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RESEARCH ARTICLE

Annihilators of distributions associated with algebraic local cohomology of a hypersurface

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Let f be a real-valued real analytic function in several variables. We associate with each algebraic local cohomology class u with support in $f = 0$ a distribution (generalized function) $\rho(u)$ in terms of the residue of f_+^λ with respect to λ at a negative integer. Then ρ constitutes a homomorphism of modules over the sheaf of analytic functions but not over the sheaf of differential operators in general. We compare the annihilator of $\rho(u)$ in the ring of differential operators with that of u: we give sufficient conditions, together with examples, for each inclusion between the two annihilators.

Keywords: local cohomology; generalized function; residue; annihilator

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1. Introduction

Let f be a holomorphic function defined on a complex manifold X. The algebraic local cohomology group supported by $f = 0$ is defined to be the sheaf

$$\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X$$

on X, where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X. This consists of residue classes $[af^{-k}]$ modulo $\mathcal{O}_X$ with a holomorphic function a and a non-negative integer k. Let $\mathcal{D}_X$ be the sheaf on X of differential operators with holomorphic coefficients. Then $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ has a structure of sheaves of left $\mathcal{D}_X$-modules (cf. [7]). An algorithm to compute its structure was given in [11] for the case f is a polynomial.

Now let f be a real-valued real analytic function defined on a paracompact real analytic manifold M. We may assume that f is extended to a holomorphic function on a complexification X of M. Let $\mathcal{D}b_M$ be the sheaf of distributions (generalized functions of L. Schwartz) on M. We first define an $\mathcal{O}_X$-homomorphism ρ of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ to $\mathcal{D}b_M$ as follows: For a residue class $[af^{-k}]$ with a holomorphic function a and a positive integer k, we define $\rho([af^{-k}])$ to be the residue of $af_+^{\lambda-k}$ at $\lambda = -1$. For the systematic study of the distribution f_+^λ, we refer to [5], [2], [3], [8]. If $f = 0$ is non-singular, then it is well-known (see [5]) that

$$\rho([f^{-k-1}]) = \mathrm{Res}_{\lambda=0}\, f^{\lambda-k-1} = \frac{(-1)^k}{k!}\delta^{(k)}(f) \quad (k = 0, 1, 2, \dots) \tag{1}$$

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holds with $\delta^{(k)}(f)$ being the k-th 'derivative' of the delta function $\delta(f)$ supported by the real hypersurface $f = 0$.

Our main purpose is to compare the annihilators

$$\mathrm{Ann}_{\mathcal{D}_M} u := \{P \in \mathcal{D}_M \mid Pu = 0\}, \quad \mathrm{Ann}_{\mathcal{D}_M} \rho(u) := \{P \in \mathcal{D}_M \mid P\rho(u) = 0\}$$

as sheaves of left ideals of $\mathcal{D}_M$ for a section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$, where $\mathcal{D}_M := \mathcal{D}_X|_M$ denotes the sheaf theoretic restriction of $\mathcal{D}_X$ to M. Note that the sheaf of annihilating ideals of a distribution is not necessarily coherent over $\mathcal{D}_M$ (see Example 7.5.1 of [4]). If $f = 0$ is non-singular, then it is easy to see, in view of (1), that the two annihilators coincide. However, at singular points of the hypersurface $f = 0$, the situation is more complicated in general.

In Section 2, we recall that both $u \in \mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ and $\rho(u)$ satisfy regular holonomic $\mathcal{D}_X$-modules whose characteristic varieties are contained in the set W^0_f defined by Kashiwara [6].

In Section 3, we give a sufficient condition (A) for $\mathrm{Ann}_{\mathcal{D}_M} \rho(u) \supset \mathrm{Ann}_{\mathcal{D}_M} u$ to hold (Theorem 3.1) and a sufficient condition (B) for the converse inclusion (Theorem 3.4). In Section 5, we give examples for which $\mathrm{Ann}_{\mathcal{D}_M} \rho(u) = \mathrm{Ann}_{\mathcal{D}_M} u$ (Example 5.1), $\mathrm{Ann}_{\mathcal{D}_M} \rho(u) \subsetneq \mathrm{Ann}_{\mathcal{D}_M} u$ (Example 5.2) and $\mathrm{Ann}_{\mathcal{D}_M} \rho(u) \supsetneq \mathrm{Ann}_{\mathcal{D}_M} u$ (Example 5.3) hold respectively. We also give examples of normal forms of real hypersurface singularities which satisfy the condition (B).

As a related problem, we notice in Section 4 a (probably well-known) sufficient condition for the annihilator of the distribution f_+^λ to coincide with that of the analytic function f^λ.

2. Algebraic local cohomology and residues of f_+^λ

If f is holomorphic on a neighbourhood of x_0 in a complex manifold X, the *b-function* or the *Bernstein-Sato polynomial* of f at x_0 is, by definition, the monic polynomial $b_{f,x_0}(s)$ in a variable s of the least degree such that a formal functional equation $P(s)f^{s+1} = b_{f,x_0}(s)f^s$ holds with some $P(s) \in (\mathcal{D}_X)_{x_0}[s]$, where $(\mathcal{D}_X)_{x_0}$ denotes the stalk of the sheaf $\mathcal{D}_X$ at x_0. If $f(x_0) = 0$, then $b_{f,x_0}(s)$ is divisible by $s + 1$ and $\tilde{b}_{f,x_0}(s) := b_{f,x_0}(s)/(s + 1)$ is called the *reduced b-function* of f at x_0. It was proved by Kashiwara [6] that $b_{f,x_0}(s)$ exists and its roots are negative rational numbers. If f is a polynomial, then there is an algorithm to compute $b_{f,x_0}(s)$ and $P(s)$ (see [10]).

In what follows, we let f be a real-valued real analytic function defined on a paracompact real analytic manifold M. Then for a complex number λ with non-negative real part (Re $\lambda \geq 0$), the distribution f_+^λ is defined to be the locally integrable function

$$f_+^\lambda(x) := \begin{cases} f(x)^\lambda = \exp(\lambda \log f(x)) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

on M and is holomorphic with respect to λ for Re $\lambda > 0$. In particular, $f_+^0 = Y(f)$ is the Heaviside function associated with f. By using the functional equation

$$b_{f,x_0}(\lambda) f_+^\lambda = P(\lambda) f_+^{\lambda+1}, \tag{2}$$

which follows from the formal one above, we can extend f_+^λ to a $\mathcal{D}b_M(M)$-valued meromorphic function of λ on the whole complex plane $\mathbb{C}$.

The real analytic function f can be extended to a holomorphic function on a complexification (a complex neighbourhood) X of M.

Definition 2.1: We define a sheaf homomorphism

$$\rho \; : \; \mathcal{H}^1_{[f=0]}(\mathcal{O}_X) \longrightarrow \mathcal{D}b_M$$

by the residue

$$\rho([af^{-k}]) := \mathrm{Res}_{\lambda=0}\, af_+^{\lambda-k}$$

at $\lambda = 0$ for a section a of $\mathcal{O}_X$ and a non-negative integer k.

Proposition 2.2: *The map ρ is well-defined and gives rise to a homomorphism of sheaves of $\mathcal{A}_M$-modules, where $\mathcal{A}_M := \mathcal{O}_X|_M$ denotes the sheaf of real analytic functions. Moreover, the support of $\rho(u)$ is contained in the set $\{x \in U \mid f(x) = 0\}$ for any section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ over an open set U of M.*

Proof: To prove that ρ is well-defined, it suffices to show that $\rho([af^{-k}]) = \rho([(af)f^{-k-1}])$ holds for any section a of $\mathcal{O}_X$. This follows from

$$\mathrm{Res}_{\lambda=0}\,(af)f_+^{\lambda-k-1} = \mathrm{Res}_{\lambda=0}\, af_+^{\lambda-k}.$$

Moreover, we have $f^k\rho([af^{-k}]) = \mathrm{Res}_{\lambda=0}\,(af_+^{\lambda}) = 0$. Hence the support of $\rho([af^{-k}])$ is contained in $f = 0$. It is easy to see that ρ is a homomorphism of $\mathcal{A}_M$-modules. □

Let us show that both u and $\rho(u)$ satisfy regular holonomic $\mathcal{D}_X$-modules. (See [8], [4] for regular holonomic systems.) First we recall the set W_f^0 defined by Kashiwara [6]:

Definition 2.3: For a holomorphic function f on X, the set $\widetilde{W}_f$ is defined to be the closure of the set

$$\{(\sigma; x, \sigma d\log f(x)) \in \mathbb{C} \times T^*X \mid f(x) \neq 0\}$$

in $\mathbb{C} \times T^*X$, where T^*X denotes the cotangent bundle of X. Let $\varpi : \mathbb{C} \times T^*X \to T^*X$ be the canonical projection and set

$$W_f^0 := \varpi(\widetilde{W}_F \cap (\{0\} \times T^*X)).$$

If f is a polynomial, then the defining equations of $\widetilde{W}_f$, and hence those of W_f^0 as well, can be computed as the ideal quotient by f of the radical of the ideal generated by $f\xi_i - \sigma\partial f/\partial x_i$ $(i = 1, \ldots, n)$. The theorem below should be classical, to which we would like to give a 'constructive' proof. (See Theorem 7.6.1 of [4] for a complex version.)

Theorem 2.4: *Let u be a section of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ defined on a neighbourhood of $x_0 \in M$ in X. Then*

(1) $\mathcal{D}_X u = \mathcal{D}_X/\mathrm{Ann}_{\mathcal{D}_X} u$ is a regular holonomic $\mathcal{D}_X$-module whose characteristic variety is contained in $W_f^0 \cap \{f = 0\}$.

(2) There exists a coherent sheaf $\mathcal{I}$ of left ideals of $\mathcal{D}_X$ defined on a neighbourhood of x_0 in X such that $\mathcal{D}_X/\mathcal{I}$ is regular holonomic with the characteristic variety contained in $W_f^0 \cap \{f = 0\}$ and that $\mathcal{I}|_M$ is a subsheaf of $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$.

Proof: Set $\mathcal{N} = \mathcal{D}_X[s]f^s$. Then $\mathcal{N}(\lambda) := \mathcal{N}/(s-\lambda)\mathcal{N}$ is a regular holonomic $\mathcal{D}_X$-module whose characteristic variety is contained in W_f^0 for any $\lambda \in \mathbb{C}$ (Theorem 2.2 of [8]). Set $u = [af^{-k}]$ with $a \in \mathcal{O}_X$. The surjective $\mathcal{D}_X$-homomorphism of $\mathcal{N}(-1)$ to $\mathcal{O}_X[f^{-1}]$ implies that $\mathcal{O}_X[f^{-1}]/\mathcal{O}_X$ and its submodule $\mathcal{D}_X u$ are regular holonomic and their characteristic varieties are contained in $W_f^0 \cap \{f = 0\}$.

Let

$$af_+^{\lambda-k} = \sum_{j=-m}^{\infty} \lambda^j v_j \qquad (v_j \in \mathcal{D}b_M)$$

be the Laurent expansion around $\lambda = 0$. By using the functional equation (2), we can find a nonzero polynomial $b(s)$ and $Q(s) \in \mathcal{D}_X[s]$ such that

$$b(\lambda)f_+^{\lambda-k} = Q(\lambda)f_+^{\lambda}.$$

Factoring $b(s)$ as $b(s) = c(s)s^l$ with $c(0) \neq 0$, we obtain

$$\rho(u) = v_{-1} = \frac{a}{(l-1)!}\lim_{\lambda\to 0}\left(\frac{\partial}{\partial\lambda}\right)^{l-1}(c(\lambda)^{-1}Q(\lambda)f_+^{\lambda}) = \sum_{j=0}^{l-1} Q_j(Y(f)(\log f)^j)$$

with $Q_j \in \mathcal{D}_X$.

For a non-negative integer k, let us introduce a free $\mathcal{O}_X[f^{-1}]$-module

$$\mathcal{L}_k := \mathcal{O}_X[f^{-1}]f^s \oplus \mathcal{O}_X[f^{-1}](f^s \log f) \oplus \cdots \oplus \mathcal{O}_X[f^{-1}](f^s(\log f)^k).$$

Then $\mathcal{L}_k$ has a natural structure of left $\mathcal{D}_X[s]$-module. Let $\mathcal{N}_k$ be the left $\mathcal{D}_X[s]$-submodule of $\mathcal{L}_k$ generated by $f^s(\log f)^j$ for $j = 0, \ldots, k$. Then it is easy to see that $\mathcal{N}_0$ and $\mathcal{N}_k/\mathcal{N}_{k-1}$ are isomorphic to $\mathcal{N}$ as left $\mathcal{D}_X[s]$-module. Set

$$\mathcal{N}_k(\lambda) := \mathcal{N}_k/(s-\lambda)\mathcal{N}_k \quad \text{for } \lambda \in \mathbb{C}.$$

Then $\mathcal{N}_0(\lambda)$ and $\mathcal{N}_k(\lambda)/\mathcal{N}_{k-1}(\lambda)$ are isomorphic to $\mathcal{N}(\lambda)$ as left $\mathcal{D}_X$-module. Hence $\mathcal{N}_k(\lambda)$ is regular holonomic and its characteristic variety is contained in W_f^0.

Now set

$$\tilde{u} := Q_0[f^s]_0 \oplus Q_1[f^s \log f]_0 \oplus \cdots \oplus Q_{l-1}[f^s(\log f)^{l-1}]_0 \ \in \mathcal{N}_{l-1}(0),$$

where $[f^s(\log f)^j]_0$ denotes the modulo class of $f^s(\log f)^j$ in $\mathcal{N}_{l-1}(0)$. Then $\mathcal{D}_X\tilde{u}$ is regular holonomic as a left $\mathcal{D}_X$-submodule of $\mathcal{N}_{l-1}(0)$. There exists a surjective homomorphism of $\mathcal{D}_M\tilde{u}$ to $\mathcal{D}_M\rho(u)$ of left $\mathcal{D}_M$-modules which sends $\tilde{u}$ to $\rho(u)$. In particular, $\mathrm{Ann}_{\mathcal{D}_M}\tilde{u}$ annihilates $\rho(u)$. This completes the proof. □

Remark 1: In the same way as the proof above, one can show that every coefficient (as a distribution) of the Laurent expansion of f_+^{λ} with respect to λ about an arbitrary point in the complex plane satisfies a regular holonomic system whose characteristic variety is contained in W_f^0.

Remark 2: If f is a polynomial, then the proof above combined with an algorithm to compute the structure of $\mathcal{N}_k$ (Algorithm 2 of [12]) yields an algorithm to compute $\mathrm{Ann}_{\mathcal{D}_M}\tilde{u}$, which is a subsheaf of $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$. However, we do not know any general algorithm to compute $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$ exactly.

3. Comparison of annihilators via ρ

Theorem 3.1: *Let f be a real-valued real analytic function defined on M. Assume*

(A) For any positive integer k, $\lambda = -k$ is at most a simple pole of f_+^λ as a distribution on M with the meromorphic parameter λ.

Then ρ is a homomorphism of sheaves of left $\mathcal{D}_M$-modules. In particular, $\mathrm{Ann}_{\mathcal{D}_M} u$ is a subsheaf of $\mathrm{Ann}_{\mathcal{D}_M} \rho(u)$ for any section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_U$.

Proof: Let P and a be germs of $\mathcal{D}_X$ and of $\mathcal{O}_X$ respectively at a point y_0 of U. There exist $p(x,s) \in (\mathcal{O}_X)_{y_0}[s]$ and a non-negative integer l such that $P(af^s) = p(x,s)f^{s-l}$ holds formally. Then we have

$$\begin{aligned}\rho(P[af^{-k}]) &= \rho([p(x,-k)f^{-k-l}]) = \mathrm{Res}_{\lambda=0}\, p(x,-k)f_+^{\lambda-k-l} \\ &= \mathrm{Res}_{\lambda=0}\, p(x,\lambda-k)f_+^{\lambda-k-l} = \mathrm{Res}_{\lambda=0}\, P(af_+^{\lambda-k}) \\ &= P\,\mathrm{Res}_{\lambda=0}\, af_+^{\lambda-k} = P\rho([af^{-k}])\end{aligned}$$

since $\lambda = 0$ is at most a simple pole of $f_+^{\lambda-k-l}$. □

Corollary 3.2: *Assume*

(A′) $\tilde{b}_{f,x_0}(-k)$ does not vanish for any positive integer k with $x_0 \in M$.

Then we have an inclusion

$$\mathrm{Ann}_{(\mathcal{D}_X)_{x_0}} u \subset \mathrm{Ann}_{(\mathcal{D}_X)_{x_0}} \rho(u)$$

for any germ u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$.

Proof: By the functional equation (2), it is easy to see that (A) is satisfied with M replaced by a neighbourhood of x_0. Hence ρ is a $\mathcal{D}_X$-homomorphism on a neighbourhood of x_0. This implies the inclusion. □

Lemma 3.3: *Let f be a real-valued real analytic function defined on a neighbourhood of $x_0 \in M$ such that its differential df does not vanish at x_0. Suppose that $a(x,\lambda)$ is an analytic function in x, λ defined on a neighbourhood of $(x_0, 0) \in M \times \mathbb{C}$ and that*

$$\mathrm{Res}_{\lambda=0}\, a(x,\lambda)f_+^{\lambda-k} = 0$$

holds on a neighbourhood of x_0 with a non-negative integer k. Then $a(x,0)$ is divisible by f^k in the stalk $(\mathcal{O}_X)_{x_0}$.

Proof: By a real analytic local coordinate system $x = (x_1, \dots, x_n)$, we may assume $f(x) = x_1$ and $x_0 = 0$ with $M = \mathbb{R}^n$. In view of the functional equation

$$(x_1)_+^\lambda = \frac{1}{(\lambda+1)\cdots(\lambda+k)}\partial_1^k (x_1)_+^{\lambda+k},$$

$\lambda = -k$ is a simple pole of $(x_1)_+^\lambda$ with residue

$$\mathrm{Res}_{\lambda=-k}\,(x_1)_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!}\partial_1^k Y(x_1).$$

Hence we have

$$\begin{aligned}a(x,0)\partial_1^k Y(x_1) &= (-1)^{k-1}(k-1)!\,\mathrm{Res}_{\lambda=-k}\,a(x,0)(x_1)_+^{\lambda}\\&= (-1)^{k-1}(k-1)!\,\mathrm{Res}_{\lambda=0}\,a(x,0)(x_1)_+^{\lambda-k}\\&= (-1)^{k-1}(k-1)!\,\mathrm{Res}_{\lambda=0}\,a(x,\lambda)(x_1)_+^{\lambda-k} = 0\end{aligned}$$

on an open neighbourhood U of 0 in M. This implies

$$\begin{aligned}0 &= \langle a(x,0)\partial_1^k Y(x_1), \varphi(x)\rangle = \langle Y(x_1), (-\partial_1)^k(a(x,0)\varphi(x))\rangle\\&= \int_{\mathbb{R}^{n-1}}\left(\int_0^\infty (-\partial_1)^k(a(x,0)\varphi(x))\,dx_1\right)dx_2\cdots dx_n\\&= \int_{\mathbb{R}^{n-1}}\Big[(-\partial_1)^{k-1}(a(x,0)\varphi(x))\Big]_{x_1=0}dx_2\cdots dx_n\\&= (-1)^{k-1}\sum_{i=0}^{k-1}\binom{k-1}{i}\int_{\mathbb{R}^{n-1}}\partial_1^i a(0,x',0)\partial_1^{k-1-i}\varphi(0,x')\,dx_2\cdots dx_n\end{aligned}$$

for any C^∞ function $\varphi(x)$ with compact support in U, where we use the notation $x' = (x_2,\ldots,x_n)$. Choose an open interval I containing 0 and an open neighbourhood V' of 0 in $\mathbb{R}^{n-1}$ such that $I\times V'$ is a subset of U. Let $\chi(x_1)$ be a C^∞ function of x_1 with compact support in I such that $\chi(x_1) = 1$ on a neighbourhood of 0. Setting $\varphi(x) = x_1^l\chi(x_1)\psi(x')$ with an arbitrary C^∞ function $\psi(x')$ with compact support in V' and an integer l with $0 \le l \le k-1$, we get

$$\int_{\mathbb{R}^{n-1}}\partial_1^{k-1-l}a(0,x',0)\psi(x')\,dx_2\cdots dx_n = 0.$$

Hence $\partial_1^i a(0,x',0) = 0$ holds for $x' \in V'$ and $0 \le i \le k-1$. Thus $a(x,0)$ is divisible by x_1^k. □

Theorem 3.4: *Let f be a real-valued real analytic function defined on M such that $f(x_0) = 0$ with $x_0 \in M$. We assume*

(B) There exists a real analytic local coordinate system $x = (x_1,x') = (x_1,x_2,\ldots,x_n)$ of M around x_0 such that x_0 corresponds to the origin and f is written in the form

$$f(x) = c(x)(x_1^m + a_1(x')x_1^{m-1} + \cdots + a_m(x'))$$

with $m \ge 1$, where $c(x)$ and $a_j(x')$ are real-valued real analytic functions defined on a neighbourhood of the origin such that $c(0) \neq 0$ and $a_j(0) = 0$ for $1 \le j \le m$. Moreover, for any neighbourhood U of the origin in $\mathbb{R}^n$, there exists $y_0' \in \mathbb{R}^{n-1}$ such that $(0,y_0') \in U$ and the equation

$$x_1^m + a_1(y_0')x_1^{m-1} + \cdots + a_m(y_0') = 0$$

in x_1 has m distinct real roots.

Under this condition,

$$\mathrm{Ann}_{(\mathcal{D}_X)_{x_0}}\rho(u) \subset \mathrm{Ann}_{(\mathcal{D}_X)_{x_0}}u$$

holds for any germ u *of* $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$ *at* x_0.

Proof: Set $u = [af^{-k}]$ with $a \in (\mathcal{O}_X)_{x_0}$ and a non-negative integer k. Let P be a germ of $\mathcal{D}_X$ at x_0 and suppose $P\rho(u)$ vanishes on a neighbourhood of x_0. There exist $p(x,s) \in (\mathcal{O}_X)_{x_0}[s]$ and a non-negative integer l such that $P(af^s) = p(x,s)f^{s-l}$. This implies

$$P\rho(u) = \mathrm{Res}_{\lambda=0}\, P(af_+^{\lambda-k}) = \mathrm{Res}_{\lambda=0}\, p(x, \lambda - k) f_+^{\lambda-k-l} = 0 \tag{3}$$

on a neighbourhood of x_0.

By the Weierstrass preparation theorem, we can find $q(x) \in (\mathcal{O}_X)_{x_0}$ and $r_i(x') \in \mathbb{C}\{x'\}$ such that

$$p(x,-k) = q(x)\tilde{f}(x)^{k+l} + r(x) \quad \text{with } \tilde{f}(x) := x_1^m + a_1(x')x_1^{m-1} + \cdots + a_m(x'),$$
$$r(x) = \sum_{i=0}^{m(k+l)-1} r_i(x')x_1^i.$$

There exist an open connected neighbourhood V' of the origin in $\mathbb{C}^{n-1}$ and an open connected neighbourhood V_1 of 0 in $\mathbb{C}$ such that $a_j(x')$ and $r_j(x')$ are analytic on V', $c(x)$ and $q(x)$ are analytic on $V := V_1 \times V'$, $c(x)$ does not vanish on V, the set $\{x_1 \in \mathbb{C} \mid \tilde{f}(x_1, x') = 0\}$ is contained in V_1 if $x' \in V'$, and $P\rho(u) = 0$ holds on $V \cap M$.

Take a $y_0' \in V' \cap \mathbb{R}^{n-1}$ such that the equation

$$\tilde{f}(x_1, y_0') = x_1^m + a_1(y_0')x_1^{m-1} + \cdots + a_m(y_0') = 0$$

in x_1 has distinct real roots $x_1 = \xi_j$ $(j = 1, \ldots, m)$. Since f is non-singular at (ξ_j, y_0'), Lemma 3.3 and (3) imply that $p(x,-k)$ is divisible by f^{k+l} in $(\mathcal{O}_X)_{(\xi_j, y_0')}$ for each $j = 1, \ldots, m$. Hence there exists an analytic function $e(x)$ defined on a neighbourhood of $V_1 \times \{y_0'\}$ such that $p(x,-k) = e(x)f(x)^{k+l}$. Consequently

$$r(x) = e(x)f(x)^{k+l} - q(x)\tilde{f}(x)^{k+l} = (e(x)c(x)^{k+l} - q(x))\tilde{f}(x)^{k+l} \tag{4}$$

holds on a neighbourhood of $V_1 \times \{y_0'\}$.

Let us show that $r(x)$ is divisible by $\tilde{f}(x)^{k+l}$ in the ring $\mathbb{C}\{x' - y_0'\}[x_1]$, where $\mathbb{C}\{x' - y_0'\}$ denotes the ring of convergent power series in $x' - y_0'$. Since $\partial \tilde{f}/\partial x_1$ does not vanish at (ξ_j, y_0'), the implicit function theorem assures the existence of real-valued real analytic functions $\varphi_j(x')$ defined on a neighbourhood of y_0' such that $\varphi_j(y_0') = \xi_j$ and

$$\tilde{f}(x) = \prod_{j=1}^{m}(x_1 - \varphi_j(x'))$$

holds on a neighbourhood of $V_1 \times \{y_0'\}$. Division in $\mathbb{C}\{x' - y_0'\}[x_1]$ yields

$$r(x) = \tilde{q}_1(x)(x_1 - \varphi_1(x'))^{k+l} + \sum_{i=0}^{k+l-1} \tilde{r}_i(x')x_1^i$$

with $\tilde{q}_1(x) \in \mathbb{C}\{x' - y_0'\}[x_1]$ and $\tilde{r}_i(x') \in \mathbb{C}\{x' - y_0'\}$. On the other hand, $r(x)$ is divisible by $(x_1 - \varphi_1(x'))^{k+l}$ in $(\mathcal{O}_X)_{(\xi_1, y_0')}$ by (4). This implies, by virtue of the uniqueness statement of the Weierstrass preparation theorem (applied with x_1 replaced by $x_1 - \xi_1$), that $\tilde{r}_i(x') = 0$ for $0 \le i \le k+l-1$. In the same way, we can show that $\tilde{q}_1(x)$ is divisible by $(x_1 - \varphi_2(x'))^{k+l}$ in $\mathbb{C}\{x' - y_0'\}[x_1]$. Repeating this argument, we conclude that $r(x)$ is divisible by $\tilde{f}(x)^{k+l}$ in $\mathbb{C}\{x' - y_0\}[x_1]$. Hence $r(x) = 0$ holds on a neighbourhood of $V_1 \times \{y_0'\}$, and hence on V, since the degree of $r(x)$ in x_1 is smaller than that of $\tilde{f}(x)^{k+l}$. Thus we have $Pu = [p(x, -k)f^{-k-l}] = [q(x)c(x)^{-k-l}] = 0$ on V. This completes the proof. □

Corollary 3.5: *Assume (A) and (B) for any $x_0 \in M$. Then ρ induces an injective sheaf homomorphism $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) \to \mathcal{D}b_M$ of left $(\mathcal{D}_X)_{x_0}$-modules. In particular,*

$$\mathrm{Ann}_{(\mathcal{D}_X)_{x_0}}\rho(u) = \mathrm{Ann}_{(\mathcal{D}_X)_{x_0}}u$$

holds for any section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)_{x_0}$.

4. Comparison of annihilators of f^λ and of f_+^λ

Let f be a real-valued real analytic function defined on a neighbourhood of $x_0 \in M$. We present some elementary facts on the annihilator of f_+^λ, which will be useful in the computation of the annihilator of $\rho([f^{-k}])$ as well.

Regarding s as an indeterminate and f^s as a 'formal' function, set

$$\mathrm{Ann}_{\mathcal{D}_X[s]}f^s := \{P(s) \in \mathcal{D}_X[s] \mid P(s)f^s = 0\},$$

which is a coherent sheaf of left ideals of $\mathcal{D}_X[s]$ (cf. [6]).

Lemma 4.1: *Let $P(s)$ be a germ of $\mathrm{Ann}_{\mathcal{D}_X[s]}f^s$ at x_0. If $\lambda = \lambda_0 \in \mathbb{C}$ is not a pole of f_+^λ as a distribution near $x = x_0$, then $P(\lambda_0)f_+^{\lambda_0} = 0$ holds.*

Proof: If Re λ is sufficiently large, then $P(\lambda)f_+^\lambda$ is locally integrable near $x = x_0$. Moreover, it vanishes where $f(x) \neq 0$ by the assumption. Hence $P(\lambda)f_+^\lambda$ vanishes near $x = x_0$ if Re λ is sufficiently large. The assertion follows from the uniqueness of analytic continuation. □

Lemma 4.2: *Assume that λ_0 is not a pole of f_+^λ near $x = x_0$ and that for any neighbourhood V of x_0 in M there exists $y \in V$ such that $f(y) > 0$. Under these two assumptions, if $P \in (\mathcal{D}_X)_{x_0}$ satisfies $Pf_+^{\lambda_0} = 0$, then $Pf^{\lambda_0} = 0$ holds as a multi-valued analytic function.*

Proof: Since $f^{\lambda_0} = e^{2\pi\sqrt{-1}k\lambda_0}f_+^{\lambda_0}$ holds with some integer k where $f(x)$ is positive, we have $Pf^{\lambda_0} = 0$ as multi-valued analytic function in view of the second assumption. □

Proposition 4.3: *Assume that $\lambda = \lambda_0$ is not a pole of f_+^λ near $x = x_0$ and that $b_{f,x_0}(\lambda_0 - \nu) \neq 0$ holds for any positive integer ν. Assume moreover that for any neighbourhood V of x_0 in M there exists $y \in V$ such that $f(y) > 0$. Then the following three conditions on $P \in (\mathcal{D}_X)_{x_0}$ are equivalent:*

(1) $Pf_+^{\lambda_0} = 0$.
(2) $Pf^{\lambda_0} = 0$ *holds as a multi-valued analytic function.*
(3) *There exists a germ $Q(s)$ of $\mathrm{Ann}_{\mathcal{D}_X[s]}f^s$ such that $P = Q(\lambda_0)$.*

Proof: By the preceding two lemmas we have implications (3) $\Rightarrow$ 1 and (1) $\Rightarrow$ (2). The equivalence of (2) and (3) follows from Proposition 6.2 of [6] in view of the assumption on $b_{f,x_0}(s)$. □

5. Examples

Example 5.1 Set $f = x_1^2x_2^2+x_3^p$ with $M = \mathbb{R}^3$ and an odd integer $p \geq 3$. Then the reduced b-function $\tilde{b}_{f,0}(s)$ of f at the origin does not have integral roots as is seen by Example 4.20 of Yano [13]. It is easy to see (e.g., by the weighted homogeneity) that the reduced b-function of f at a point other than the origin is a factor of $\tilde{b}_{f,0}(s)$. Hence the assumption (A′) is satisfied for any $x_0 \in M$.

By a coordinate transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$, $y_3 = x_3$, f takes the form

$$f = (y_1^2 - y_2^2)^2 + y_3^p = y_1^4 - 2y_1^2y_2^2 + y_2^4 + y_3^p.$$

Hence the equation $f = 0$ in y_1 has four distinct real roots if and only if $y_3 < 0$ and $y_2^4+y_3^p > 0$. Hence the assumption (B) is satisfied at each point $x_0 = (x_{01}, x_{02}, x_{03})$ belonging to the the singular loci $x_1 = x_3 = 0$ or $x_2 = x_3 = 0$. It is easy to see that the assumption (B) is always satisfied at a non-singular point. In conclusion we have $\mathrm{Ann}_{\mathcal{D}_M} u = \mathrm{Ann}_{\mathcal{D}_M}\rho(u)$ for any section u of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)|_M$ in view of Corollary 3.5.

We conjecture that $\mathrm{Ann}_{\mathcal{D}_X}[f^{-1}]$ is generated by

$$x_1^2x_2^2 + x_3^p, \quad px_3^{p-1}\partial_1 - 2x_1x_2^2\partial_3, \quad px_3^{p-1}\partial_2 - 2x_1^2x_2\partial_3,$$
$$px_1\partial_1 + 2x_3\partial_3 + 2p, \quad px_2\partial_2 + 2x_3\partial_3 + 2p$$

for any integer $p \geq 1$. We have verified it for $1 \leq p \leq 290$ by using an algorithm in [11] with a computer algebra system Risa/Asir ([9]). For example, if $p = 3$, then the characteristic cycle of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{D}_X[f^{-1}]$ is given by

$$2T^*_{\{x_1=x_2=x_3=0\}}\mathbb{C}^3 + T^*_{\{x_1=x_3=0\}\setminus\{0\}}\mathbb{C}^3 + T^*_{\{x_2=x_3=0\}\setminus\{0\}}\mathbb{C}^3 + T^*_{Y'}\mathbb{C}^3$$

with $Y' := \{(x_1, x_2, x_3) \mid x_1^2x_2^2 + x_3^3 = 0\} \setminus \{(x_1, x_2, x_3) \mid x_1x_2 = x_3 = 0\}$.

Example 5.2 Set $f = x_1x_2$ with $M = \mathbb{R}^2$ and consider a section $u := [(x_1x_2)^{-1}]$ of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X)$. The assumption (B) is satisfied at every point of $f = 0$ as is seen by the coordinate transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$. Hence $\mathrm{Ann}_{\mathcal{D}_M}\rho(u) \subset \mathrm{Ann}_{\mathcal{D}_M} u$ holds by Theorem 3.4. It is easy to see that $\mathrm{Ann}_{\mathcal{D}_X} u$ is the left ideal of $\mathcal{D}_X$ generated by x_1x_2, $x_1\partial_1 + 1$, $x_2\partial_2 + 1$. In fact, suppose $P \in \mathcal{D}_X$ annihilates u on a neighbourhood of $0 = (0, 0)$. We can write P in the form

$$P = Q_0(x_1, x_2; \partial_1, \partial_2)x_1x_2 + Q_1(x_1; \partial_1, \partial_2)x_1 + Q_2(x_2; \partial_1, \partial_2)x_2 + R(\partial_1, \partial_2).$$

Then on a neighbourhood of a point $(x_1, 0)$ with $|x_1| > 0$ sufficiently small, we have

$$(Q_1(x_1; \partial_1, \partial_2) + R(\partial_1, \partial_2)x_1^{-1})[x_2^{-1}] = 0.$$

In view of the Laurent expansion with respect to x_1, this implies that $R = 0$ and that Q_1 is written in the form $Q_1 = Q_1'\partial_1$. Likewise Q_2 is written in the form

$Q_2 = Q_2'\partial_2$. Hence P belongs to the left ideal generated by x_1x_2, $\partial_1 x_1$, $\partial_2 x_2$.

On the other hand, the annihilator of $\rho(u)$ is generated by x_1x_2 and $x_1\partial_1 - x_2\partial_2$. This can be shown as follows: Assume $P \in (\mathcal{D}_X)_0$ annihilates $\rho(u)$. Then P is written in the form

$$P = Q_0(x_1, x_2; \partial_1, \partial_2)x_1x_2 + Q_1(x_1, x_2; \partial_1, \partial_2)\partial_1 x_1 + Q_2(x_1, x_2; \partial_1, \partial_2)\partial_2 x_2,$$

which can be verified in the same way as above, or else follows from Theorem 3.4. Since

$$\partial_1 x_1 \rho(u) = \partial_2 x_2 \rho(u) = 2\delta(x_1, x_2) \tag{5}$$

holds, where $\delta(x_1, x_2) = \delta(x_1)\delta(x_2)$ is the Dirac delta function with support at the origin, $P\rho(u) = 0$ implies

$$(Q_1(x_1, x_2; \partial_1, \partial_2) + Q_2(x_1, x_2; \partial_1, \partial_2))\delta(x_1)\delta(x_2) = 0.$$

Let us rewrite Q_1 and Q_2 in the form

$$Q_i(x_1, x_2; \partial_1, \partial_2) = S_i(x_1, x_2; \partial_1, \partial_2)x_1 + T_i(x_2; \partial_1, \partial_2)x_2 + R_i(\partial_1, \partial_2) \quad (i = 1, 2).$$

Then $R_1 + R_2 = 0$ follows from $(R_1 + R_2)\delta(x_1)\delta(x_2) = 0$. Summing up we get

$$P = (Q_0 + T_1\partial_1 + S_2\partial_2)x_1x_2 + S_1x_1\partial_1 x_1 + T_2x_2\partial_2 x_2 + R_1(\partial_1 x_1 - \partial_2 x_2).$$

This shows that P belongs to the left ideal generated by x_1x_2 and $x_1\partial_1 - x_2\partial_2$ since

$$x_1\partial_1 x_1 = x_1(\partial_1 x_1 - \partial_2 x_2) + \partial_2 x_1 x_2.$$

Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{D}_X$ generated by x_1x_2 and $x_1\partial_1 - x_2\partial_2$. Then $\mathcal{I}|_M$ coincides with the sheaf $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$ of annihilating ideals on $M = \mathbb{R}^2$, which is consequently coherent over $\mathcal{D}_M$. In fact, they coincide at the origin by the above argument. At another point, say, $(0, x_2)$ with $x_2 \neq 0$, it is also easy to see that $\mathcal{I}$ coincides with the annihilator of $\rho(u)$. The characteristic cycle of $\mathcal{D}_X u$ is

$$T^*_{\{x_1=x_2=0\}}\mathbb{C}^2 + T^*_{\{x_1=0, x_2\neq 0\}}\mathbb{C}^2 + T^*_{\{x_2=0, x_1\neq 0\}}\mathbb{C}^2,$$

while that of $\mathcal{D}_X/\mathcal{I}$ is

$$2T^*_{\{x_1=x_2=0\}}\mathbb{C}^2 + T^*_{\{x_1=0, x_2\neq 0\}}\mathbb{C}^2 + T^*_{\{x_2=0, x_1\neq 0\}}\mathbb{C}^2.$$

Finally, let us verify (5). By using the functional equation

$$\partial_1\partial_2(x_1x_2)_+^{s+1} = (s+1)^2(x_1x_2)_+^s,$$

we get

$$\rho(u) = \mathrm{Res}_{\lambda=0}\,(x_1x_2)_+^{\lambda-1} = \partial_1\partial_2(Y(x_1x_2)\log(x_1x_2)).$$

Let $\varphi(x) = \varphi(x_1, x_2)$ be a C^∞ function with compact support. Then we have

$$\begin{aligned}&\langle \partial_1\partial_2(Y(x_1x_2)\log(x_1x_2)), \varphi(x)\rangle \\ &\quad = \int_0^\infty \int_0^\infty (\log x_1 + \log x_2)\left\{(\partial_1\partial_2\varphi)(x_1, x_2) + (\partial_1\partial_2\varphi)(-x_1, -x_2)\right\} dx_1 dx_2.\end{aligned}$$

We rewrite the integral involving $\log x_1$ as follows:

$$\begin{aligned}&\int_0^\infty \int_0^\infty (\log x_1)\left\{(\partial_1\partial_2\varphi)(x_1, x_2) + (\partial_1\partial_2\varphi)(-x_1, -x_2)\right\} dx_1 dx_2 \\ &= -\int_0^\infty (\log x_1)\left\{(\partial_1\varphi)(x_1, 0) - (\partial_1\varphi)(-x_1, 0)\right\} dx_1 \\ &= -\int_0^\infty (\log x_1)\partial_1\left\{\varphi(x_1, 0) + \varphi(-x_1, 0) - 2\varphi(0, 0)\right\} dx_1 \\ &= \int_0^\infty \frac{\varphi(x_1, 0) + \varphi(-x_1, 0) - 2\varphi(0, 0)}{x_1} dx_1.\end{aligned}$$

Hence if we define the distribution $v(t)$ on $\mathbb{R}$ by

$$\langle v(t), \psi(t)\rangle = \int_0^\infty \frac{\psi(t) + \psi(-t) - 2\psi(0)}{t} dt$$

for any C^∞ function $\psi(t)$ with compact support, we have

$$\rho(u) = v(x_1)\delta(x_2) + \delta(x_1)v(x_2).$$

It is easy to see that $tv(t) = Y(t) - Y(-t)$, from which (5) follows.

Example 5.3 Set $f = x_1(x_2^2 + x_3^2 + x_4^2)$ with $M = \mathbb{R}^4$ and set $u := [f^{-1}]$. Then f^s satisfies a functional equation

$$\frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)f^{s+1} = (s+1)^2\left(s + \frac{3}{2}\right)f^s. \tag{6}$$

Let

$$f_+^\lambda = (\lambda+1)^{-2}v_{-2}(x) + (\lambda+1)^{-1}v_{-1}(x) + v_0(x) + \cdots$$

be the Laurent expansion around $\lambda = -1$. Then we have

$$\begin{aligned}v_{-2}(x) &= \frac{1}{2}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)Y(x_1) = 0, \\ v_{-1}(x) &= \frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)\left\{\lim_{\lambda\to -1}\frac{\partial}{\partial\lambda}\left(\left(\lambda + \frac{3}{2}\right)^{-1} f_+^{\lambda+1}\right)\right\} \\ &= \frac{1}{4}\partial_1(\partial_2^2 + \partial_3^2 + \partial_4^2)\left\{-4Y(x_1) + 2Y(x_1)(\log x_1 + \log(x_2^2 + x_3^2 + x_4^2)\right\} \\ &= \delta(x_1)(x_2^2 + x_3^2 + x_4^2)^{-1}.\end{aligned}$$

Thus $\lambda = -1$ is a simple pole of f_+^λ and so is $\lambda = -k$ for any positive integer k in view of (6). Hence (A) is satisfied.

Let us show that $\mathrm{Ann}_{\mathcal{D}_X} u$ is generated by

$$x_1(x_2^2 + x_3^2 + x_4^2), \quad x_1\partial_1 + 1, \quad x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + 2,$$
$$x_2\partial_3 - x_3\partial_2, \quad x_2\partial_4 - x_4\partial_2, \quad x_3\partial_4 - x_4\partial_3.$$

We have only to find $P \in \mathcal{D}_X$ other than f which annihilates the rational function $f^{-1} = x_1^{-1}g^{-1}$ with $g := x_2^2 + x_3^2 + x_4^2$. In fact, if $P[f^{-1}] = 0$ then $a := Pf^{-1}$ is analytic, which means that $P - af$ annihilates f^{-1}. The annihilator of g^{-1} as analytic function is generated by the operators listed above except the first two since g is homogeneous and $g = 0$ has an isolated singularity (see e.g. Theorem 2.19 of [13]). The annihilator of x_1^{-1} is generated by $x_1\partial_1 + 1$. The assertion follows from these observations.

On the other hand, the annihilator of $\rho(u) = v_{-1}(x) = \delta(x_1)g^{-1}$ is generated by the operators

$$x_1, \quad x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + 2, \quad x_2\partial_3 - x_3\partial_2, \quad x_2\partial_4 - x_4\partial_2, \quad x_3\partial_4 - x_4\partial_3. \quad (7)$$

We can verify this as follows: First note that g^λ is locally integrable for $\mathrm{Re}\,\lambda > -3/2$. Since the b-function of g is $(s+1)(s+3/2)$, Theorem 4.3 guarantees that for a differential operator P in the variables $x' = (x_2, x_3, x_4)$, we have $Pg_+^{-1} = 0$ as a distribution if and only if Pg^{-1} holds as an analytic function. It follows that $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$ is generated by the operators listed in (7).

In conclusion, $\mathrm{Ann}_{\mathcal{D}_M}\rho(u)$ is coherent on $M = \mathbb{R}^4$, of which $\mathrm{Ann}_{\mathcal{D}_M} u$ is a proper subsheaf. The characteristic cycle of $\mathcal{H}^1_{[f=0]}(\mathcal{O}_X) = \mathcal{D}_X u$ is

$$T^*_{\{0\}}\mathbb{C}^4 + T^*_{\{x_2=x_3=x_4=0\}\setminus\{0\}}\mathbb{C}^4 + T^*_{\{x_1=x_2^2+x_3^2+x_4^2=0\}\setminus\{0\}}\mathbb{C}^4$$
$$+ T^*_{\{x_1=0, x_2^2+x_3^2+x_4^2\neq 0\}}\mathbb{C}^4 + T^*_{\{x_2^2+x_3^2+x_4^2=0, x_1\neq 0, (x_2,x_3,x_4)\neq(0,0,0)\}}\mathbb{C}^4,$$

whereas that of $\mathcal{D}_X/\mathcal{I}$ with $\mathcal{I}$ being the sheaf of left ideals of $\mathcal{D}_X$ generated by the operators in (7) is

$$T^*_{\{0\}}\mathbb{C}^4 + T^*_{\{x_1=x_2^2+x_3^2+x_4^2=0\}\setminus\{0\}}\mathbb{C}^4 + T^*_{\{x_1=0, x_2^2+x_3^2+x_4^2\neq 0\}}\mathbb{C}^4.$$

Example 5.4 Among the normal forms of real hypersurface singularities in $M = \mathbb{R}^n$ (see [1]), at least the following ones satisfy the condition (B) at the origin, where $q(x_k, \dots, x_n)$ denotes a non-degenerate quadratic form in the variables $x_k, \dots, x_n$ and a is a real constant:

(1) $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 \quad (1 \leq p \leq n-1)$,

(2) $D_4^- : \ x_1^2 x_2 - x_2^3 + q(x_3, \dots, x_n)$,

(3) $E_7 : \ x_1^3 + x_1 x_2^3 + q(x_3, \dots, x_n)$,

(4) $P_8^\pm : \ x_1^3 + a x_1^2 x_3 \pm x_1 x_3^2 + x_2^2 x_3 + q(x_4, \dots, x_n)$ with $-a^2 \pm 4 < 0$,

(5) $J_{10}^\pm : \ x_1^3 + a x_1^2 x_2^2 \pm x_1 x_2^4 + q(x_3, \dots, x_n)$ with $-a^2 \pm 4 < 0$,

(6) $J_{10+k}^\pm : \ x_1^3 \pm x_1^2 x_2^2 + a x_2^{6+k} + q(x_3, \dots, x_n)$ with $k \geq 1$ and ($\pm a < 0$ or k: odd),

(7) $P_{8+k}^\pm : \ x_1^3 \pm x_1^2 x_3 + x_2^2 x_3 + a x_3^{k+3} + q(x_4, \dots, x_n)$ with $k \geq 1$ and $a \neq 0$ and ($-$ or $a < 0$ or k: odd),

(8) $R_{l,m} : \ x_1(x_1^2 + x_2 x_3) \pm x_2^l \pm a x_3^m + q(x_4, \dots, x_n)$ with $a \neq 0, \ m \geq l \geq 5$,

(9) $\tilde{R}_m^- : \ x_1(-x_1^2 + x_2^2 + x_3^2) + ax_2^m + q(x_4, \ldots, x_n)$ with $a \neq 0, \ m \geq 5$,

(10) $E_{12} : \ x_1^3 + x_2^7 \pm x_3^2 + ax_1x_2^5 + q(x_4, \ldots, x_n)$,

(11) $E_{13} : \ x_1^3 + x_1x_2^5 \pm x_3^2 + ax_2^8 + q(x_4, \ldots, x_n)$,

(12) $E_{14} : \ x_1^3 \pm x_2^8 \pm x_3^2 + ax_1x_2^6 + q(x_4, \ldots, x_n)$,

(13) $Z_{11} : \ x_1^3x_2 + x_2^5 \pm x_3^2 + ax_1x_2^4 + q(x_4, \ldots, x_n)$,

(14) $Z_{12} : \ x_1^3x_2 + x_1x_2^4 \pm x_3^2 + ax_1^2x_2^3 + q(x_4, \ldots, x_n)$,

(15) $Z_{13} : \ x_1^3x_2 \pm x_2^6 \pm x_3^2 + ax_1x_2^5 + q(x_4, \ldots, x_n)$,

(16) $W_{12} : \ \pm x_1^4 + x_2^5 \pm x_3^2 + ax_1^2x_2^3 + q(x_4, \ldots, x_n)$,

(17) $W_{13} : \ \pm x_1^4 + x_1x_2^4 \pm x_3^2 + ax_2^6 + q(x_4, \ldots, x_n)$,

(18) $Q_{11} : \ x_1^3 + x_2^2x_3 \pm x_1x_3^3 + ax_3^5 + q(x_4, \ldots, x_n)$.

Let us show that the polynomial f of $P_8^{\pm}$ satisfies the condition (B) if $-a^2 \pm 4 < 0$. The discriminant of the cubic polynomial $f(x_1, x_2, x_3, 0, \ldots, 0)$ in x_1 is

$$D(x_2, x_3) = 27x_2^4x_3^2 + (4a^3 \mp 18a)x_2^2x_3^4 + (-a^2 \pm 4)x_3^6.$$

Substituting tx_3 for x_2 we get

$$D(tx_3, x_3) = (27t^4 + (4a^3 \mp 18a)t^2 - a^2 \pm 4)x_3^6.$$

Hence we have $D(tx_3, x_3) < 0$, which assures that the cubic equation

$$f(x_1, tx_3, x_3, 0, \ldots, 0) = 0$$

in x_1 has three real roots, if $-a^2 \pm 4 < 0$, $x_3 \neq 0$, and t is sufficiently small.

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