An algorithm to compute the differential equations for the logarithm of a polynomial

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Weyl algebra

Let $D_n = \mathbb{C}\langle x, \partial_x \rangle$ be the Weyl algebra, or the ring of differential operators with polynomial coefficients. An element P of D_n is expressed as a finite sum

$$P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta$$

with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ and $a_{\alpha,\beta} \in \mathbb{C}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ are multi-indices with $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\partial_i = \partial/\partial x_i$ $(i = 1, \ldots, n)$ denote derivations.

Holonomic functions

A function $u = u(x)$ in the variables $x = (x_1, \ldots, x_n)$ is called a *holonomic function* if its *annihilator*

$$\mathrm{Ann}_{D_n} u := \{P \in D_n \mid Pu = 0\}$$

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For a left ideal I of D_n, one can define the dimension of the left D_n-module D_n/I. One way is to define it as the (usual) dimension of the *characteristic variety*, which is an algebraic subset of $\mathbb{C}^{2n}$. It is known that $n \leq \dim D_n/I \leq 2n$ if $I \neq D_n$. I is holonomic by definition if $\dim D_n/I = n$ or else $I = D_n$.

Integrals of holonomic functions

If $u(x_1, \dots, x_n)$ is holonomic, then the integral

$$v(x_1, \dots, x_{n-d}) := \int_C u(x_1, \dots, x_n)\, dx_{n-d+1} \cdots dx_n,$$

where C is a d-cycle in $\mathbb{C}^d$, or $C = \mathbb{R}^d$, or C is a domain of $\mathbb{R}^d$ defined by polynomial inequalities, is also holonomic if the integral is 'well-defined'. Moreover, a holonomic ideal $I \subset \mathrm{Ann}_{D_{n-d}} v$ is computable by using the D-module theoretic integration algorithm (O-Takayama (1999) for C without boundary, and O (to appear in JSC) for C with boundary defined by polynomials).

Examples of holonomic functions

$\mathrm{Ann}_{D_n} u$ is precisely computable for the following u:

(1) $f_1^{\lambda_1} \cdots f_m^{\lambda_m}$ with $f_1, \dots, f_m \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ and $\lambda_1, \dots, \lambda_m$; especially a rational function (O 1997 for $m = 1$, O-Takayama 1999, Briançon-Maisonobe 2002 for $m \geq 1$).

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(2) $\exp(g)$ with $g \in \mathbb{C}(x)$ (by 'localization algorithm' of O-Takayama-Walther 2000, or, more generally, 'Weyl closure algorithm' by Tsai 2002).

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(3) $f^\lambda(g_0 + g_1 \log f + \cdots + g_m (\log f)^m)$ with $f, g_0, \dots, g_m \in \mathbb{C}[x]$, $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$. (this talk).

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Computing $\mathrm{Ann}_{D_n} f^\lambda (\log f)^m$

Let $f \in K[x] = K[x_1, \ldots, x_n]$ be a non-constant polynomial with a computable subfield K of $\mathbb{C}$. Then the annihilator of $f^s (\log f)^m$ with a parameter s can be computed as follows:

Step 1: Compute $\mathrm{Ann}_{D_n[s]} f^s$.

For $f \in K[x]$, consider the function $f^s = f(x)^s$ with a parameter s. Then $\mathrm{Ann}_{D_n[s]} f^s$ can be computed by an algorithm of O (1997) or of Briançon-Maisonobe (2002).

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Example: Set $f = x^3 - y^2$. Then $\mathrm{Ann}_{D_n[s]} f^s$ is generated by

$$2y\partial_x + 3x^2\partial_y, \quad 2x\partial_x + 3y\partial_y - 6s.$$

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Remark: $f(x)^s$ is not a holonomic function in (x, s); for a fixed λ, $f(x)^\lambda$ is holonomic in x.

Computing $\mathrm{Ann}_{D_n[s]}(f^s, f^s \log f, \ldots, f^s(\log f)^m)$

Step 2: Differentiation with respect to s

Let G_1 be a set of generators of $\mathrm{Ann}_{D_n[s]} f^s$ and

$$e_0 = (1, 0, \ldots, 0), \ldots, e_m = (0, \ldots, 0, 1)$$

be the canonical basis of $\mathbb{C}^{m+1}$. For $P(s) \in G_1$ and $j \geq n$ set

$$P(s)^{(j)} := \sum_{\nu=0}^{j} \binom{j}{\nu} \frac{\partial^{j-\nu} P(s)}{\partial s^{j-\nu}} e_\nu .$$

Then $G_2 := \{P(s)^{(j)} \mid P(s) \in G,\ 0 \leq j \leq m\}$ generates the 'annihilating module'

$$\mathrm{Ann}_{D_n[s]}(f^s, f^s \log f, \ldots, f^s(\log f)^m)$$

$$:= \{(P_0, P_1, \ldots, P_m) \in (D_n)^{m+1} \mid \sum_{j=0}^{m} P_j(f^s(\log f)^j)) = 0\}.$$

Computing $\mathrm{Ann}_{D_n[s]}(f^s, f^s \log f, \ldots, f^s(\log f)^m)$

In fact, differentiating the equation $P(s)f^s = 0$ j times w.r.t. s, we get

$$\sum_{\nu=0}^{j} \binom{j}{\nu} \frac{\partial^{j-\nu} P(s)}{\partial s^{j-\nu}} (f^s (\log f)^\nu) = 0.$$

Step 2: Differentiation with respect to s (an example)

Example: Set $f = x^3 - y^2$. Then $\mathrm{Ann}_{D_n[s]}(f^s, f^s \log f)$ is generated by

$$\begin{array}{ll} (2y\partial_x + 3x^2\partial_y,\ 0), & (0,\ 2y\partial_x + 3x^2\partial_y), \\ (2x\partial_x + 3y\partial_y - 6s,\ 0), & (-6,\ 2x\partial_x + 3y\partial_y - 6s) \end{array}$$

in $(D_2[s])^2$, which follows from differentiating the generators

$$2y\partial_x + 3x^2\partial_y, \quad 2x\partial_x + 3y\partial_y - 6s$$

of $\mathrm{Ann}_{D_n[s]} f^s$.

Specialization of the parameter s

The annihilator of $f^\lambda(\log f)^m$ for a fixed $\lambda \in \mathbb{C}$ can be computed as follows:

Step 3: roots of the b-function

Let $b_f(s)$ be the b-function, i.e, the Bernstein-Sato polynomial of f. There exists $Q(s) \in D_n[s]$ such that

$$Q(s)f^{s+1} = b_f(s)f^s.$$

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Example: For $f = x^3 - y^2$, we have

$$\underbrace{\left(\frac{1}{27}\partial_x^3 + \frac{1}{8}y\partial_y^3 + (-\frac{1}{2}s - \frac{3}{8})\partial_y^2\right)}_{Q(s)} f^{s+1} = \underbrace{(s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)}_{b(s)} f^s.$$

Specialization of the parameter s

Theorem: Let G_2 be a set of generators of $\mathrm{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m)$ and $\lambda \in \mathbb{C}$. If $b_f(\lambda - \nu) \neq 0$ for $\nu = 1, 2, 3, \dots,$ then

$$G_3 := \{P(\lambda) \mid P(s) \in G_2\}$$

generates the annihilator of $(f^\lambda, \dots, f^\lambda(\log f)^m)$, which is a left submodule of the free module $(D_n)^{m+1}$.

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Remark: In order to verify the condition

$$b_f(\lambda - \nu) \neq 0 \quad (\nu = 1, 2, 3, \dots),$$

one does not need the entire $b_f(s)$; one can employ the check root algorithm of Levandovskyy-Morales (2008), which is much faster, together with a bound of the roots of $b_f(s)$.

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(3) Set $\lambda_0 := \lambda - \nu_0$ and $G_2 := G_1|_{s=\lambda_0}$ (substitute λ_0 for s in each element of G_1).

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(4) If $\nu_0 > 0$, then let G_3 be a set of generators of the module quotient $\langle G_2 \rangle : f^{\nu_0} = \langle G_2 \rangle : (f^{\nu_0}, \dots, f^{\nu_0})$, where $\langle G_2 \rangle$ denotes the left module generated by G_2.

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(5) If $\nu_0 = 0$, then set $G_3 := G_2$.

Specialization of the parameter s

Algorithm (continued)

(6) Compute a Gröbner basis G_4 of the module generated by G_3 with respect to a term order $\prec$ for $(D_n)^{m+1}$ such that $Me_j \prec M'e_k$ for any monomials M and M' if $k < j$. Let G_5 be the set of the last component of each element of G_4.

Output: G_3 generates $\mathrm{Ann}_{D_n}(f^\lambda, \dots, f^\lambda(\log f)^m)$; G_5 generates $\mathrm{Ann}_{D_n} f^\lambda(\log f)^m$.

Examples of $\mathrm{Ann}_{D_n} f^{\lambda}(\log f)^m$

Example: Set $f = x^3 - y^2$. If $\lambda \neq k,\ \frac{1}{6} + k,\ -\frac{1}{6} + k$ for $k = 0, 1, 2, \ldots,$ then $\mathrm{Ann}_{D_2} f^{\lambda} \log f$ is generated by

$$
\begin{aligned}
&2y\partial_x + 3x^2\partial_y,\\
&4x^2\partial_x^2 + 12yx\partial_y + (-24\lambda + 4)x\partial_x + 9y^2\partial_y^2\\
&\qquad\qquad + (-36\lambda + 9)y\partial_y + 36\lambda^2.
\end{aligned}
$$

On the other hand, e.g., $\mathrm{Ann}_{D_2} \log f = (\mathrm{Ann}_{D_2} f^{-1} \log f) : f$ is generated by

$$
2y\partial_x + 3x^2\partial_y, \quad 2x\partial_y\partial_x + 3y\partial_y^2 + 3\partial_y.
$$

Summary of the algorithm

Starting with the annhilator of f^s, we get the annihilator of $f^\lambda(\log f)^m$ for a fixed λ following the diagram

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\begin{array}{lcl}
\mathrm{Ann}_{D_n[s]} f^s & & \\
\downarrow \text{ differentiation w.r.t } s & & \\
\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m) & \xrightarrow{\text{elimination}} & \mathrm{Ann}_{D_n[s]} f^s(\log f)^m \\
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$\mathrm{Ann}_{D_n} f^\lambda(g_0 + g_1 \log f + \cdots + g_m(\log f)^m)$ with $g_i \in K[x]$ can also be computed. We have implemented the algorithms in a computer algebra system Risa/Asir (Noro et al.).

Timing data

Computation time for $\mathrm{Ann}_{D_n}(\log f)^m$
(1.7 GHz Intel Core i5 processor with 4 GB memory)

f	$m = 2$	$m = 4$	$m = 8$	$m = 16$
$xy^2 + z^2$	0.02s	0.04s	0.14s	2.1s
$xy^2 + z^2 + 1$	0.04s	0.31s	20.8s	–
$x^3 + xy^2 + z^2$	0.04s	0.12s	1.6s	586s

The most time-consuming part is the elimination in the free module $(D_n)^{m+1}$.

Application to integration: Example 1

Let us give an example which shows an advantage of exact computation of the annilator:

Set $\boxed{f = x^2 + 1}$ with a single variable x.

Then $\boxed{\mathrm{Ann}_{D_1}(\log f)^2}$ is generated by

$$P_1 := x^2(x^2+1)^2\partial_x^3 + (3x^5 - 3x)\partial_x^2 + (x^4+3)\partial_x,$$

$$P_2 := x(x^2+1)^2\partial_x^4 + (9x^4+8x^2-1)\partial_x^3 + 16x^3\partial_x^2 + 4x^2\partial_x,$$

$$P_3 := (x^2+1)^2\partial_x^5 + 14x(x^2+1)\partial_x^4 + (52x^2+16)\partial_x^3 + 52x\partial_x^2 + 8\partial_x.$$

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&\qquad + 52x\partial_x^2 + 8\partial_x.
\end{aligned}
$$

On the other hand, the annihilator of $(\log f)^2$ in the ring of differential operators *with rational function coefficients* is generated by P_1 since we have

$$
xP_2 = \partial_x P_1, \qquad x^3 P_3 = (x\partial_x^2 - \partial_x)P_1.
$$

Application to integration: Example 1 (continued)

Consider the integral

$$u(t) := \int_{-\infty}^{\infty} \exp(-tx^2 + x)(\log(x^2+1))^2 \, dx$$

for $t > 0$. It is easy to compute the annihilator of the integrand from that of $(\log(x^2+1))^2$, which is generated by P_1, P_2, P_3. Then by the D-module theoretic integration algorithm (cf. Appendix B of the paper) we get a differential equation $\boxed{Pu(t) = 0}$ with a differential operator P of order 7 as follows (it takes about 2.5s by using the library file 'nk_restriction.rr' of Risa/Asir):

The operator P with $Pu(t) = 0$:

$$-64t^6(192t^9 - 288t^8 + 328t^6 - 8064t^5 - 1830t^4 - 483t^3 + 3349t^2 + 768t + 29)\boxed{\partial_t^7}$$
$$+(49152t^{15} - 313344t^{14} + 368640t^{13} + 97792t^{12} - 2536704t^{11} + 11644032t^{10} + 3132864t^9 + 1733440t^8 - 5460112t^7 - 1505008t^6 - 89760t^5 - 1392t^4)\partial_t^6$$
$$+(-73728t^{15} + 921600t^{14} - 2632704t^{13} + 2234880t^{12} + 4754688t^{11} - 43612160t^{10} + 79860992t^9 + 22867200t^8 + 26400784t^7 - 44105880t^6 - 14456652t^5 - 1133452t^4 - 32880t^3 - 348t^2)\partial_t^5$$
$$+(49152t^{15} - 1069056t^{14} + 5357568t^{13} - 9096704t^{12} + 2064384t^{11} + 58446592t^{10} - 246172736t^9 + 183521920t^8 + 34777024t^7 + 151880952t^6 - 128846576t^5 - 51768580t^4 - 4768431t^3 - 179833t^2 - 3378t - 29)\partial_t^4$$

to be continued...

The operator P with $Pu(t) = 0$ continued:

$$+(-12288t^{15} + 534528t^{14} - 4429824t^{13} + 11753984t^{12} - 12734976t^{11} - 27311744t^{1}0 + 246524096t^{9} - 484399040t^{8} + 99537376t^{7} - 103646640t^{6} + 341872136t^{5} - 100839440t^{4} - 62318032t^{3} - 6128000t^{2} - 216952t - 2712)\partial_t^3$$
$$+(-92160t^{14} + 1474560t^{13} - 5936640t^{12} + 10110720t^{11} - 2156928t^{10} - 90855104t^{9} + 346183872t^{8} - 261435344t^{7} + 16030976t^{6} - 241838336t^{5} + 253619008t^{4} + 32414466t^{3} - 14223770t^{2} - 1457516t - 30002)\partial_t^2$$
$$+(-138240t^{13} + 990720t^{12} - 2292480t^{11} + 3181824t^{10} + 7660416t^{9} - 71860224t^{8} + 118294032t^{7} - 11163768t^{6} + 46032612t^{5} - 109104420t^{4} + 25631736t^{3} + 18262788t^{2} + 1518344t + 29760)\partial_t$$
$$-23040t^{12} + 69120t^{11} - 161280t^{1}0 + 125760t^{9} + 1468800t^{8} - 6962016t^{7} + 2546936t^{6} - 449808t^{5} + 5489268t^{4} - 5472139t^{3} - 2277397t^{2} - 177522t - 3417.$$

Application to integration: Example 1 (continued)

Remark: If we use only

$$P_1 = x^2(x^2+1)^2\partial_x^3 + (3x^5 - 3x)\partial_x^2 + (x^4+3)\partial_x,$$

which is the one with minimal order among the generators P_1, P_2, P_3 of $\mathrm{Ann}_{D_1}(\log(x^2+1))^2$, then we get $Qu(t) = 0$ with a differential operator Q of order 9. The equation $Qu(t) = 0$ is weaker than $Pu(t) = 0$.

An alternative algorithm for the integral

An alternative way to compute this integral is to first compute differential equations for the integral

$$v(s,t) := \int_{-\infty}^{\infty} \exp\left(-tx^2 + x\right)(x^2+1)^s \, dx$$

with a parameter s and then differentiate with respect to s. This gives relations among $v(s,t)$, $\partial_s v(s,t)$ and $\partial_s^2 v(s,t)$. Then by substitution $s = 0$ and elimination, we get an equation for

$$\partial_s^2 v(0,t) = u(t) = \int_{-\infty}^{\infty} \exp\left(-tx^2 + x\right)(\log(x^2+1))^2 \, dx$$

of order 9, which is weaker than $Pu = 0$.

Example 2

Set

$$u(x) := \int_{\mathbb{R}^2} \exp(-y^2 - z^2)(\log(xy^2 + z^2 + 1))^2 \, dydz$$

for $x > 0$. Then by the integration algorithm, we get a differential equation $Pu(x) = 0$ with

$$\begin{aligned} P = {} & 16x^4(x-1)^2\partial_x^7 + 16x^2(x-1)(29x^2 - 17x - 2)\partial_x^6 \\ & + (4504x^4 - 5336x^3 + 896x^2 + 240x + 16)\partial_x^5 \\ & + (17712x^3 - 14220x^2 + 540x + 288)\partial_x^4 \\ & + (27153x^2 - 12348x - 441)\partial_x^3 \\ & + (12915x - 2205)\partial_x^2 + 945\partial_x. \end{aligned}$$

The computation of the integral takes about 4.3 seconds.

Example 3: $(\log f)^m$ as a generalized function

Set

$$u(x) := \int_{\mathbb{R}^2} e^{-y^2-z^2} (\log(xy^2+z^2))^2 \, dydz$$

for $x > 0$. Since $f := xy^2 + z^2$ vanishes if $y = z = 0$, we must regard $(\log f)^2$ as a distribution (generalized function) on $\mathbb{R}^2$ with respect to (y, z) with a parameter x. A holonomic system for $(\log f)^2$ regarded as such is obtained by the substitution $s = 0$ from the annihilator of $f^s(\log f)^2$ in $D_3[s]$, which is weaker than the annihilator of $(\log f)^2$ as analytic function. From this we get $Pu(x) = 0$ with

$$\begin{aligned} P &= 8x^3(x-1)^3\partial_x^6 + 12x^2(x-1)^2(13x-7)\partial_x^5 \\ &+ (926x^4 - 1926x^3 + 1218x^2 - 218x)\partial_x^4 \\ &+ (1911x^3 - 3107x^2 + 1369x - 125)\partial_x^3 \\ &+ (1155x^2 - 1360x + 325)\partial_x^2 + (105x - 75)\partial_x. \end{aligned}$$

Appendix: Integration ideal

Let I be a holonomic ideal of D_{n+d} which annihilates a function $u(x,t)$ in $(x,t) = (x_1,\dots,x_n,t_1,\dots,t_d)$, where D_{n+d} denotes the Weyl algebra in (x,t). Set

$$v(x) := \int_{\mathbb{R}^d} u(x,t)\,dt_1\cdots dt_d.$$

The **integration ideal** of I is the left ideal

$$\int I\,dt := (\partial_{t_1} D_{n+d} + \cdots + \partial_{t_d} D_{n+d} + I) \cap D_n$$

of D_n. Then $Pv(x) = 0$ holds for all $P \in \int I\,dt$. Moreover, $D_n / \int I\,dt$ is holonomic.

Appendix: Integration algorithm

Input: A set G_0 of generators of I.

(1) Compute a Gröbner basis G_1 of I with respect to a monomial order which is compatible with the weight vector $w = (0, \dots, 0, 1, \dots, 1; 0, \dots, 0, -1, \dots, -1)$ for the variables $(x, t, \partial_x, \partial_t)$.

(2) Compute the b-function of I with respect to w, which is a nonzero univariate polynomial $b(s)$ of the minimum degree such that $b(-\partial_{t_1} t_1 - \cdots - \partial_{t_d} t_d) + P$ belongs to I with some $P \in D_{n+d}$ of order ≤ -1 with respect to the weight vector w.

(3) Let k_1 be the maximum integral root of $b(s) = 0$ if any; if there is none or else $k_1 < 0$, then set $G := \{1\}$ and quit.

Appendix: Integration algorithm (continued)

(4) For $P \in G_1$ and $\alpha \in \mathbb{N}^d$ such that $\mathrm{ord}_w(P) + |\alpha| \leq k_1$, one has

$$t^\alpha P = \sum_{j=1}^{d} \partial_{t_j} Q_j + \sum_{|\beta| \leq k_1} R_\beta t^\beta$$

with $Q_j \in D_{n+d}$ and unique $R_\beta \in D_n$. Set $\chi(t^\alpha P) := \sum_{|\beta| \leq k_1} R_\beta t^\beta$. Let N be the left D_n-submodule of $\bigoplus_{|\beta| \leq k_1} D_n t^\beta$ generated by

$$\{\chi(t^\alpha P) \mid P \in G_1,\ |\alpha| + \mathrm{ord}_w(P) \leq k_1\}.$$

(5) Compute a set G of generators of the ideal $N \cap D_n$.

Output: G generates $\int I\, dt$ and $D_n / \int I\, dt$ is holonomic.

Thank you for your attention.

Merci beaucoup!