nu]$ with $\nu = 1, 2, 3, \dots$ and $g \in \mathbb{C}[x]$.

- $H_Y^d(\mathbb{C}[x])$ has a natural structure of left D_n -module and is holonomic as such.
- The simplest example of the local cohomology group is

$$H_{\{0\}}^1(\mathbb{C}[x]) = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$$

with $x = x_1$ (one variable), which is spanned by the classes $[x^{-k}]$ with $k = 1, 2, 3, \dots$ as a $\mathbb{C}$ -vector space. As a left D_1 -module, it is generated only by $[x^{-1}]$ since $\partial_x^k [x^{-1}] = (-1)^k k! [x^{-k-1}]$.

- There are algorithms (U. Walther, Oaku-Takayama) to compute the local cohomology group as a D -module, in particular, the annihilator (the holonomic system) for each cohomology class.

Integrands of Feynman amplitudes

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in $\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_1, \dots, \mathbf{k}_{l_{N-1}}$. We regard the integrand

$$\prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \\ \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_1, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}0)^{-1}$$

of the Feynman amplitude as the local cohomology class

$$\left[\prod_{l \in J} (\mathbf{k}_l^2 - m_l^2)^{-1} \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_1, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1} \right]$$

associated with polynomials $\mathbf{k}_l^2 - m_l^2$ with $l \in J$ and $\psi_l - m_l^2$ with $l \in J^c$.

The holonomic system for the local cohomology class

$$\left[\prod_{I \in J} (\mathbf{k}_I^2 - m_I^2)^{-1} \prod_{I \in J^c} (\psi_I(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_I^2)^{-1} \right]$$

can be computed by general algorithms; in case $m_I > 0$, it is generated by first order operators, which are much easier to compute.

Integrals of D -modules

In general, let M be a finitely generated left module over D_{n+d} , the ring of differential operators in the variables

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_d).$$

Then the integral of M w.r.t. y is the left D_n -module

$$\int M dy := M / (\partial_{y_1} M + \dots + \partial_{y_d} M).$$

Theorem (J. Bernstein)

If M is a holonomic D_{n+d} -module, then $\int M dy$ is a holonomic D_n -module.

- Let $\mathcal{F}$ be a $\mathbb{C}$ -vector space of (generalized) functions in (x, y) such that $f \in \mathcal{F}$ implies $Pf \in \mathcal{F}$ for any $P \in D_{n+d}$.
- Let $\mathcal{F}'$ be a $\mathbb{C}$ -vector space of (generalized) functions in x such that $f \in \mathcal{F}'$ implies $Pf \in \mathcal{F}'$ for any $P \in D_n$.

Suppose that for $u(x, y) \in \mathcal{F}$, the integral $\int u(x, y) dy$ is well-defined and belongs to $\mathcal{F}'$ so that

$$P \int u(x, y) dy = \int Pu(x, y) dy \quad (\forall P \in D_n)$$

$$\int \partial_{y_j} u(x, y) dy = 0 \quad (j = 1, \dots, d)$$

for any $u \in \mathcal{F}$. Then there is a natural $\mathbb{C}$ -linear map

$$\mathrm{Hom}_{D_{n+d}}(M, \mathcal{F}) \longrightarrow \mathrm{Hom}_{D_n}\left(\int M dy, \mathcal{F}'\right).$$

More concretely, let $M = D_{n+d}/I$ with a left ideal I of D_{n+d} . Then

$$\int M dy = D_{n+d}/(I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d})$$

as left D_n -module. If M is holonomic, there exists $m \in \mathbb{N}$ such that the left D_n -module $\int M dy$ is generated by the residue classes of y^α with $\alpha \in \mathbb{N}^d$ such that

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \leq m.$$

Then the homomorphism

$$\mathrm{Hom}_{D_{n+d}}(M, \mathcal{F}) \longrightarrow \mathrm{Hom}_{D_n}\left(\int M dy, \mathcal{F}'\right)$$

is given by the map, for $u(x, y)$ such that $Pu = 0$ for any $P \in I$,

$$u(x, y) \longmapsto \left(\int y^\alpha u(x, y) dy \mid \alpha \in \mathbb{N}^d, |\alpha| \leq m \right)$$

and for $|\beta| > m$, there exist $P_{\beta,\alpha} \in D_n$ such that

$$\int y^\beta u(x, y) dy = \sum_{|\alpha| \leq m} P_{\beta,\alpha} \int y^\alpha u(x, y) dy.$$

This relation amounts to

$$y^\beta - \sum_{|\alpha| \leq m} P_{\beta,\alpha} y^\alpha \in I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d}.$$

In particular, $\int u(x, y) dy$ is annihilated by the ideal

$$J = D_n \cap (I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d})$$

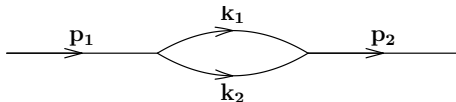
of D_n , which is called the integration ideal of I .

There is an algorithm to compute $\int M dy$ as well as J precisely (Oaku-Takayama). In the algorithm, the bound m is given as the largest integer root of what is called the b -function (an analogue of the indicial polynomial for ODE).

There are also heuristic algorithms to compute an ‘approximation’ of J , which might be smaller than J , such as ‘creative telescoping’ method.

Example 1 (as a toy case)

Let us study the Feynman diagram G below:



Then the Feynman integral is written in the form

$$\begin{aligned} F_G(\mathbf{p}_1, \mathbf{p}_2) &= \int_{\mathbb{R}^4} \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2) \delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) \\ &\quad \times (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1} (\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1} d\mathbf{k}_1 d\mathbf{k}_2 \\ &= \delta(\mathbf{p}_1 - \mathbf{p}_2) \tilde{F}_G(\mathbf{p}_1) \end{aligned}$$

with the amplitude

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1} ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1} d\mathbf{k}_1.$$

In view of the invariance under Lorentz transformations, let us set $\mathbf{p}_1 = (x, 0, \dots, 0)$.

In case $\nu = 2$, the integration ideal of the annihilator of the local cohomology class

$$[(k_{10}^2 - k_{11}^2 - m_1^2)^{-1}((x - k_{10})^2 - k_{11}^2 - m_2^2)^{-1}]$$

is generated by

$$(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x + 2x(x^2 - m_1^2 - m_2^2).$$

The solutions (the kernel) of this operator are constant multiples of

$$(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2}(x + m_1 + m_2)^{-1/2}(x - m_1 - m_2)^{-1/2}.$$

In case of $\nu = 4$, by using the 3-dimensional polar coordinates, we have

$$\begin{aligned}\tilde{F}_G(\mathbf{p}_1) &= \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1} 0)^{-1} ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1} 0)^{-1} d\mathbf{k}_1 \\ &= 2\pi \int_{\mathbb{R}^2} (k_{10}^2 - r^2 - m_1^2 + \sqrt{-1} 0)^{-1} \\ &\quad \times ((x - k_{10})^2 - r^2 - m_2^2 + \sqrt{-1} 0)^{-1} r_+^2 dk_{10} dr.\end{aligned}$$

The integration ideal of the annihilator of the cohomology class

$$[r^2(k_{10}^2 - r^2 - m_1^2)^{-1}((x - k_{10})^2 - r^2 - m_2^2)^{-1}]$$

is generated by

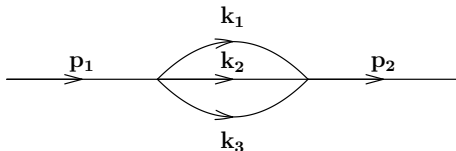
$$\begin{aligned}x(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x \\ - 2((m_1^2 + m_2^2)x^2 - m_1^4 + 2m_2^2m_1^2 - m_2^4).\end{aligned}$$

The solutions (the kernel) of this operator are constant multiples of

$$x^{-2}(x - m_1 + m_2)^{1/2}(x + m_1 - m_2)^{1/2}(x + m_1 + m_2)^{1/2}(x - m_1 - m_2)^{1/2}.$$

Example 2 (Cf. Adams-Bogner-Weinzierl 2015)

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2) \tilde{F}_G(\mathbf{p}_1)$$

with

$$\begin{aligned} \tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^4} & (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1} (\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1} \\ & \times ((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2 + \sqrt{-1}0)^{-1} d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned}$$

We work in the 2-dimensional space-time ($\nu = 2$) and compute holonomic systems for $\tilde{F}_G((x, 0))$ by assigning some special values to m_1, m_2, m_3 since the computation for general m_1, m_2, m_3 (as parameters) is intractable.

First let us set $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ so that $(-m_1 + m_2 + m_3)^2$, $(m_1 - m_2 + m_3)^2$, $(m_1 + m_2 - m_3)^2$ are distinct. Then $\tilde{F}_G((x, 0))$ is annihilated by the differential operator

$$\begin{aligned} & 30x(x-1)(x+1)(x-3)(x+3)(x-5)(x+5)(x-7)(x+7)\underline{\partial_x^3} \\ & + (-2x^{12} + 191x^{10} - 5340x^8 + 35954x^6 + 273082x^4 \\ & \quad - 2071305x^2 + 661500)\underline{\partial_x^2} \\ & + (-10x^{11} + 675x^9 - 12108x^7 + 15454x^5 + 936462x^3 \\ & \quad - 2692665x)\underline{\partial_x} \\ & - 8x^{10} + 372x^8 - 3300x^6 - 36028x^4 + 457932x^2 - 356760. \end{aligned}$$

The singular points $x = 0, \pm 1, \pm 3, \pm 5, \pm 7$ are all regular and the indicial equations are all $s^2(s-1)$.

Next set $m_1 = m_2 = m_3 = 1$. Then $\tilde{F}_G((x, 0))$ is annihilated by

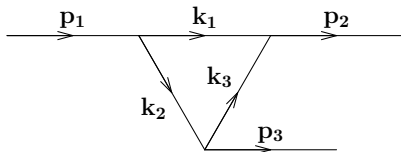
$$x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2 .

See Adams-Bogner-Weinzierl (2015) for complete computation with arbitrary m_1, m_2, m_3 by a different (and more efficient) method.

Example 3

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$\begin{aligned} \tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2) &= \int_{\mathbb{R}^\nu} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1} \\ &\times ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1} ((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2 + \sqrt{-1}0)^{-1} d\mathbf{k}_1. \end{aligned}$$

We work in the 2-dimensional space-time ($\nu = 2$). In view of the invariance under Lorentz transformation, we set $\mathbf{p}_1 = (x, 0)$, and $\mathbf{p}_2 = (y, z)$.

First let us set $m_1 = m_2 = m_3 = 1$. We compute the integration ideal I of the annihilator of the integrand regarded as a local cohomology class.

Since I is too complicated (with tens of generators), we will present only the characteristic variety of $M := D_3/I$.

The characteristic variety of $M := D_3/I$ is given by

$$\begin{aligned}\text{Char}(M) = & T_{\{f=0\}}^* \mathbb{C}^3 \cup T_{\{x=0\}}^* \mathbb{C}^3 \cup T_{\{x=f_0=0\}}^* \mathbb{C}^3 \cup T_{\{x=y^2-z^2-4=0\}}^* \mathbb{C}^3 \\ & \cup T_{\{x=y+z=0\}}^* \mathbb{C}^3 \cup T_{\{x=y-z=0\}}^* \mathbb{C}^3 \cup T_{\{x=y=z=0\}}^* \mathbb{C}^3\end{aligned}$$

with

$$\begin{aligned}f(x, y, z) &= (y - z)(y + z)x^2 - 2(y - z)(y + z)yx \\ &\quad + (y - z)^2(y + z)^2 + 4z^2, \\ f_0(y, z) &= f(0, y, z),\end{aligned}$$

where we denote by $T_Z^* \mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$.

Only the first component $T_{\{f=0\}}^* \mathbb{C}^3$ should be physically significant. Especially, it roughly means that the Feynman amplitude is analytic outside the surface $f = 0$, which has rather complicated singularities.

Definition of the conormal bundle

In general, if a complex analytic submanifold Z is defined by

$$f_1(x, y, z) = \cdots = f_d(x, y, z) = 0$$

with holomorphic functions $f_1, \dots, f_d$ such that $df_1 \wedge \cdots \wedge df_d \neq 0$, then the conormal bundle of Z is defined to be the submanifold

$$T_Z^*\mathbb{C}^3 = \{(x, y, z; c_1 df_1 + \cdots + c_d df_d) \mid c_1, c_2, c_3 \in \mathbb{C}\}$$

of the cotangent bundle

$$T^*\mathbb{C}^3 = \{(x, y, z; \xi dx + \eta dy + \zeta dz) \mid \xi, \eta, \zeta \in \mathbb{C}\}.$$

In fact, $T_Z^*\mathbb{C}^3$ is a Lagrangean submanifold of $T^*\mathbb{C}^3$.

Singularities of the surface $f = 0$

Let us investigate the singularities of the complex surface

$$Z = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\},$$

$$f = (y - z)(y + z)x^2 - 2(y - z)(y + z)yx + (y - z)^2(y + z)^2 + 4z^2.$$

Following N. Honda and T. Kawai, we rewrite f as

$$f = yzx^2 - yz(y + z)x + y^2z^2 + (y - z)^2$$

by change of coordinates $(y + z, y - z) \rightarrow (y, z)$.

Then the singular locus (the set of the singular points) of Z is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$.

The projection $Z \ni (x, y, z) \mapsto (y, z)$ defines a double covering on $\{(x, y) \mid xy \neq 0\}$ branched along the union of curves $y - z = 0$ and $yz - 4 = 0$.

The stratification of Z with respect to the (local) b -function $b_{f,p}(s)$ of f at a point p is

strata	$b_{f,p}(s)$
$\{(0, 0, 0)\}$	$(s + 1)^3(2s + 3)$
$\{(2, 0, 0), (-2, 0, 0), (2, 2, 2), (-2, -2, -2)\}$	$(s + 1)^2(2s + 3)$
$\{x = y = z\} \cup \{y = z = 0\}$ $\setminus \{(0, 0, 0), (\pm 2, 0, 0), \pm(2, 2, 2)\}$	$(s + 1)^2$
$\{f = 0\} \setminus (\{x = y = z\} \cup \{y = z = 0\})$	$s + 1$

In comparison, that of $g := x^2 - y^2z$ (Whitney umbrella) is

strata	$b_{g,p}(s)$
$\{(0, 0, 0)\}$	$(s + 1)^2(2s + 3)$
$\{x = y = 0\} \setminus \{(0, 0, 0)\}$	$(s + 1)^2$
$\{g = 0\} \setminus \{x = y = 0\}$	$s + 1$

Local b -function (Bernstein-Sato polynomial)

In general, let $f(x) = f(x_1, \dots, x_n)$ be a holomorphic function defined on a neighborhood of $p \in \mathbb{C}^n$. Let $\mathcal{D}_p$ be the ring of differential operators whose coefficients are holomorphic on a neighborhood of p . Then the local b -function (Bernstein-Sato polynomial) of f at p is the monic polynomial $b_{f,p}(s) \in \mathbb{C}[s]$ of the least degree such that

$$P(s)f^{s+1} = b_{f,p}(s)f^s \quad (\exists P(s) \in \mathcal{D}_p[s]).$$

By the definition, $b_{f,p}(s)$ is a complex analytic invariant of f at p .

In case $m_1 = 1, m_2 = 2, m_3 = 3$

In case $m_1 = 1, m_2 = 2, m_3 = 3$, we succeeded in computation of the integration ideal I of the integrand as a local cohomology class. The characteristic variety of $M := D_3/I$ is given by

$$\text{Char}(M) = T_{\{g=0\}}^* \mathbb{C}^3 \cup T_{\{x=y-z=0\}}^* \mathbb{C}^3,$$

with

$$g(x, y, z) = (y - z)(y + z)x^4 - 2y(y^2 - z^2 - 8)x^3 \\ + (y^4 + (-2z^2 - 22)y^2 + z^4 + 26z^2 + 64)x^2 + 6y(y^2 - z^2 - 8)x + 9(y^2 - z^2).$$

The decomposition of $\text{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir for prime and primary decomposition of polynomial ideals developed by M. Noro.

The singularities of the surface $g = 0$

Again by change of coordinates $(y + z, y - z) \rightarrow (y, z)$, we rewrite g as

$$g = yzx^4 - (y + z)(yz - 8)x^3 + ((z^2 + 1)y^2 - 24zy + z^2 + 64)x^2 + 3(y + z)(yz - 8)x + 9yz.$$

The set of the singular points of the surface $g = 0$ is given by the curve

$$zx^2 - (z^2 - 8)x - 3z = y - z = 0.$$

The local b -function is $(s + 1)^2(2s + 3)$ at the 8 points

$$\pm(1, 2, 2), \quad \pm(1, -4, -4), \quad \pm(3, -2, -2), \quad \pm(3, 4, 4);$$

$(s + 1)^2$ on the curve $zx^2 - (z^2 - 8)x - 3z = y - z = 0$ other than the 8 points above.

The local b -function of g at the 8 points is the same as that of the Whitney umbrella at the origin.

Microlocal point of view

We work in the d -dimensional Euclidean space $\mathbb{R}^n$ together with its complexification $\mathbb{C}^n$. We use complex coordinates

$$z = (z_1, \dots, z_n) \in \mathbb{C}^n \text{ with } z_k = x_k + \sqrt{-1} y_k.$$

We mean by an open cone Γ in $\mathbb{R}^n$ an open set of $\mathbb{R}^n$ such that $y \in \Gamma$ implies $cy \in \Gamma$ for any $c > 0$. For two open cones Γ and Γ' , we denote $\Gamma \supseteq \Gamma'$ if $\Gamma \cup \{0\} \supset \overline{\Gamma'}$ holds.

Let U be an open set of $\mathbb{R}^n$, and Γ an open cone in $\mathbb{R}^n$. We say that a function $F(z)$ is holomorphic on $U + \sqrt{-1}\Gamma 0$ and denote $F \in \mathcal{O}(U + \sqrt{-1}\Gamma 0)$ if, for any compact subset K of U and any open cone Γ' such that $\Gamma' \subseteq \Gamma$, there exists $\varepsilon > 0$ so that $F(z)$ is holomorphic (complex analytic) on the set

$$\{x + \sqrt{-1}y \mid x \in K, y \in \Gamma', |y| < \varepsilon\}.$$

Hyperfunctions

A hyperfunction (of Mikio Sato) $u(x)$ on $U \subset \mathbb{R}^n$ is defined as the sum of ‘the boundary values’

$$u(x) = \sum_{k=1}^N F_k(x + \sqrt{-1} \Gamma_k 0)$$

with open cones Γ_k and $F_k \in \mathcal{O}(U + \sqrt{-1} \Gamma_k 0)$. Here $F_k(x + \sqrt{-1} \Gamma_k 0)$ means the formal (cohomological) limit of $F_k(x + \sqrt{-1} y)$ as $y \in \Gamma_k$ tends to zero. This limit often makes sense as a distribution of L. Schwartz.

Abstractly, the space of the hyperfunctions on $\mathbb{R}^n$ is defined to be the local cohomology group $H_{\mathbb{R}^n}^n(\mathcal{O}_{\mathbb{C}^n})$, where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf on $\mathbb{C}^n$ of holomorphic functions.

Micro-analyticity

The dual (or polar) cone Γ° of an open cone Γ is defined to be the closed cone

$$\Gamma^\circ = \{\xi \in \mathbb{R}^n \mid \langle \xi, y \rangle \geq 0 \quad (\forall y \in \Gamma)\},$$

where $\langle \xi, y \rangle$ is the Euclidean inner product. Let

$$\sqrt{-1} T^*\mathbb{R}^n = \{(x, \sqrt{-1} \langle \xi, dx \rangle) \mid x, \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n$$

be the purely imaginary cotangent bundle of $\mathbb{R}^n$. We identify $(x, \sqrt{-1} \langle \xi, dx \rangle)$ with $(x, \sqrt{-1} \xi)$.

Definition

A hyperfunction $u(x)$ is said to be micro-analytic at $(p, \sqrt{-1}\xi_0)$ if it is written, on a neighborhood U of p , in the form

$$u(x) = \sum_{k=1}^N F_k(x + \sqrt{-1}\Gamma_k 0), \quad F_k \in \mathcal{O}(U + \sqrt{-1}\Gamma_k 0)$$

with open convex cones Γ_k such that $\xi_0 \notin \Gamma_k^\circ$.

Definition

The analytic wave-front set $WA(u)$ of a hyperfunction $u(x)$ on U is defined to be the closed subset

$$WA(u) = \{(p, \sqrt{-1}\xi_0) \mid u \text{ is not micro-analytic at } (p, \sqrt{-1}\xi_0)\}$$

of $\sqrt{-1}T^*U$.

Microfunctions

Definition

A germ of microfunction at $(p, \sqrt{-1} \xi_0 0)$ is a hyperfunction $u(x)$ defined on a neighborhood of p modulo hyperfunctions which are micro-analytic at $(p, \sqrt{-1} \xi_0 0)$. This defines the sheaf $\mathcal{C}$ of microfunctions on $\sqrt{-1} T^*\mathbb{R}^n$.

Let us denote by $\mathcal{A}$ the sheaf on $\mathbb{R}^n$ of real analytic functions, by $\mathcal{B}$ that of hyperfunctions, and by $\pi : \sqrt{-1} T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ the projection. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \pi_* \mathcal{C} \longrightarrow 0.$$

This means that the sheaf $\mathcal{C}$ describes the cotangential decomposition of the analytic singularity $\mathcal{B}/\mathcal{A}$ of hyperfunctions.

Feynman amplitudes as microfunctions

As was pointed out by Sato-Kawai-Kashiwara in the 1970's, the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p})$ associated with an external diagram G with positive masses is well-defined as a microfunction on the set

$$\sqrt{-1} \, T^* \mathbb{R}^{\nu(n-1)} \setminus \varpi(\Lambda(G) \setminus \Lambda_+(G))$$

and its support (analytic wave-front set) is contained in $\varpi(\Lambda_+(G))$. These sets are called Landau (-Nakanishi) varieties and defined as follows:

We set

$$\begin{aligned}
 \Lambda(G) = & \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \dots, \mathbf{u}_{n-1}; \alpha_1, \dots, \alpha_N) \\
 & \in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\
 & \alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \quad (1 \leq j \leq N - n + 1), \\
 & \alpha_l(\psi_l^2 - m_l^2) = 0 \quad (l \in J^c), \\
 & \alpha_{l_j} \mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \quad (1 \leq j \leq N - n + 1), \\
 & \mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \quad (1 \leq r \leq n - 1), \\
 & \alpha_l \geq 0 \quad (1 \leq l \leq N)\}
 \end{aligned}$$

with

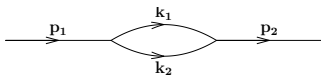
$$\psi_l = \sum_{r=1}^{n-1} a_{lr} \mathbf{p}_r + \sum_{j=1}^{N-n+1} b_{lj} \mathbf{k}_{l_j},$$

$$\begin{aligned}
\Lambda_+(G) = & \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \dots, \mathbf{u}_{n-1}; \alpha_1, \dots, \alpha_N) \\
& \in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\
& \alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \leq j \leq N - n + 1), \\
& \alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\
& \alpha_{l_j} \mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \leq j \leq N - n + 1), \\
& \mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \leq r \leq n - 1), \\
& \alpha_l > 0 \ (1 \leq l \leq N)\},
\end{aligned}$$

and ϖ is the projection of $\Lambda(G)$ to

$$\begin{aligned}
& \sqrt{-1} \ T^* \mathbb{R}^{\nu(n-1)} \\
& = \{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}; \sqrt{-1} (\langle \mathbf{u}_1 d, \mathbf{p}_1 \rangle + \dots + \langle \mathbf{u}_{n-1}, d \mathbf{p}_{n-1} \rangle))\}.
\end{aligned}$$

For example, for the graph G below



$$\Lambda(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1, \alpha_1, \alpha_2) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0 \\ \alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2 \mathbf{p}_1, \quad \alpha_1, \alpha_2 \geq 0\}$$

$$\Lambda_+(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1, \alpha_1, \alpha_2) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0 \\ \alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2 \mathbf{p}_1, \quad \alpha_1, \alpha_2 > 0\},$$

from which, we can confirm that

$$\varpi(\Lambda(G) \setminus \Lambda_+(G)) = \{(\mathbf{p}_1, \sqrt{-1} \langle \mathbf{u}_1, d\mathbf{p}_1 \rangle) \mid \mathbf{u}_1 = \mathbf{0}\},$$

$$\varpi(\Lambda_+(G)) = \sqrt{-1} T_{\{\mathbf{p}_1^2 - (m_1 + m_2)^2 = 0\}}^* \mathbb{R}^2 \cup \sqrt{-1} T_{\{\mathbf{p}_1^2 - (m_1 - m_2)^2 = 0\}}^* \mathbb{R}^2.$$

This implies that the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as an element of $\mathcal{B}(\mathbb{R}^2)/\mathcal{A}(\mathbb{R}^2)$.

Theorem

Let $\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})$ be the Feynman amplitude associated with an external diagram with positive masses. Let I be the annihilator of the integrand of the Feynman amplitude as a local cohomology class.

Then as a microfunction $\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})$ is annihilated by the integration ideal

$$D_{\nu(n-1)} \cap \left(I + \sum_{l=1}^{N-n+1} \sum_{j=1}^{\nu} D_{\nu N} \partial_{k_{lj}} \right)$$

as a microfunction on the set

$$\sqrt{-1} \, T^* \mathbb{R}^{\nu(n-1)} \setminus \varpi(\Lambda(G) \setminus \Lambda_+(G)).$$