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Algorithms for b-Functions, Restrictions, and Algebraic Local Cohomology Groups of D-Modules

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1. INTRODUCTION

Our purpose is to present algorithms for computing some invariants and functors attached to algebraic D-modules by using Gröbner bases for differential operators. Let K be an algebraically closed field of characteristic zero and let X be a Zariski open set of K^n with a positive integer n. We fix a coordinate system $x = (x_1, \dots, x_n)$ of X and write $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i := \partial/\partial x_i$. We denote by $\mathcal{D}_X$ the sheaf of algebraic linear differential operators on X (cf. [3], [6]).

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module and u a section of $\mathcal{M}$. Suppose that $f = f(x) \in K[x]$ is an arbitrary nonconstant polynomial of n variables. If $\mathcal{M}$ is holonomic, then for each point p of $Y := \{x \in X \mid f(x) = 0\}$, there exists a germ $P(x, \partial, s)$ of $\mathcal{D}_X[s]$ at p and a polynomial $b(s) \in K[s]$ of one variable so that

$$P(x, \partial, s)(f^{s+1}u) = b(s) f^s u \tag{1.1}$$

holds with an indeterminate s (cf. [17]). More precisely, (1.1) means that there exists a nonnegative integer m so that

$$Q := f^{m-s}(b(s) - P(s, \partial, s)f) f^s \in \mathcal{D}_X[s]$$

satisfies $Qu = 0$ in $\mathcal{M}[s] := K[s] \otimes_K \mathcal{M}$. The monic polynomial $b(s)$ of the least degree that satisfies (1.1), if any, is called the (generalized) *b-function* for f and u at p. The b-function in this sense was first studied by Kashiwara [17] (cf. also [44]). Some of its applications were given by

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Kashiwara and Kawai [20]. In particular, when $\mathcal{M}$ coincides with the sheaf $\mathcal{O}_X$ of regular functions and $u = 1$, we get the classical b-function (or the Bernstein–Sato polynomial) of f. An algorithm for computing the Bernstein–Sato polynomial has been given in [33] (see also [44, 34] for examples, and [38, 4, 25] for algorithms in some special cases).

Suppose that a presentation (i.e., generators and the relations among them) of a coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and a section u of $\mathcal{M}$ are given. Then we are concerned with algorithms for solving the following problems:

(A1) to determine whether there exists and to find, if it does, the b-function for f and u;

(A2) to obtain presentations of the algebraic local cohomology groups $\mathcal{H}^j_{[Y]}(\mathcal{M})$ $(j = 0, 1)$ as left $\mathcal{D}_X$-modules (cf. [17] for the definition);

(A3) to obtain a presentation of the localization $\mathcal{M}(*Y) = \mathcal{M}[f^{-1}]$ of $\mathcal{M}$ by f as a left $\mathcal{D}_X$-module;

(A4) to obtain a presentation of the left $\mathcal{D}_X[s]$-module

$$\sum_{i=1}^{r} \mathcal{D}_X[s](f^s \otimes u_i),$$

where $u_1, \ldots, u_r$ are generators of $\mathcal{M}$ and $f^s \otimes u_i$ is regarded as a section of $(\mathcal{O}_X[s, f^{-1}]f^s) \otimes_{\mathcal{O}_X} \mathcal{M}$.

It turns out that these problems are closely related with one another not only from theoretical but also from algorithmic point of view: Solutions to (A2)–(A4) need the existence of and some information on the b-functions for f and $u_1, \ldots, u_r$; one can solve the problem (A3) by using a solution to (A4) by specializing the parameter s to an appropriate negative integer. As an application, for two polynomials $f_1, f_2 \in K[s]$, we can obtain a presentation of the left $\mathcal{D}_X$-module $\mathcal{D}_X(f_1^{s_1} f_2^{s_2})$ for generic constants $s_1, s_2 \in K$.

Kashiwara [17] proved that $\mathcal{H}^j_{[Y]}(\mathcal{M})$ and $\mathcal{M}(*Y)$ are holonomic if so $\mathcal{M}$ is. In this case (more generally, under a weaker condition that the b-functions for f and $u_1, \ldots, u_r$ exist, which can be determined algorithmically), we can solve the problems (A1)–(A4) completely except that we need the condition $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ to solve the latter part of (A1), (A3), and (A4); even if this condition fails, we can obtain certain information (estimates 'from above') on solutions of these problems. We solve the problem (A4) by generalizing a method developed in [34] for computing a presentation of $\mathcal{D}_X[s]f^s$. Note that Ginsburg [14] (see also [5]) gave formulas which connect the characteristic cycles of $\mathcal{H}^j_{[Y]}(\mathcal{M})$ and $\mathcal{M}(*Y)$ with that of $\mathcal{M}$, and which can also serve as algorithms at least in algebraic case for computing those characteristic cycles via Gröbner bases in the polynomial ring (combined with an algorithm to compute the characteristic cycle of $\mathcal{M}$ (cf. [30])), under the condition that $\mathcal{M}$ is regular holonomic.

Our algorithms for (A1) and (A2) are actually obtained as applications of algorithms for more general problems as follows: Now let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\tilde{X}}$-module with $\tilde{X} := K \times X$. Let $u_1, \dots, u_r$ be generators of $\mathcal{M}$. We identify X with the hyperplane $\{(t,x) \in \tilde{X} \mid t = 0\}$ of $\tilde{X}$. Then the b-function of $\mathcal{M}$ along X at $p \in X$ is the monic polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$\big(b(t\partial_t) + tP_i(t, x, t\partial_t, \partial)\big)u_i = 0 \qquad (i = 1, \dots, r)$$

with germs $P_i(t, x, t\partial_t, \partial)$ of $\mathcal{D}_{\tilde{X}}$ at p, where we write $\partial_t := \partial/\partial t$. $\mathcal{M}$ is called *specializable* along X at p if such $b(s)$ exists. On the other hand, the *restriction* (also called the induced system or the tangential system) of $\mathcal{M}$ to X is the complex of left $\mathcal{D}_X$-modules:

$$\mathcal{M}_X^\bullet : 0 \longrightarrow \mathcal{M} \stackrel{t}{\longrightarrow} \mathcal{M} \longrightarrow 0.$$

It was proved by Laurent-Schapira [24] (and by Kashiwara [17]) that if $\mathcal{M}$ is specializable along X (or holonomic), then the cohomology groups of $\mathcal{M}_X^\bullet$ are coherent left $\mathcal{D}_X$-modules (holonomic systems, respectively).

In the classical case $K = \mathbf{C}$ (the field of complex numbers), if X is noncharacteristic for $\mathcal{M}$ (cf. [19]), or $\mathcal{M}$ is Fuchsian along X (cf. [23]), we have an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{M}, \mathcal{O}_{\tilde{X}}^{\mathrm{an}})\big|_X \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_X^\bullet, \mathcal{O}_X^{\mathrm{an}})$$

in the derived category, where $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{O}_{\tilde{X}}^{\mathrm{an}}$ denote the sheaves of holomorphic functions on X and on $\tilde{X}$ respectively, and $\mathbf{R}\mathcal{H}om$ means the right derived functor of the functor of taking homomorphisms between sheaves (cf. [15]). This isomorphism is a generalization of the classical Cauchy–Kowalevskaja theorem.

Assume now that a presentation of a coherent left $\mathcal{D}_{\tilde{X}}$-module $\mathcal{M}$ is given. Then we obtain a complete algorithm for solving the problem

(B1) to determine whether $\mathcal{M}$ is specializable along X and to find, if so, the b-function of $\mathcal{M}$ along X.

This algorithm is obtained by generalizing a method of Gröbner basis computation (the Buchberger algorithm [7]) in the Weyl algebra with respect to the so-called V-filtration ([18], [27]) developed in [31, 32, 33] (cf. also [1]). We have solved (B1) for the case $r = 1$ in [33]. Here we generalize an algorithm of [33] so that we can compute the b-function as a function of the point of X for arbitrary $r \geq 1$.

Under the condition that $\mathcal{M}$ is specializable along X, we also get an algorithm to solve the problem

(B2) to obtain presentations of the cohomology groups of $\mathcal{M}_X^\bullet$ as left $\mathcal{D}_X$-modules.

It seems that no complete algorithm for (B2) used to be known (see [41, 42, 32] for partial algorithms). Note that $\mathcal{M}$ is specializable if $\mathcal{M}$ is holonomic ([21]). Algorithms for (A1) and (A2) are obtained by applying the algorithms for (B1) and (B2) to the module $(\mathcal{D}_{\tilde{X}}\delta(t-f(x)) \otimes_{\mathcal{O}_X} \mathcal{M}$ for a given $\mathcal{D}_X$-module $\mathcal{M}$, where $\delta(t-f(x))$ denotes the modulo class of $(t-f(x))^{-1}$ in $\mathcal{O}_{\tilde{X}}[(t-f(x))^{-1}]$. Thus we can solve (A2) under the condition that $(\mathcal{D}_{\tilde{X}}\delta(t-f(x))) \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X, and (A1), (A3), (A4) under the additional assumption $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. We can also show that $(\mathcal{D}_{\tilde{X}}\delta(t-f(x))) \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X if and only if there exists the b-function for f and each generator of $\mathcal{M}$ in the sense of (1.1).

When $K = \mathbf{C}$, we can consider the problems explained so far with $\mathcal{D}_X$ replaced by the sheaf $\mathcal{D}_X^{\mathrm{an}}$ of *analytic* differential operators. Then our algorithms yield correct solutions also in this analytic case if the left $\mathcal{D}_X^{\mathrm{an}}$-module $\mathcal{M}^{\mathrm{an}}$ in question is written in the form $\mathcal{M}^{\mathrm{an}} = \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}$ with a coherent $\mathcal{D}_X$-module $\mathcal{M}$ whose presentation is given explicitly.

We have implemented the algorithms in the present paper by using computer algebra systems Kan [43], developed by Takayama of Kobe University, and Risa/Asir [29], developed by Noro et al. at Fujitsu Laboratories Limited. We use Kan for Gröbner basis computation in Weyl algebras, and Risa/Asir for Gröbner basis computation, factorization, and primary decomposition in polynomial rings.

2. V-FILTRATION AND INVOLUTORY GENERATORS

Let $\tilde{X}$ be a Zariski open subset of $K \times K^n$ with the coordinate system $(t, x) = (t, x_1, \dots, x_n)$. We denote by $\partial_t = \partial/\partial t$ and $\partial = (\partial_1, \dots, \partial_n)$ the corresponding derivations with $\partial_i = \partial/\partial x_i$. Put $X := \tilde{X} \cap (\{0\} \times K^n)$. Then X can be identified with a Zariski open subset of K^n. Let $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$ be the sheaves of regular functions on X and on $\tilde{X}$ respectively. We denote by $\mathcal{D}_{\tilde{X}}$ and $\mathcal{D}_X$ the sheaves of rings of algebraic linear differential operators on $\tilde{X}$ and on X respectively. Let $\mathcal{D}_{\tilde{X}}|_X$ be the sheaf theoretic restriction of $\mathcal{D}_{\tilde{X}}$ to X. Put $\mathcal{J}_X := \mathcal{O}_{\tilde{X}} t$. Then for each integer k we put

$$F_k(\mathcal{D}_{\tilde{X}}) := \left\{ P \in \mathcal{D}_{\tilde{X}}|_X \mid P(\mathcal{J}_X)^j \in (\mathcal{J}_X)^{j-k} \text{ for any } j \geq 0 \right\}.$$

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\tilde{X}}$-module. We assume that $\mathcal{M}$ has a presentation $\mathcal{M} = (\mathcal{D}_{\tilde{X}})^r / \mathcal{N}$ on $\tilde{X}$, where $\mathcal{N}$ is a left $\mathcal{D}_{\tilde{X}}$-submodule of $(\mathcal{D}_{\tilde{X}})^r$. Then let us put

$$F_k(\mathcal{N}) := \mathcal{N} \cap F_k(\mathcal{D}_{\tilde{X}})^r, \qquad F_k(\mathcal{M}) := F_k(\mathcal{D}_{\tilde{X}})^r / F_k(\mathcal{N})$$

for each integer $k \in \mathbf{Z}$. These are called V-filtrations ([18, 27]). The graded ring and modules associated with these filtrations are defined by

$$\operatorname{gr}(\mathcal{D}_{\tilde{X}}) := \bigoplus_{k \in \mathbf{Z}} F_k(\mathcal{D}_{\tilde{X}}) / F_{k-1}(\mathcal{D}_{\tilde{X}}),$$

$$\operatorname{gr}(\mathcal{N}) := \bigoplus_{k \in \mathbf{Z}} F_k(\mathcal{N}) / F_{k-1}(\mathcal{N}),$$

$$\operatorname{gr}(\mathcal{M}) := \bigoplus_{k \in \mathbf{Z}} F_k(\mathcal{M}) / F_{k-1}(\mathcal{M}).$$

Then $\operatorname{gr}(\mathcal{M})$ is a coherent left $\operatorname{gr}(\mathcal{D}_{\tilde{X}})$-module. Note that $\operatorname{gr}(\mathcal{D}_{\tilde{X}})$ is isomorphic to $\mathcal{D}_X[t, \partial_t]$, which consists of the sections of $\mathcal{D}_{\tilde{X}}|_X$ that are polynomials in t.

For a nonzero section P of $(\mathcal{D}_{\tilde{X}})^r|_X$, let $k = \operatorname{ord}_F(P)$ be the minimum $k \in \mathbf{Z}$ such that $P \in F_k(\mathcal{D}_{\tilde{X}})^r$. Then let $\hat{\sigma}(P)$ be the modulo class of P in

$$F_k(\mathcal{D}_{\tilde{X}})^r / F_{k-1}(\mathcal{D}_{\tilde{X}})^r \simeq (\mathcal{D}_X[t\partial_t] S_k)^r,$$

where $S_k := \partial_t^k$ if $k \geq 0$ and $S_k := t^{-k}$ otherwise. Moreover, we define $\psi(P)(s) \in (\mathcal{D}_X[s])^r$ so that $\hat{\sigma}(S_{-k}P) = \psi(P)(t\partial_t)$ holds.

DEFINITION 2.1. Let U be a Zariski open subset of X. A subset $\mathbf{G}$ of $\Gamma(U, \mathcal{N}|_X)$ is called a set of *F-involutory generators* of $\mathcal{N}$ on U if $\mathbf{G}$ generates $\mathcal{N}|_X$ as a left $\mathcal{D}_{\tilde{X}}|_X$-module on U and if $\hat{\sigma}(\mathbf{G}) := \{\hat{\sigma}(P) \mid P \in \mathbf{G}\}$ generates $\operatorname{gr}(\mathcal{N})$ as a left $\operatorname{gr}(\mathcal{D}_{\tilde{X}})$-module.

The following two propositions are immediate consequences of the definitions:

PROPOSITION 2.2. *Let $\mathbf{G} = \{P_1, \ldots, P_m\} \subset \Gamma(U, \mathcal{N}|_X)$ be a set of generators of $\mathcal{N}|_X$ on a Zariski open set $U \subset X$. Then $\mathbf{G}$ is a set of F-involutory generators of $\mathcal{N}$ on U if and only if for an arbitrary nonzero element P of the stalk $\mathcal{N}_p$ of $\mathcal{N}$ at $p \in U$, and for an arbitrary integer j, there exist $Q_1, \ldots, Q_m \in \mathcal{N}_p$ so that* $\operatorname{ord}_F(Q_i P_i) \leq \operatorname{ord}_F(P)$ $(i = 1, \ldots, m)$ *and*

$$P - Q_1 P_1 - \cdots - Q_m P_m \in F_j(\mathcal{D}_{\tilde{X}})_p^r.$$

PROPOSITION 2.3. *Let $\mathbf{G}$ be a set of F-involutory generators of $\mathcal{N}$. Denote by $\psi(\mathcal{N})$ the left $\mathcal{D}_X[s]$-submodule of $(D_X[s])^r$ generated by $\{\psi(P) \mid P \in \mathcal{N}\}$. Then $\psi(\mathcal{N})$ is generated by $\psi(\mathbf{G}) := \{\psi(P) \mid P \in \mathbf{G}\}$.*

3. GRÖBNER BASES WITH RESPECT TO THE V-FILTRATION

The purpose of this section is to show that a set of F-involutory generators of a given submodule $\mathcal{N}$ of $(\mathcal{D}_{\tilde{X}})^r$ can be provided by a Gröbner basis in the Weyl algebra with respect to an appropriate term ordering, which can be computed by the Buchberger algorithm [7]. See e.g. [2, 9, 10] for Gröbner bases of polynomial rings. The fact that the Buchberger algorithm applies to the Weyl algebra (the ring of differential operators with polynomial coefficients) was observed by Galligo [12] (cf. also [8, 40]).

Let us denote by A_n and A_{n+1} the Weyl algebras on the n variables x and on the $n+1$ variables (t, x) respectively with coefficients in K (cf. [3]). Let r be a positive integer and put $L := \mathbf{N}^{2+2n} = \mathbf{N} \times \mathbf{N} \times \mathbf{N}^n \times \mathbf{N}^n$ with $\mathbf{N} =: \{0, 1, \ldots\}$. An element P of $(A_{n+1})^r$ is written in a finite sum

$$P = \sum_{i=1}^{r} \sum_{(\mu, \nu, \alpha, \beta) \in L} a_{\mu\nu\alpha\beta i} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i \tag{3.1}$$

with $a_{\mu\nu\alpha\beta i} \in K$, $e_1 := (1, 0, \ldots, 0), \ldots, e_r := (0, \ldots, 0, 1)$, $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^{\beta} := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$.

Let $\prec_F$ be a total order on $L \times \{1, \ldots, r\}$ which satisfies

(O1) $(\alpha, i) \prec_F (\beta, j)$ implies $(\alpha + \gamma, i) \prec_F (\beta + \gamma, j)$ for any $\alpha, \beta, \gamma \in L$ and $i, j \in \{1, \ldots, r\}$;

(O2) if $\nu - \mu < \nu' - \mu'$, then $(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j)$ for any $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, $\mu, \nu, \mu', \nu' \in \mathbf{N}$ and any $i, j \in \{1, \ldots, r\}$;

(O3) $(\mu, \mu, \alpha, \beta, i) \succcurlyeq_F (0, 0, 0, 0, i)$ for any $\mu \in \mathbf{N}, \alpha, \beta \in \mathbf{N}^n$, $i \in \{1, \ldots, r\}$.

Note that $\prec_F$ is not a well order (linear ordering). However, throughout the present paper, every order is supposed to satisfy (O1). Let P be a nonzero element of $(A_{n+1})^r$ which is written in the form (3.1). Then the *leading exponent* $\operatorname{lexp}_F(P) \in L \times \{1, \ldots, r\}$ of P with respect to $\prec_F$ is defined as the maximum element

$$\max\{(\mu, \nu, \alpha, \beta, i) \mid a_{\mu\nu\alpha\beta i} \neq 0\}$$

with respect to the order $\prec_F$. The set of leading exponents $E_F(N)$ of a subset N of $(A_{n+1})^r$ is defined by

$$E_F(N) := \{\operatorname{lexp}_F(P) \mid P \in N \setminus \{0\}\}.$$

DEFINITION 3.1. A finite set $\mathbf{G}$ of generators of a left A_{n+1}-submodule N of $(A_{n+1})^r$ is called an FW-*Gröbner basis* of N if we have

$$E_F(N) = \bigcup_{P \in \mathbf{G}} (\operatorname{lexp}_F(P) + L),$$

where we write

$$(\alpha, i) + L = \{(\alpha + \beta, i) \mid \beta \in L\}$$

for $\alpha \in L$ and $i \in \{1, \ldots, r\}$.

LEMMA 3.2. *For any integer k, the order $\prec_F$ restricted to the set $\{(\mu, \nu, \alpha, \beta, i) \in L \mid \nu - \mu \geq k\}$ is a well-order.*

Proof. The conditions (O1) and (O3) imply that $\prec_F$ is a well-order restricted to $\{(\mu, \nu, \alpha, \beta, i) \in L \mid \nu - \mu = k\}$. This implies the lemma combined with (O2). ∎

PROPOSITION 3.3. *Let $\mathbf{G}$ be an FW-Gröbner basis of a left A_{n+1}-submodule N of $(A_{n+1})^r$. Then $\mathbf{G}$ is a set of F-involutory generators of the left $\mathcal{D}_{\tilde{X}}$-submodule $\mathcal{N} := \mathcal{D}_{\tilde{X}} N$ of $(\mathcal{D}_{\tilde{X}})^r$ on X.*

Proof. Put $\mathbf{G} = \{P_1, \ldots, P_m\}$. Let P be a nonzero element of $\mathcal{N}_p$ with $p \in X$. Then by definition there exists $a(x) \in K[x]$ such that $a(p) \neq 0$ and $a(x)P \in N$. We have $\operatorname{lexp}(P) \in E_F(N)$ since $\mathbf{G}$ is an FW-Gröbner basis of N. Hence there exist $i \in \{1, \ldots, m\}$ and a monomial $Q \in A_{n+1}(K)$ such that $\operatorname{lexp}_F(P - QP_i) \prec_F \operatorname{lexp}_F(P)$. Let j be an arbitrary integer. Repeating this process a finite number of times, we can find, by virtue of the preceding lemma, $Q_1, \ldots, Q_m \in A_{n+1}(K)$ so that $\operatorname{lexp}_F(Q_i P_i) \preceq_F \operatorname{lexp}_F(aP)$ if $Q_i \neq 0$ and that

$$a(x)P - Q_1 P_1 - \cdots - Q_m P_m \in F_j(\mathcal{D}_{\tilde{X}})^r.$$

This completes the proof in view of Proposition 2.2. ∎

Since the order $\prec_F$ is not a well-order, the Buchberger algorithm for computing Gröbner bases does not work directly. We use the homogenization with respect to the V-filtration in order to bypass this difficulty (cf. [31, 32, 33, 1]). The following arguments generalize those in [33], where the case with $r = 1$ is treated. Since this generalization is straightforward, we omit the proof.

DEFINITION 3.4. For $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \mathbf{N}$, $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, and $i, j \in \{1, \ldots, r\}$, an order $\prec_H$ on $L_1 \times \{1, \ldots, r\}$ with $L_1 := \mathbf{N} \times L$ is defined so that we have $(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$ if and only if one of the following conditions holds:

(1) $\lambda < \lambda'$;

(2) $\lambda = \lambda'$, $(\mu + l, \nu, \alpha, \beta, i) \prec_F (\mu' + l', \nu', \alpha', \beta', j)$ with $l, l' \in \mathbf{N}$ such that $\nu - \mu - l = \nu' - \mu' - l'$;

(3) $(\lambda, \nu, \alpha, \beta, i) = (\lambda', \nu', \alpha', \beta', j)$, $\mu < \mu'$.

This definition is independent of the choice of l, l' in view of the condition (O1).

LEMMA 3.5. (1) $\prec_H$ *is a well-order.*

(2) *If* $\nu - \mu - \lambda = \nu' - \mu' - \lambda'$, *then we have* $(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$ *if and only if* $(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j)$.

For a nonzero element $P = P(x_0)$ of $(A_{n+1}[x_0])^r$, let us denote by $\mathrm{lexp}_H(P) \in L_1 \times \{1, \dots, r\}$ the leading exponent of P with respect to $\prec_H$.

DEFINITION 3.6. An element P of $(A_{n+1}[x_0])^r$ of the form

$$P = \sum_{i=1}^{r} \sum_{\lambda, \mu, \nu, \alpha, \beta} a_{\lambda\mu\nu\alpha\beta i} x_0^{\lambda} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i$$

is said to be *F-homogeneous* of order m if $a_{\lambda\mu\nu\alpha\beta i} = 0$ whenever $\nu - \mu - \lambda \neq m$.

DEFINITION 3.7. For an element P of $(A_{n+1})^r$ of the form (3.1), put $m := \min\{\nu - \mu \mid a_{\mu\nu\alpha\beta i} \neq 0$ for some $\mu, \nu \in \mathbf{N}$, $\alpha, \beta \in \mathbf{N}^n$, and $i \in \{1, \dots, r\}\}$. Then the *F-homogenization* $P^h \in (A_{n+1}[x_0])^r$ of P is defined by

$$P^h := \sum_{i=1}^{r} \sum_{\mu, \nu, \alpha, \beta} a_{\mu\nu\alpha\beta i} x_0^{\nu - \mu - m} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i$$

with a parameter x_0 which commutes with all the other variables and derivations. P^h is F-homogeneous of order m.

LEMMA 3.8. *If* $P \in A_{n+1}[x_0]$ *and* $Q \in (A_{n+1}[x_0])^r$ *are both F-homogeneous, then so is* PQ.

LEMMA 3.9. *We have* $(PQ)^h = P^h Q^h$ *for* $P \in A_{n+1}[x_0]$ *and* $Q \in (A_{n+1}[x_0])^r$.

LEMMA 3.10. *For* $P_1, \dots, P_k \in (A_{n+1})^r$, *put* $P = P_1 + \cdots + P_k$. *Then there exist* $l, l_1, \dots, l_k \in \mathbf{N}$ *so that*

$$x_0^{l} P^h = x_0^{l_1} (P_1)^h + \cdots + x_0^{l_k} (P_k)^h.$$

Let us define $\varpi \colon L_1 \times \{1, \dots, r\} \to L \times \{1, \dots, r\}$ by

$$\varpi(\lambda, \mu, \nu, \alpha, \beta, i) = (\mu, \nu, \alpha, \beta, i).$$

LEMMA 3.11. (1) *If* $P(x_0) \in (A_{n+1}[x_0])^r$ *is F-homogeneous, then we have* $\operatorname{lexp}_F(P(1)) = \varpi(\operatorname{lexp}_H(P(x_0)))$.

(2) *For any* $P \in (A_{n+1})^r$, *we have* $\operatorname{lexp}_F(P) = \varpi(\operatorname{lexp}_H(P^h))$.

PROPOSITION 3.12. *Let* $\tilde{N}$ *be a left* $A_{n+1}[x_0]$*-submodule of* $(A_{n+1}[x_0])^r$ *generated by F-homogeneous operators. Then there exists an H-Gröbner basis (i.e. a Gröbner basis with respect to* $\prec_H$*) of* $\tilde{N}$ *consisting of F-homogeneous operators. Moreover, such an H-Gröbner basis can be computed by the Buchberger algorithm.*

PROPOSITION 3.13. *Let N be a left* A_{n+1}*-submodule of* $(A_{n+1})^r$ *generated by* $P_1, \dots, P_d \in (A_{n+1})^r$. *Let us denote by* N^h *the left* $A_{n+1}[x_0]$*-submodule of* $(A_{n+1}[x_0])^r$ *generated by* $(P_1)^h, \dots, (P_d)^h$. *Let* $\mathbf{G} = \{Q_1(x_0), \dots, Q_k(x_0)\}$ *be an H-Gröbner basis of* N^h *consisting of F-homogeneous operators. Then* $\mathbf{G}(1) := \{Q_1(1), \dots, Q_k(1)\}$ *is an FW-Gröbner basis of N.*

These two propositions, combined with Proposition 3.3, provide us with an algorithm of computing a finite set of F-involutory generators of $\mathcal{N} = \mathcal{D}_{\tilde{X}} N$ on X.

4. THE b-FUNCTION OF A D-MODULE

We retain the notation in the preceding section. Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\tilde{X}}$-module on $\tilde{X}$. We assume that a left A_{n+1}-submodule N of $(A_{n+1})^r$ is given explicitly so that $\mathcal{M} = \mathcal{D}_{\tilde{X}} \otimes_{A_{n+1}} M$ holds with $M := (A_{n+1})^r/N$. Set $\mathcal{N} := \mathcal{D}_{\tilde{X}} \otimes_{A_{n+1}} N \subset (\mathcal{D}_{\tilde{X}})^r$. Let $F_k(\mathcal{N})$, $F_k(\mathcal{M})$ be the V-filtrations of $\mathcal{N}$ and $\mathcal{M}$ respectively defined in Section 2 and put

$$\operatorname{gr}_k(\mathcal{D}_{\tilde{X}}) := F_k(\mathcal{D}_{\tilde{X}})/F_{k-1}(\mathcal{D}_{\tilde{X}}),$$
$$\operatorname{gr}_k(\mathcal{N}) := F_k(\mathcal{N})/F_{k-1}(\mathcal{N}),$$
$$\operatorname{gr}_k(\mathcal{M}) := F_k(\mathcal{M})/F_{k-1}(\mathcal{M}).$$

In particular, $\operatorname{gr}_0(\mathcal{M})$ and $\operatorname{gr}_0(\mathcal{N})$ are left $\operatorname{gr}_0(\mathcal{D}_{\tilde{X}})$-modules and we can identify $\operatorname{gr}_0(\mathcal{D}_{\tilde{X}})$ with $\mathcal{D}_X[t\partial_t]$.

DEFINITION 4.1. The *b-function* $b(s,p) \in K[s]$ of $\mathcal{M}$ along X (with respect to the V-filtration $\{F_k(\mathcal{M})\}$) at $p \in X$ is the monic polynomial $b(s,p) \in K[s]$ of the least degree, if any, that satisfies

$$b(t\partial_t, p)\operatorname{gr}_0(\mathcal{M})_p = 0. \tag{4.1}$$

If such $b(s,p)$ exists, $\mathcal{M}$ is called *specializable* along X at p. If $\mathcal{M}$ is not specializable at p, we put $b(s,p) = 0$.

It is known that if $\mathcal{M}$ is holonomic, then $\mathcal{M}$ is specializable at any $p \in X$ ([21]). In the sequel, we describe an algorithm for computing $b(s,p) \in K[s]$ as a function of $p \in X$.

PROPOSITION 4.2. *Put* $\mathcal{J} := \psi(\mathcal{N}) \cap (\mathcal{O}_X[s])^r$, *which is an* $\mathcal{O}_X[s]$-*submodule of* $(\mathcal{O}_X[s])^r$. *Let* $\mathrm{Ann}((\mathcal{O}_X[s])^r/\mathcal{J}) \subset \mathcal{O}_X[s]$ *be the annihilator ideal for* $(\mathcal{O}_X[s])^r/\mathcal{J}$. *Then the ideal* $\mathrm{Ann}((\mathcal{O}_X[s])^r/\mathcal{J})_p \cap K[s]$ *of* $K[s]$ *is generated by* $b(s,p)$ *for each* $p \in X$.

Proof. By the identification $t\partial_t = s$, we have an isomorphism $\psi(\mathcal{N}) \simeq \mathrm{gr}_0(\mathcal{N})$ as left $\mathcal{D}_X[s]$-modules (cf. [33]). Hence we get an isomorphism

$$\mathrm{gr}_0(\mathcal{M}) \simeq \mathcal{D}_X[s]^r/\psi(\mathcal{N}).$$

We have an inclusion $\mathcal{O}_X[s]^r/\mathcal{J} \subset \mathcal{D}_X[s]^r/\psi(\mathcal{N})$ and $\mathcal{O}_X[s]^r/\mathcal{J}$ generates $\mathcal{D}_X[s]^r/\psi(\mathcal{N})$ over $\mathcal{D}_X[s]$. Since $K[s]$ is the center of $\mathcal{D}_X[s]$, this proves the assertion of the proposition. ∎

A set of generators of $\psi(\mathcal{N})$ on X can be computed by using Propositions 2.3, 3.12, 3.13. Hence our first task here is to compute a set of generators of $\mathcal{J}$. Let $\prec_D$ be a total order on $L_0 \times \{1,\dots,r\}$ with $L_0 := \mathbf{N}^{1+2n}$ which satisfies (O1) with L replaced by L_0 and

(O4) $(\alpha, i) \succ_D (0, i)$ for any $\alpha \in L_0 \setminus \{0\}$ and $i \in \{1,\dots,r\}$;

(O5) $|\beta| < |\beta'|$ implies $(\mu, \alpha, \beta, i) \prec_D (\mu', \alpha', \beta', j)$ for any $\mu, \mu' \in \mathbf{N}$, $\alpha, \alpha', \beta, \beta' \in \mathbf{N}^n$, $i, j \in \{1,\dots,r\}$.

Note that the order $\prec_D$ is a well-order.

PROPOSITION 4.3. *Let* $\mathbf{G}_1$ *be a finite subset of* $(A_n[s])^r$ *which generates* $\psi(\mathcal{N})$ *as a left* $\mathcal{D}_X[s]$-*module on* X. *Let* $\mathbf{G}_2$ *be a Gröbner basis with respect to* $\prec_D$ *of the submodule of* $(A_n[s])^r$ *generated by* $\mathbf{G}_1$. *Put* $\mathbf{G}_3 := \mathbf{G}_2 \cap K[s,x]^r$. *Then* $\mathcal{J}$ *is generated by* $\mathbf{G}_3$ *on* X *as an* $\mathcal{O}_X[s]$-*module.*

Proof. This proposition follows immediately from the fact that $\prec_D$ is an order for eliminating ∂. This order can be also used for the computation of the characteristic variety of a D-module (cf. [30]). ∎

The final step will be devoted to the computation of $b(s,p)$ with a set of generators of $\mathcal{J}$ as an input. For $i = 1,\dots,r$, put

$$\mathcal{J}^{(i)} := \left\{ f = (f_1,\dots,f_r) \in \mathcal{J} \mid f_j = 0 \text{ if } j > i \right\}.$$

Then $\mathcal{J}^{(i)}/\mathcal{J}^{(i-1)}$ can be regarded as an ideal of $\mathcal{O}_X[s]$ whose generators can be computed via a Gröbner basis with respect to an order $\prec$ on $\mathbf{N}^{1+n} \times \{1,\dots,r\}$ satisfying $(\alpha, i) \prec (\beta, j)$ for any $\alpha, \beta \in \mathbf{N}^{1+n}$ if $i < j$.

So far we have used only the Buchberger algorithm, which does not require field extension, for computing Gröbner bases with respect to various orders. Hence we do not need to assume that K is algebraically closed from the viewpoint of algorithms. Thus, in the rest of this section, we assume that K is an arbitrary field of characteristic zero so that the inputs are defined over K. Since we will make use of primary decomposition, which is sensitive to field extension, we will have to pay attention to the coefficient fields.

Let $\overline{K}$ be the algebraic closure of K and suppose that X is a Zariski open subset of $\overline{K}^n$. We denote by $\mathcal{O}_X$ the sheaf of regular functions on X. In particular, $\mathcal{O}_X$ is a sheaf of $\overline{K}$-algebras. In general, for an ideal Q of $K[s,x]$ and $p \in \overline{K}^n$, let us denote by $b(s,Q,p) \in K[s]$ a generator of the ideal $K[s] \cap \mathcal{O}_X[s]_p Q$. We may assume that $b(s,Q,p)$ is monic if it is not zero. Put

$$\mathbf{V}_X(Q) := \{x \in X \mid f(x) = 0 \text{ for any } f \in Q \cap K[x]\}.$$

Note that $\mathbf{V}_X(Q)$ can be computed by eliminating s by means of a Gröbner basis of Q.

LEMMA 4.4. *In the above notation, the ideal $\mathcal{O}_X[s]_p Q \cap \overline{K}[s]$ of $\overline{K}[s]$ is also generated by $b(s,Q,p)$.*

Proof. Let $b(s,Q,p)$ be of degree d. Then it suffices to show that $\deg f \geq d$ for any nonzero element f of $\mathcal{O}_X[s]_p Q \cap \overline{K}[s]$. Then there exist $f_1,\dots,f_m \in Q$ and $a_1,\dots,a_m, q \in \overline{K}[x]$ so that

$$q(x)f(s) = \sum_{j=1}^{m} a_j(x) f_j(s,x)$$

and $q(p) \neq 0$. Let $\pi\colon \overline{K} \to K$ be a projection, i.e., a K-linear map whose restriction to K is an identity. We may assume, by multiplying elements of $\overline{K}$ to q and f, that $\pi(q(p)) \neq 0$ and that f is monic. Since we have

$$\pi(q)\pi(f) = \sum_{j=1}^{m} \pi(a_j) f_j(s,x),$$

which implies that $\pi(f) \in \mathcal{O}_X[s]_p Q \cap K[s]$, we know that $b(s,Q,p)$ divides $\pi(f)$. Since the degree of $\pi(f)$ is equal to that of f, we are done. ∎

PROPOSITION 4.5. *Assume that Q is a primary ideal of $K[s,x]$ and let $h(s,Q)$ be a generator of the ideal $Q \cap K[s]$ of $K[s]$.*

(1) *Case $h(s,Q) \neq 0$: In this case there exists an irreducible polynomial $h_0(s,Q) \in K[s]$ and $\nu_0 \in \mathbf{N}$ so that $h(s,Q) = h_0(s,Q)^{\nu_0}$. Put*

$$\mathbf{V}_X^{\nu}(Q) := \left\{x \in X \mid f(x) = 0 \textit{ for any } f \in K[x] \cap \left(Q : h_0(s,Q)^{\nu}\right)\right\}$$

for each $\nu \in \mathbf{N}$, *where* : *denotes the ideal quotient in* $K[s,x]$. *Then we have a decreasing sequence of algebraic sets*

$$X \supset \mathbf{V}_X(Q) = \mathbf{V}_X^0(Q) \supset \mathbf{V}_X^1(Q) \supset \cdots \supset \mathbf{V}_X^{\nu_0}(Q) = \varnothing$$

of X. *If* $p \in \mathbf{V}_X^{\nu-1}(Q) \setminus \mathbf{V}_X^{\nu}(Q)$, *then we have* $b(s,Q,p) = h_0(s,Q)^\nu$ *for* $\nu = 0, \dots, \nu_0$, *where we put* $\mathbf{V}_X^{-1}(Q) := X$.

(2) *Case* $h(s,Q) = 0$: *In this case we have* $b(s,Q,p) = 0$ *if* $p \in \mathbf{V}_X(Q)$ *and* $b(s,Q,p) = 1$ *otherwise*.

Proof. First assume $h(s,Q) \neq 0$. The existence of $h_0(s,Q)$ and ν_0 as above follows from the fact that $Q \cap K[s]$ is a primary ideal of $K[s]$. In order to prove the assertion of (1), it suffices to show that $b(s,Q,p) = h_0(s,Q)^\nu$ with some $\nu \in \mathbf{N}$, which may depend on $p \in X$. This follows from the fact that $b(s,Q,p)$ divides $h(s,Q)$ in $K[s]$ by definition.

Next assume $h(s,Q) = 0$. Suppose $p \in \mathbf{V}_X(Q)$ and $b(s,Q,p) \neq 0$. Then there exists $a(x) \in K[x]$ such that $a(p) \neq 0$ and $a(x)b(s,Q,p) \in Q$. It follows that there exists $\mu \in \mathbf{N}$ so that $b(s,Q,p)^\mu \in Q$ since $a(x) \notin Q$ in view of the condition $p \in \mathbf{V}_X(Q)$. This contradicts the assumption $Q \cap K[s] = 0$. If $p \notin \mathbf{V}_X(Q)$, there exists $a(x) \in Q \cap K[x]$ such that $a(p) \neq 0$. This implies $b(s,Q,p) = 1$. ∎

Note that $h(s,Q)$ and the ideal quotient $Q \colon h_0(s,Q)^\nu$ can be computed also by Gröbner bases ([9, 2, 10]).

PROPOSITION 4.6. *Under the above assumptions and notation*, *let* J_i *be an ideal of* $K[s,x]$ *such that* $\mathcal{O}_X[s]J_i = \mathcal{J}^{(i)}/\mathcal{J}^{(i-1)}$ *for* $i = 1, \dots, r$. *Let*

$$J_i = Q_{i,1} \cap \cdots \cap Q_{i,m_i}$$

be a primary decomposition of J_i *in* $K[s,x]$. *Then the b-function* $b(s,p)$ *of* $\mathcal{M}$ *at* $p \in X$ *is the least common multiple of* $b(s,Q_{i,j},p)$*'s where* (i,j) *runs over the set* $\{(i,j) \mid 1 \leq i \leq r,\ 1 \leq j \leq m_i\}$.

Proof. It is easy to see that

$$\begin{aligned}\operatorname{Ann}\big(\mathcal{O}_X[s]^r/\mathcal{J}\big) \cap K[s] &= \bigcap_{i=1}^{r} \operatorname{Ann}\big(\mathcal{O}_X[s]/(\mathcal{J}^{(i)}/\mathcal{J}^{(i-1)})\big) \cap K[s] \\ &= \bigcap_{i=1}^{r} \left(\mathcal{O}_X[s]J_i \cap K[s]\right).\end{aligned}$$

Hence the assertion of the proposition follows from

$$\mathcal{O}_X[s]J_i = \mathcal{O}_X[s]Q_{i,1} \cap \cdots \cap \mathcal{O}_X[s]Q_{i,m_i}.$$

This completes the proof. ∎

Thus by combining Propositions 4.2, 4.3, 4.5 and 4.6, we have obtained an algorithm to compute the b-function $b(s,p)$ of $\mathcal{M}$ as a function of $p \in X$. In particular, note that $b(s,p)$ belongs to $K[s]$ for any $p \in X$. Let us assume that X is defined over K, i.e., there exists an ideal I_X of $K[x]$ so that $\overline{K}^n \setminus X$ is the set of the zeros of I_X in $\overline{K}^n$. Then the following theorem provides us with an algorithm to determine whether $\mathcal{M}$ is specializable along X at *every* point of $p \in X$, and to compute the set $\{s \in \overline{K} \mid b(s,p) = 0$ for some $p \in X\}$. This will be needed in order to compute the restriction and the algebraic local cohomology groups globally on X in the subsequent sections (cf. Proposition 5.2 below). Let us denote by rad Q' the radical of an ideal $Q' \subset K[x]$.

THEOREM 4.7. *Let J_i and Q_{ij} be as in the preceding proposition.*

(1) *$\mathcal{M}$ is specializable along X at each point of X if and only if the condition*

$$Q_{ij} \cap K[s] \neq \{0\} \quad or \quad \operatorname{rad}\bigl(Q_{ij} \cap K[x]\bigr) \supset I_X \tag{4.2}$$

holds for each $i = 1,\dots,r$ and $j = 1,\dots,m_i$.

(2) *Assume that* (4.2) *holds for each i and j. Let $b_{ij}(s)$ be a generator of $Q_{ij} \cap K[s]$ if $\operatorname{rad}(Q_{ij} \cap K[x]) \not\supset I_X$, and put $b_{ij}(s) := 1$ if $\operatorname{rad}(Q_{ij} \cap K[x]) \supset I_X$. Let $b(s)$ be the least common multiple of $b_{ij}(s)$'s with $1 \le i \le r$ and $1 \le j \le m_i$. Then the b-function $b(s,p)$ of $\mathcal{M}$ divides $b(s)$ for any $p \in X$. Moreover, for any irreducible factor $g(s)$ of $b(s)$, there exists some $p \in X$ so that $g(s)$ divides $b(s,p)$.*

(3) *Assume $X = \overline{K}^n$. Then $\mathcal{M}$ is specializable along X at each point of X if and only if $J_i \cap K[s] \neq 0$ for any $i = 1,\dots,r$. In this case let $b_i(s)$ be a generator of $J_i \cap K[s]$ and let $b(s)$ be the least common multiple of $b_1(s),\dots,b_r(s)$. Then $b(s)$ is the least common multiple of $b(s,p)$'s where p runs over X.*

Proof. (1) and the first assertion of (2) follow immediately from Propositions 4.5 and 4.6 since the condition $\operatorname{rad}(Q_{ij} \cap K[s]) \supset I_X$ is equivalent to $\mathbf{V}_X(Q_{ij}) = \emptyset$. To verify the latter assertion of (2), assume $\operatorname{rad}(Q_{ij} \cap K[x]) \not\supset I_X$. Then $h_0(s,Q_{ij})$ divides $b(s,Q_{ij},p)$ if and only if p belongs to $\mathbf{V}_X(Q_{ij})$, which is not empty. On the other hand, if $\operatorname{rad}(Q_{ij} \cap K[x]) \supset I_X$, then we have $b_{ij}(s) = 1$ and $b(s,Q_{ij},p) = 1$ for any $p \in X$.

(3) Assume that $\mathcal{M}$ is specializable at any $p \in X = \overline{K}^n$. Suppose $J_i \cap K[s] = \{0\}$ for some i. Then we have $Q_{ij} \cap K[s] = \{0\}$ for some j. Then $\mathbf{V}_X(Q_{ij})$ is not empty. In fact, if $\mathbf{V}_X(Q_{ij})$ would be an empty set, then we should have $1 \in Q_{ij} \cap K[x]$, and hence $1 \in Q_{ij}$, which contradicts the assumption. This means that $\mathcal{M}$ is not specializable on $\mathbf{V}_X(Q_{ij})$. Hence we must have $J_i \cap K[s] \neq \{0\}$ for $i = 1,\dots,r$. This proves the first statement

of (3). Note that $\mathcal{O}_X[s]J_i$ is a sheaf of ideals of $\mathcal{O}_X[s]$ and we have $\Gamma(X, \mathcal{O}_X[s]J_i) = \bar{K}[s,x]J_i$ for $X = \bar{K}^n$. Hence for $f(s) \in K[s]$ in general, we have $f(s) \in K[s,x]J_i$ if and only if $f(s) \in \mathcal{O}_X[s]_p J_i$, or equivalently $b(s, J_i, p)$ divides $f(s)$, for any $p \in X$. This proves the latter part of (3). ∎

Algorithms for primary decomposition are known at least if the coefficient field is algebraic and finite over $\mathbf{Q}$. See, e.g. [2, 11, 39] for recent developments. Note that we do not need primary decomposition in order to compute $b(s,p)$ for a fixed p (cf. [33]). There is also a simple algorithm for determining whether the condition $\operatorname{rad}(Q_{ij} \cap K[x]) \supset I_X$ holds (cf. [9]).

5. THE RESTRICTION OF A D-MODULE

We retain the notation of the preceding section. In particular, let $b(s,p)$ be the b-function of $\mathcal{M}$ at $p \in X$. The (D-module theoretic) *restriction* of $\mathcal{M}$ to X is the complex

$$\mathcal{M}_X^\bullet : 0 \longrightarrow \mathcal{M} \stackrel{t}{\longrightarrow} \mathcal{M} \longrightarrow 0$$

of left $\mathcal{D}_X$-modules, where the homomorphism t denotes the one defined by $t(u) = tu$ for each $u \in \mathcal{M}$. We regard the right $\mathcal{M}$ to be placed at the degree 0 in considering the cohomology groups of $\mathcal{M}_X^\bullet$. Put $\mathcal{D}_{X \to \tilde{X}} := \mathcal{D}_{\tilde{X}}/t\,\mathcal{D}_{\tilde{X}}$. Then $\mathcal{D}_{X \to \tilde{X}}$ is a $(\mathcal{D}_X, \mathcal{D}_{\tilde{X}})$-bimodule, and $\mathcal{M}_X^\bullet$ is isomorphic to $\mathcal{D}_{X \to \tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathcal{M}$ in the derived category, where $\overset{\mathbf{L}}{\otimes}$ denotes the left derived functor of $\otimes$ (cf. [15]). Let us denote by $\mathcal{M}_X := \mathcal{H}^0(\mathcal{M}_X^\bullet) = \mathcal{M}/t\,\mathcal{M}$ the 0-th cohomology group of the complex $\mathcal{M}_X^\bullet$.

LEMMA 5.1. *The homomorphism* $t\colon \operatorname{gr}_{k+1}(\mathcal{M})_p \to \operatorname{gr}_k(\mathcal{M})_p$ *is bijective if* $b(k,p) \neq 0$ *for* $p \in X$.

Proof. We write $b(s) = b(s,p)$ for simplicity. First, let us prove that t is injective. Let u be a section of $F_{k+1}(\mathcal{M})$ and denote by $\bar{u}$ its residue class in $\operatorname{gr}_{k+1}(\mathcal{M})$. Assume $t\bar{u} = 0$ in $\operatorname{gr}_k(\mathcal{M})$. Note that $b(t\partial_t)\operatorname{gr}_0(\mathcal{M}) = 0$ implies $b(t\partial_t + k)\operatorname{gr}_k(\mathcal{M}) = 0$ for any $k \in \mathbf{Z}$. Hence we have

$$0 = b(t\partial_t + k + 1)\bar{u} = b(\partial_t t + k)\bar{u} = b(k)\bar{u}.$$

Since $b(k) \neq 0$, we get $\bar{u} = 0$.

Next, let us prove that t is surjective. Let $\bar{u}$ be an arbitrary element of $\operatorname{gr}_k(\mathcal{M})$. Then we have $b(t\partial_t + k)\bar{u} = 0$. We can take $c(t,\partial_t) \in K[t]\langle \partial_t \rangle$ so that $b(t\partial_t + k) = tc(t,\partial_t) + b(k)$. Hence we get

$$\bar{u} = -b(k)^{-1} tc(t,\partial_t)\bar{u},$$

which implies that t is surjective. ∎

PROPOSITION 5.2. *Assume that* $\mathcal{M}$ *is specializable along* X *at each point of* X. *Let* $k_0 \leq k_1$ *be integers such that the b-function* $b(s,p)$ *of* $\mathcal{M}$ *satisfies* $b(k,p) \neq 0$ *for any* $p \in X$ *and for any integer* k *such that* $k < k_0$ *or* $k > k_1$. *Then* $\mathcal{M}_X^\bullet$ *is quasi-isomorphic to the complex*

$$0 \longrightarrow F_{k_1+1}(\mathcal{M})/F_{k_0}(\mathcal{M}) \xrightarrow{t} F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) \longrightarrow 0$$

of left $\mathcal{D}_X$*-modules on* X. *In particular*, t: $\mathcal{M} \to \mathcal{M}$ *is bijective if* $b(k,p) \neq 0$ *for any* $p \in X$ *and* $k \in \mathbf{Z}$.

Proof. First, let us show that two homomorphisms

$$F_{k_0}(\mathcal{M}) \xrightarrow{t} F_{k_0-1}(\mathcal{M}), \tag{5.1}$$

$$\mathcal{M}/F_{k_1+1}(\mathcal{M}) \xrightarrow{t} \mathcal{M}/F_{k_1}(\mathcal{M}) \tag{5.2}$$

are bijective. To prove the injectivity of (5.1), suppose an element $u \in F_k(\mathcal{M}) \setminus F_{k-1}(\mathcal{M})$ with some $k \leq k_0$ satisfies $tu = 0$. Then we have $\bar{u} = 0$ in $\mathrm{gr}_k(\mathcal{M})$ since $t: \mathrm{gr}_k(\mathcal{M}) \to \mathrm{gr}_{k-1}(\mathcal{M})$ is bijective. This contradicts the assumption.

Next let us show that (5.1) is surjective. Let $u \in F_k(\mathcal{M})$ with $k \leq k_0 - 1$. Then there exists $v_0 \in F_{k+1}(\mathcal{M})$ so that $u - tv_0 \in F_{k-1}(\mathcal{M})$. We can take $v_1 \in F_k(\mathcal{M})$ so that $u - tv_0 - tv_1 \in F_{k-2}(\mathcal{M})$. Hence by induction, we can take $v_0, v_1, \dots, v_j \in F_{k+1}$ so that

$$u - t(v_0 + v_1 + \cdots + v_j) \in F_{-1}(\mathcal{M}) = tF_0(\mathcal{M}).$$

It follows that (5.1) is surjective.

Let us show the injectivity of (5.2). Let $u \in \mathcal{M}$ satisfy $tu \in F_{k_1}(\mathcal{M})$. We may assume $u \in F_k(\mathcal{M}) \setminus F_{k-1}(\mathcal{M})$ with some $k \geq k_1 + 2$. Then we have $t\bar{u} = 0$ in $\mathrm{gr}_{k-1}(\mathcal{M})$, which implies $\bar{u} = 0$ in $\mathrm{gr}_k(\mathcal{M})$. This contradicts the assumption. The surjectivity of (5.2) can be proved in the same way as for (5.1).

Now let us turn to the proof of the proposition. First note that the bijectivity of (5.2) implies that the vertical chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_{k_1+1}(\mathcal{M}) & \xrightarrow{t} & F_{k_1}(\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M} & \xrightarrow{t} & \mathcal{M} & \longrightarrow & 0 \end{array} \tag{5.3}$$

is a quasi-isomorphism (i.e. induces isomorphisms between the corresponding cohomology groups). In the same way, the bijectivity of (5.1) implies that the chain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{k_1+1}(\mathcal{M}) & \stackrel{t}{\longrightarrow} & F_{k_1}(\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{k_1+1}(\mathcal{M})/F_{k_0}(\mathcal{M}) & \stackrel{t}{\longrightarrow} & F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) & \longrightarrow & 0 \end{array} \qquad (5.4)$$

is also a quasi-isomorphism. Combining (5.3) and (5.4), we get the result. ∎

REMARK 5.3. The optimal k_0, k_1 in Proposition 5.2 can be determined by $b(s)$ defined in Theorem 4.7.

The following proposition provides a sufficient condition for the -1th cohomology group $\mathcal{H}^{-1}(\mathcal{M}_X^\bullet)$ to vanish.

PROPOSITION 5.4. *Assume that there exists* $b_0(s) \in K[s]$ *and* $m \in \mathbf{N}$ *so that*

$$b_0(t\partial_t)\,\partial_t^m \operatorname{gr}_0(\mathcal{M})_p = 0.$$

Assume, moreover, $b_0(k) \neq 0$ *for any* $k \in \mathbf{Z}$. *Then the homomorphism* $t\colon \mathcal{M}_p \to \mathcal{M}_p$ *is injective.*

Proof. Since

$$t^m b_0(t\partial_t)\,\partial_t^m = b_0(t\partial_t - m)\,t\partial_t(t\partial_t - 1)\cdots(t\partial_t - m + 1), \qquad (5.5)$$

we have only to show that $t\colon \operatorname{gr}_{k+1}(\mathcal{M}) \to \operatorname{gr}_k(\mathcal{M})$ is injective for $0 \leq k \leq m-1$ taking into account the proof of Proposition 5.2. Assume that an element $\bar{v}$ of $\operatorname{gr}_{k+1}(\mathcal{M})$ satisfies $t\bar{v} = 0$. There exists $\bar{u} \in \operatorname{gr}_0(\mathcal{M})$ such that $\bar{v} = \partial_t^{k+1}\bar{u}$. Then we have

$$\begin{aligned} 0 = b_0(t\partial_t)\,\partial_t^m \bar{u} &= \partial_t^{m-k-1} b_0(t\partial_t - m + k + 1)\,\partial_t^{k+1}\bar{u} \\ &= \partial_t^{m-k-1} b_0(\partial_t t - m + k)\,\partial_t^{k+1}\bar{u} = \partial_t^{m-k-1} b_0(-m+k)\,\bar{v} \end{aligned}$$

in view of $t\partial_t^{k+1}\bar{u} = 0$. Hence we have

$$\partial_t^{m-k-1}\bar{v} = t\bar{v} = 0 \qquad (5.6)$$

since $b_0(-m+k) \neq 0$. From (5.6) we get

$$\left[t, \partial_t^{m-k-1}\right]\bar{v} = -(m-k-1)\,\partial_t^{m-k-2}\bar{v} = 0.$$

Proceeding in the same way, we obtain $\bar{v} = 0$. ∎

An algorithm to determine if there exists, and to find if any, such $b_0(s)$ as in the preceding proposition is given as follows: Let $b(s, p)$ be the

b-function of $\mathcal{M}$ at p. In view of (5.5), we may assume

$$s(s-1)\cdots(s-m+1)b_0(s-m) = b(s,p) \tag{5.7}$$

by choosing a minimal $b_0(s)$ satisfying the assumption. There exists $b_0(s) \in K[s]$ which satisfies (5.7) and $b_0(k) \neq 0$ for any $k \in \mathbf{Z}$ if and only if $b(s,p)$ has $0,1,\ldots,m-1$ as simple roots and has no other integral roots. If such is the case, we can determine if $b_0(s)$ satisfies the condition of Proposition 5.4 by using the following two lemmas.

LEMMA 5.5. *Let N be a left A_{n+1}-submodule of $(A_{n+1})^r$ whose generators are given explicitly. Suppose also that $Q \in (A_{n+1})^r$ is given. Then there is an algorithm to obtain a finite set of generators of the left ideal $N\colon Q = \{P \in A_{n+1} \mid PQ \in N\}$ of A_{n+1}.*

Proof. Let $\{Q_1,\ldots,Q_m\}$ be a set of generators of N and put

$$S(Q,Q_1,\ldots,Q_m) := \left\{(U,U_1,\ldots,U_m) \in (A_{n+1})^{m+1} \mid UQ + U_1Q_1 + \cdots + U_mQ_m = 0\right\}.$$

Then by computing a Gröbner basis of the left A_{n+1}-module generated by $Q,Q_1,\ldots,Q_m$, we get a set of generators $\{U_1,\ldots,U_d\}$ of $S(Q,Q_1,\ldots,Q_m)$ (cf. [10, 40]). Let $\pi\colon (A_{n+1})^{m+1} \to A_{n+1}$ be the projection to the first component. Then it is easy to see that $N\colon Q$ is generated by $\pi(U_1),\ldots,\pi(U_d)$. ∎

LEMMA 5.6. *Let N be the left A_{n+1}-submodule of $(A_{n+1})^r$ as above and let $\hat{\sigma}(N)$ be the left $A_n[t,\partial_t]$-submodule generated by $\{\hat{\sigma}(P) \mid P \in N\}$. Then we have $b_0(t\partial_t)\partial_t^m\,\mathrm{gr}_0(\mathcal{M})_p = 0$ if and only if the ideal*

$$\bigcap_{i=1}^{r} \left(\hat{\sigma}(N)\colon b(t\partial_t)\,\partial_t^m e_i\right)$$

of A_{n+1} contains some $a(x) \in K[x]$ such that $a(p) \neq 0$.

Now we shall give an algorithm to compute $\mathcal{M}_X$. Let P be an element of $F_m(\mathcal{D}_{\tilde{X}})^r$. Then we can write P in the form

$$P = \sum_{i=1}^{r}\sum_{k=0}^{m} P_{ik}(t\partial_t,x,\partial)\,\partial_t^k e_i + R$$

uniquely with $P_{ik} \in \mathcal{D}_X[t\partial_t]$ and $R \in F_{-1}(\mathcal{D}_{\tilde{X}})^r$. Then we put

$$\rho(P,k_0) := \sum_{i=1}^{r}\sum_{k=k_0}^{m} P_{ik}(0,x,\partial)\,\partial_t^k e_i$$

for each integer k_0 with $0 \le k_0 \le m$.

THEOREM 5.7. *Assume that* $\mathcal{M}$ *is specializable along* X *and let* k_0, k_1 *be as in Proposition* 5.2. *Redefine* k_0 *to be* 0 *if* $k_0 < 0$. (*We have* $k_0 = 0$ *and* $k_1 = m - 1$ *under the assumption of Proposition* 5.4.) *Let* $\mathbf{G}$ *be a finite set of F-involutory generators of* $\mathcal{N}$ *on* X. *Then we have an isomorphism*

$$\mathcal{M}_X \simeq \left(\bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} \mathcal{D}_X \, \partial_t^k e_i \right) \Big/ \mathcal{N}_X$$

of left $\mathcal{D}_X$*-modules, where* $\mathcal{N}_X$ *is the left* $\mathcal{D}_X$*-module generated by a finite set*

$$\mathbf{G}_X := \left\{ \rho\left(\partial_t^j P, k_0\right) \mid P \in \mathbf{G}, j \in \mathbf{N}, k_0 \le j + \operatorname{ord}_F(P) \le k_1 \right\}.$$

In particular, we have $\mathcal{M}_X = 0$ *if* $b(\nu, p) \neq 0$ *for any* $\nu \in \mathbf{N}$ *and* $p \in X$.

Proof. Put $\mathbf{G} = \{P_1, \ldots, P_d\}$. By Proposition 5.2, we have an isomorphism

$$\mathcal{M}_X \simeq F_{k_1}(\mathcal{M}) / \left(tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M})\right).$$

Put

$$\mathcal{D}^{(k_0,k_1)} := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} \mathcal{D}_X \, \partial_t^k e_i .$$

Define a $\mathcal{D}_X$-homomorphism φ: $\mathcal{D}^{(k_0,k_1)} \to F_{k_1}(\mathcal{M})$ by

$$\varphi\left(\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_k(x, \partial) \partial_t^k e_i \right) = \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_k(x, \partial) \partial_t^k u_i$$

for $P_k(x, \partial) \in \mathcal{D}_X$. We shall prove

$$\varphi^{-1}\left(tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M})\right) = \mathcal{N}_X .$$

Assume $P = \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_k(x, \partial)\partial_t^k e_i$ belongs to $\varphi^{-1}(tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M}))$. Then there exist $B \in F_{k_1+1}(\mathcal{D}_{\tilde{X}})^r$, $R \in F_{k_0-1}(\mathcal{D}_{\tilde{X}})^r$, and $Q_1, \ldots, Q_d \in (\mathcal{D}_{\tilde{X}})^r$ so that

$$P - tB - R = \sum_{j=1}^{d} Q_j P_j \tag{5.8}$$

and $\operatorname{ord}_F(Q_j P_j) \le k_1$ in view of Proposition 2.2. Put $m_j := \operatorname{ord}_F(P_j)$. We may assume that Q_j are written in the form

$$Q_j = \sum_{k=0}^{k_1 - m_j} Q_{jk}(t\partial_t, x, \partial)\partial_t^k + R_j$$

with $R_j \in F_{-1}(\mathcal{D}_{\tilde{X}})^r$ and $Q_{jk} \in \mathcal{D}_X[t\partial_t]$. Then from (5.8) we get

$$\begin{aligned} P &= \rho(P, k_0) \\ &= \rho\left(\sum_{j=1}^{d} Q_j P_j, k_0\right) \\ &= \rho\left(\sum_{j=1}^{d} \sum_{k=0}^{k_1 - m_j} Q_{jk}(t\partial_t, x, \partial)\partial_t^k P_j, k_0\right) \\ &= \sum_{j=1}^{d} \sum_{k=0}^{k_1 - m_j} Q_{jk}(0, x, \partial)\rho\left(\partial_t^k P_j, k_0\right). \end{aligned}$$

Here note that $\rho(\partial_t^k P_j, k_0) = 0$ if $k + m_j < k_0$. Hence we have proved $\varphi^{-1}(tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M})) \subset \mathcal{N}_X$. The converse inclusion follows from $\varphi(\mathbf{G}_X) \subset tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M})$. Since

$$\varphi(\mathcal{D}^{(k_0, k_1)}) + tF_{k_1+1}(\mathcal{M}) + F_{k_0-1}(\mathcal{M}) = F_{k_1}(\mathcal{M}),$$

we are done. ▮

In order to interpret the preceding theorem more concretely, let $u_1, \ldots, u_r$ be the modulo classes of $e_1, \ldots, e_r$ in $\mathcal{M}$. Then as is seen by the proof of the preceding theorem, $\mathcal{M}_X \simeq \mathcal{D}_{X \to \tilde{X}} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{M}$ is generated by $1 \otimes (\partial_t^k u_i)$ with $k_0 \le k \le k_1$ and $1 \le i \le r$ as left $\mathcal{D}_X$-module. Moreover, for $P_{ik} \in \mathcal{D}_X$, we have

$$\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{ik}\left(1 \otimes \partial_t^k u_i\right) = 0$$

if and only if $\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{ik} e_i \in \mathcal{N}_X$.

Our next aim is to give an algorithm for computing the structure of the kernel $\mathcal{H}^{-1}(\mathcal{M}_X^\bullet)$ of $t\colon \mathcal{M} \to \mathcal{M}$ as a left $\mathcal{D}_X$-module. Note that $\mathcal{H}^{-1}(\mathcal{M}_X^\bullet)$ has a structure of left $\mathcal{D}_X[t\partial_t]$-module which is compatible with that of left $\mathcal{D}_X$-module. For two integers $k_0 \le k_1$, put

$$\tilde{\mathcal{D}}^{(k_0, k_1)} := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} \mathcal{D}_X[t\partial_t]^r S_k e_i,$$

where $S_k := \partial_t^k$ if $k \ge 0$, and $S_k := t^{-k}$ if $k < 0$. Let P be a section of $F_m(\mathcal{D}_{\tilde{X}})^r$. Then we can write P uniquely in the form

$$P = \sum_{i=1}^{r} \sum_{k=-\infty}^{m} P_{ik}(t\partial_t, x, \partial) S_k e_i \tag{5.9}$$

with $P_{ik} \in \mathcal{D}_X[t\partial_t]$. Then we define

$$\tau(P, k_0) := \sum_{i=1}^{r} \sum_{k=k_0}^{m} P_{ik}(t\partial_t, x, \partial) S_k e_i.$$

PROPOSITION 5.8. *Let* $\mathbf{G}$ *be a finite set of F-involutory generators of* $\mathcal{N}$ *on* X. *Then, for any integers* $k_0 \le k_1$, *we have an isomorphism*

$$F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) \simeq \tilde{\mathcal{D}}^{(k_0,k_1)}/\mathcal{G}^{(k_0,k_1)}$$

of left $\mathcal{D}_X[t\partial_t]$*-modules, where* $\mathcal{G}^{(k_0,k_1)}$ *is a left* $\mathcal{D}_X[t\partial_t]$*-module generated by a finite set*

$$\mathbf{G}^{(k_0,k_1)} := \left\{\tau\left(S_j P, k_0\right) \mid P \in \mathbf{G}, j \in \mathbf{Z}, k_0 \le j + \operatorname{ord}_F(P) \le k_1\right\}.$$

Proof. Let us define a left $\mathcal{D}_X$-homomorphism $\tilde{\varphi}\colon \tilde{\mathcal{D}}^{(k_0,k_1)} \to F_{k_1}(\mathcal{M})$ by

$$\tilde{\varphi}(P) := \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_k(t\partial_t, x, \partial) S_k u_i$$

for $P \in \tilde{\mathcal{D}}^{(k_0,k_1)}$ of the form (5.9) with $m = k_1$, where u_i denotes the residue class of e_i in $\mathcal{M}$. Then we have only to prove that

$$\tilde{\varphi}^{-1}\left(F_{k_0-1}(\mathcal{M})\right) = \mathcal{G}^{(k_0,k_1)}.$$

It is easy to see that $\tilde{\varphi}(\mathcal{G}^{(k_0,k_1)}) \subset F_{k_0-1}(\mathcal{M})$. Suppose that $P \in \tilde{\mathcal{D}}^{(k_0,k_1)}$ of the form (5.9) with $m = k_1$ satisfies $\tilde{\varphi}(P) \in F_{k_0-1}(\mathcal{M})$. Put $\mathbf{G} = \{P_1, \ldots, P_d\}$. Then there exist $Q_1, \ldots, Q_d \in \mathcal{D}_{\tilde{X}}$ and $R \in F_{k_0-1}(\mathcal{D}_{\tilde{X}})^r$ so that

$$P = \sum_{j=1}^{d} Q_j P_j + R \tag{5.10}$$

and that $\operatorname{ord}_F(Q_j P_j) \le \operatorname{ord}_F(P) \le k_1$. Put $m_j := \operatorname{ord}_F(P_j)$. Then Q_j can be written in the form

$$Q_j = \sum_{k=k_0-m_j}^{k_1-m_j} Q_{jk}(t\partial_t, x, \partial) S_k + R_j \tag{5.11}$$

with $R_j \in F_{k_0-m_j-1}(\mathcal{D}_{\tilde{X}})^r$. From (5.10) and (5.11) we get

$$P = \tau(P, k_0) = \sum_{j=1}^{d} \sum_{k=k_0-m_j}^{k_1-m_j} Q_{jk}(t\partial_t, x, \partial)\tau\left(S_k P_j, k_0\right) \in \mathcal{G}^{(k_0,k_1)}.$$

This completes the proof. ∎

Let $\chi\colon \tilde{D}^{(k_0+1,k_1+1)} \to \tilde{D}^{(k_0,k_1)}$ be a left $D_X[t\partial_t]$-module homomorphism defined by

$$\chi\left(\sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t, x, \partial)S_{k+1}e_i\right) = \sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t - 1, x, \partial)T_k e_i$$

with

$$T_k := \begin{cases} S_k & (k \le -1) \\ t\partial_t S_k & (k \ge 0). \end{cases}$$

THEOREM 5.9. *Under the same assumptions as in Proposition* 5.2, *we have an isomorphism*

$$H^{-1}(M_X^\bullet) \simeq \chi^{-1}(G^{(k_0,k_1)})/G^{(k_0+1,k_1+1)}$$

as left $D_X[t\partial_t]$-*modules. Moreover*, $\chi^{-1}(G^{(k_0,k_1)})/G^{(k_0+1,k_1+1)}$ *is a coherent left* D_X-*module.*

Proof. First note that $\chi^{-1}(G^{(k_0,k_1)})$ is a left $D_X[t\partial_t]$-module since we have $\chi(t\partial_t P) = (t\partial_t - 1)\chi(P)$ for $P \in \tilde{D}^{(k_0+1,k_1+1)}$. Let

$$\bar{\chi}\colon \tilde{D}^{(k_0+1,k_1+1)}/G^{(k_0+1,k_1+1)} \to \tilde{D}^{(k_0,k_1)}/G^{(k_0,k_1)}$$

be the homomorphism induced by χ. Then $\bar{\chi}$ represents the homomorphism

$$t\colon F_{k_1+1}(M)/F_{k_0}(M) \to F_{k_1}(M)/F_{k_0-1}(M)$$

via the isomorphism of Proposition 5.8 since

$$t\sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t, x, \partial)S_{k+1}e_i = \sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t - 1, x, \partial)tS_{k+1}e_i$$

and $tS_{k+1} = T_k$. This implies the first assertion of the theorem.

The coherency of $\chi^{-1}G^{(k_0,k_1)}/G^{(k_0,k_1)}$ over D_X follows from the existence of the b-function (cf. [24]). See the proof of Theorem 5.11 below for an algorithmic proof of this fact. ∎

A presentation of $H^{-1}(M_X^\bullet)$ as a left coherent D_X-module can be obtained by the following algorithm. Put

$$A^{(k_0,k_1)} := \bigoplus_{i=1}^{r}\bigoplus_{k=k_0}^{k_1} A_n[t\partial_t]S_k e_i.$$

We regard $A^{(k_0,k_1)}$ as a free left $A_n[t\partial_t]$-module of rank $k_1 - k_0 + 1$.

ALGORITHM 5.10. Input: a finite set $\mathbf{G} \subset (A_{n+1})^r$ of F-involutory generators of $\boldsymbol{N}$ on X, and integers k_0, k_1 satisfying the assumption of Proposition 5.2.

(1) Let N_1 be the left $A_n[t\partial_t, z]$-submodule of

$$A^{(k_0,k_1)}[z] := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} A_n[t\partial_t, z]S_k e_i$$

which is generated by

$$\bigcup_{i=1}^{r} \bigcup_{k=k_0}^{k_1} \{(1-z)T_k e_i\} \cup \{zP \mid P \in \mathbf{G}^{(k_0,k_1)}\}$$

with an indeterminate z.

(2) Let $\mathbf{G}_1$ be a Gröbner basis of N_1 with respect to a well-order $\prec_z$ on $L \times \{1, \ldots, r\}$ for eliminating z, i.e., satisfying $(\mu, \nu, \alpha, \beta, i) \prec_z (\mu', \nu', \alpha', \beta', j)$ whenever $\mu < \mu'$; here $(\mu, \nu, \alpha, \beta, i) \in L \times \{1, \ldots, r\}$ corresponds to the monomial $z^\mu s^\nu x^\alpha \partial^\beta e_i$ with $s = t\partial_t$.

(3) Each element P of $\mathbf{G}_1 \cap A^{(k_0,k_1)}$ can be written uniquely in the form

$$P = \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} Q_{ik}(t\partial_t)T_k e_i$$

with $Q_{ik}(t\partial_t) \in A_n[t\partial_t]$. Then we define $\chi^{-1}(P) \in A^{(k_0+1,k_1+1)}$ by

$$\chi^{-1}(P) := \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} Q_{ik}(t\partial_t + 1)S_{k+1}e_i.$$

Put

$$\mathbf{G}_2 := \{\chi^{-1}(P) \mid P \in \mathbf{G}_1 \cap A^{(k_0,k_1)}\}.$$

Then $\mathbf{G}_2$ generates the left $\mathcal{D}_X[t\partial_t]$-module $\chi^{-1}(\mathcal{G}^{(k_0,k_1)})$.

(4) Suppose $\mathbf{G}_2 = \{P_1, \ldots, P_d\}$ and $\mathbf{G}^{(k_0+1,k_1+1)} = \{P_{d+1}, \ldots, P_l\}$ and put

$$S := \left\{(Q_1, \ldots, Q_l) \in A_n[t\partial_t]^l \,\middle|\, \sum_{j=1}^{l} Q_j P_j = 0\right\}.$$

Compute a set of generators $\mathbf{G}_3$ of S by means of a Gröbner basis. Let $\pi_d : A_n[t\partial_t]^l \to A_n[t\partial_t]^d$ be the projection to the first d components. Then we have an isomorphism

$$\chi^{-1}(G^{(k_0,k_1)})/G^{(k_0+1,k_1+1)} \simeq D_X[t\partial_t]^d/\big(D_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S)\big)$$

of left $D_X[t\partial_t]$-modules and $D_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S)$ is generated by $\pi_d(\mathbf{G}_3)$.

(5) Put $\mathbf{G}_4 := \{P(-1) \mid P(t\partial_t) \in \pi_d(\mathbf{G}_3)\}$ and let N_X^{-1} be the left D_X-module generated by $\mathbf{G}_4$. Then we have an isomorphism

$$\chi^{-1}(G^{(k_0,k_1)})/G^{(k_0+1,k_1+1)} \simeq D_X^d/N_X^{-1}$$

of left D_X-modules.

THEOREM 5.11. *The statements in the above algorithm are correct if M is specializable along X at each point of X.*

Proof. In steps (1) and (2), $\mathbf{G}_1 \cap A^{(k_0,k_1)}$ is a set of generators of the intersection of the left module generated by $\mathbf{G}^{(k_0,k_1)}$ and the left module generated by $T_k e_i$ with $1 \le i \le r$ and $k_0 \le k \le k_1$. In fact, the argument for the intersection of two ideals of a polynomial ring (cf. [9, 2, 10]) applies without modification. Hence $\mathbf{G}_1 \cap A^{(k_0,k_1)}$ generates $\chi(\tilde{D}^{(k_0+1,k_1+1)}) \cap G^{(k_0,k_1)}$. This implies that $\mathbf{G}_2$ generates $\chi^{-1}(G^{(k_0,k_1)})$ since χ is injective. This proves the correctness of the step (3). The step (4) is easy to verify.

Now let us verify the step (5). Since we have

$$(t\partial_t + 1)\big(\chi^{-1}(G^{(k_0,k_1)})/G^{(k_0+1,k_1+1)}\big) \simeq \partial_t t\, H^{-1}(M_X^\bullet) = 0,$$

the homomorphism

$$\rho_{-1}\colon D_X[t\partial_t]^d \to D_X^d$$

defined by $\rho_{-1}(P(t\partial_t)) = P(-1)$ induces an isomorphism

$$D_X[t\partial_t]^d/\big(D_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S)\big) \xrightarrow{\simeq} D_X^d/N_X^{-1}$$

of left D_X-modules. This completes the proof. ∎

In particular, we have proved in an algorithmic and constructive way that $H^j(M_X^\bullet)$ $(j = 0, -1)$ are coherent D_X-modules if M is specializable along X.

The following is a rather simple example for illustrating how the algorithms proceed.

EXAMPLE 5.12. Let $\mathcal{N}$ be a left ideal of $\mathcal{D}_{\tilde{X}}$ with $X := K^3$ and $\tilde{X} = K^4$ generated by

$$P_1 := x_2\partial_2 + x_3\partial_3 - a_1,$$
$$P_2 := t\partial_t + x_2\partial_2 - a_2,$$
$$P_3 := x_1\partial_1 + x_3\partial_3 - a_3,$$
$$P_4 := \partial_t\partial_3 - \partial_1\partial_2,$$

and put $\mathcal{M} := \mathcal{D}_{\tilde{X}}/\mathcal{N}$, where a_1, a_2, a_3 are regarded as parameters with values in K. In fact, this is a rather simple case of the A-hypergeometric D-module defined by Gelfand et al. [13]. The following computation (and the other examples as well) has been performed by using a computer algebra system Kan [43].

We get $\mathbf{G} := \{P_1, P_2, P_3, P_4, P_5\}$ with

$$P_5 := -x_3\partial_3^2 + (a_1 - a_2 - 1)\partial_3 + t\partial_1\partial_2$$

as a set of F-involutory generators of $\mathcal{N}$ by computing an FW-Gröbner basis in the Weyl algebra. The ideal $\mathcal{J}$ of $\mathcal{O}_X[s]$ of Proposition 4.2 in this case is generated by a single element $s^2 + a_1 s - a_2 s$. Hence the b-function along the hyperplane $X = \{t = 0\}$ is $s(s + a_1 - a_2)$ at any point of X and for any values of parameters a_1, a_2, a_3. Actually, we can find by an algorithm given in [33] that $t\partial_t^2 + (a_1 - a_2 + 1)\partial_t - x_3\,\partial_1\partial_2$ is a section of $\mathcal{N}$ on X, and the indicial polynomial of this Fuchsian operator with respect to t is the same as the above b-function.

Under the condition that $a_1 - a_2$ is not a nonzero integer, we can take $k_0 = k_1 = 0$ in Theorems 5.7 and 5.9. By Theorem 5.7, we have $\mathcal{M}_X = \mathcal{D}_X/\mathcal{N}_X$ with the left ideal $\mathcal{N}_X$ of $\mathcal{D}_X$ generated by P_1, $x_2\partial_2 - a_2$, P_3, $-x_3\partial_3^2 + (a_1 - a_2 - 1)\partial_3$. Actually, $\mathcal{N}_X$ is generated by

$$x_1\partial_1 + a_1 - a_2 - a_3, \qquad x_2\partial_2 - a_2, \qquad x_3\partial_3 - a_1 + a_2.$$

Roughly speaking (we assume $K \subset \mathbf{C}$), this means that if $u(t, x)$ is a multi-valued analytic function which is holomorphic in t and satisfies $P_i u = 0$ for $i = 1, 2, 3, 4$, then we have $u(0, x) = c x_1^{-a_1 + a_2 + a_3} x_2^{a_2} x_3^{a_1 - a_2}$ with some $c \in \mathbf{C}$.

From the F-involutory generators $P_1, \ldots, P_5$, we know that in Proposition 5.8, $\mathcal{G}^{(0,0)}$ is the left ideal of $\mathcal{D}_X[s]$ with $s = t\partial_t$ generated by

$$x_2\partial_2 + x_3\partial_3 - a_1, \qquad s + x_2\partial_2 - a_2, \qquad x_1\partial_1 + x_3\partial_3 - a_3,$$
$$s\partial_3, \qquad -x_3\partial_3^2 + (a_1 - a_2 - 1)\partial_3,$$

while $\mathcal{G}^{(1,1)}$ is the left $\mathcal{D}_X[s]$-submodule of $\mathcal{D}_X[s]\partial_t$ generated by

$$(x_2\partial_2 + x_3\partial_3 - a_1)\partial_t, \qquad (s + x_2\partial_2 - a_2 + 1)\partial_t,$$
$$(x_1\partial_1 + x_3\partial_3 - a_3)\partial_t, \qquad \partial_3\partial_t.$$

By executing Algorithm 5.10, we conclude that

$$\mathcal{H}^{-1}(\mathcal{M}_X^\bullet) = \chi^{-1}(\mathcal{G}^{(0,0)})/\mathcal{G}^{(1,1)} = 0$$

on X as long as $a_1 - a_2$ is not a nonzero integer. Actually, we can perform the Gröbner basis computation for this example with the field of rational functions $\mathbf{Q}(a_1, a_2, a_3)$ as the coefficient field; afterward we can detect the exceptional values of a_1, a_2, a_3 for which the output may fail to be correct (cf. [34]). In this case, we can verify that there are no exceptional values. Hence we have only to take into account the condition on the integral roots of the b-function. See [37] and [36] for the theoretical determination of the b-function and restrictions of some classes of A-hypergeometric D-modules.

6. ALGEBRAIC LOCAL COHOMOLOGY GROUPS

In this section, let X be a Zariski open set of K^n and put $\tilde{X} := K \times X$. We identify X with the subset $\{0\} \times X$ of K^{n+1} as in the preceding sections. In the sequel we consider a $\mathcal{D}_X$-module $\mathcal{M}$ instead of a $\mathcal{D}_{\tilde{X}}$-module. Let N be a left A_n-submodule of $(A_n)^r$ and put $M := (A_n)^r/N$ and $\mathcal{M} := \mathcal{D}_X \otimes_{A_n} M$. Then we have $\mathcal{M} = (\mathcal{D}_X)^r/\mathcal{N}$ with $\mathcal{N} := \mathcal{D}_X N$.

Let $f = f(x) \in K[x]$ be a nonconstant polynomial and put $Y := \{x \in X \mid f(x) = 0\}$. Then the algebraic local cohomology group $\mathcal{H}^j_{[Y]}(\mathcal{M})$ has a structure of left $\mathcal{D}_X$-module and vanishes for $j \neq 0, 1$ ([17]). Our purpose is to give an algorithm of computing $\mathcal{H}^j_{[Y]}(\mathcal{M})$ as a left $\mathcal{D}_X$-module. In general, for an $\mathcal{O}_X$-module $\mathcal{F}$, put

$$\Gamma_{[Y]}(\mathcal{F}) := \{u \in \mathcal{F} \mid f^k u = 0 \text{ for some } k \in \mathbf{N}\}.$$

Then $\mathcal{H}^j_{[Y]}(\mathcal{F})$ is defined as the j-th derived functor of $\Gamma_{[Y]}$.

Put $Z := \{(t, x) \in K \times X \mid t - f(x) = 0\}$. Let $\mathcal{J}_Z$ be a left ideal of $\mathcal{D}_{\tilde{X}}$ generated by $t - f(x)$, $\partial_1 + (\partial f/\partial x_1)\partial_t, \ldots, \partial_n + (\partial f/\partial x_n)\partial_t$, and put $\mathcal{B}_{[Z]} := \mathcal{D}_{\tilde{X}}/\mathcal{J}_Z$. We denote by $\delta(t - f)$ the residue class of $1 \in \mathcal{D}_{\tilde{X}}$ in $\mathcal{B}_{[Z]}$.

Put $\mathcal{L} := \mathcal{O}_X[f^{-1}, s]f^s$, where f^s is regarded as a free generator. Then $\mathcal{L}$ has a natural structure of left $\mathcal{D}_X[s]$-module. As was observed by Malgrange [26], $\mathcal{L}$ has a structure of left $\mathcal{D}_{\tilde{X}}$-module so that

$$t(g(s)f^s) = g(s+1)f^{s+1}, \qquad \partial_t(g(s)f^s) = -sg(s-1)f^{s-1} \tag{6.1}$$

for $g(s) \in \mathcal{O}_X[f^{-1}, s]$. This implies that there exists an injective homomorphism ι: $\mathcal{B}_{[Z]}|_X \to \mathcal{L}$ of left $\mathcal{D}_{\tilde{X}}$-modules such that $\iota(\delta(t - f)) = f^s$ ([26]).

LEMMA 6.1. *We have an isomorphism* $(\mathcal{B}_{[Z]})^{\bullet}_X \simeq \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[1]$ *in the derived category of left* $\mathcal{D}_X$*-modules, where* $\mathbf{R}\Gamma_{[Y]}$ *denotes the right derived functor of* $\Gamma_{[Y]}$, *and* $[1]$ *the translation functor* ([15]).

Proof. In view of Theorem 1.2 of [17], we have

$$\begin{aligned}(\mathcal{B}_{[Z]})^{\bullet}_X &\simeq \mathcal{D}_{X\to\tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathcal{B}_{[Z]} \\ &\simeq \mathcal{D}_{X\to\tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathbf{R}\Gamma_{[Z]}(\mathcal{O}_{\tilde{X}})[1] \\ &\simeq \mathbf{R}\Gamma_{[Z]}\Big(\mathcal{D}_{X\to\tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}\Big)[1] \\ &\simeq \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[1]. \quad \blacksquare\end{aligned}$$

Now let $\pi: \tilde{X} \to X$ be the projection. Then the tensor product $\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$ has a structure of sheaves of left $\mathcal{D}_{\tilde{X}}$-modules. Let π_1 and π_2 be the projections of $\tilde{X} \times X$ to $\tilde{X}$ and to X respectively defined by $\pi_1(t,x,y) = (t,x)$ and $\pi_2(t,x,y) = y$ for $t \in K$ and $x, y \in X$. Put

$$\Delta := \{(t,x,y) \in \tilde{X} \times X \mid x = y\}$$

and

$$\mathcal{D}_{\Delta\to\tilde{X}\times X} := \mathcal{D}_{\tilde{X}\times X}/((x_1 - y_1)\,\mathcal{D}_{\tilde{X}\times X} + \cdots + (x_n - y_n)\,\mathcal{D}_{\tilde{X}\times X}).$$

LEMMA 6.2. *Let* $\mathcal{F}$ *be a left* $\mathcal{D}_{\tilde{X}}$*-module. Then we have*

$$\mathcal{F} \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M} \simeq \mathcal{D}_{\Delta\to\tilde{X}\times X} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}\times X}} (\mathcal{F} \hat{\otimes} \mathcal{M})$$

with

$$\mathcal{F} \hat{\otimes} \mathcal{M} := \mathcal{D}_{\tilde{X}\times X} \otimes_{\pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes \pi_2^{-1}\mathcal{D}_X} (\pi_1^{-1}\mathcal{F} \otimes_K \pi_2^{-1}\mathcal{M}).$$

Proof. In the same way as the proof of Proposition 4.7 of [17], we have

$$\begin{aligned}\mathcal{D}_{\Delta\to\tilde{X}\times X} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}\times X}} (\mathcal{F} \hat{\otimes} \mathcal{M}) &= \mathcal{O}_\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{\tilde{X}\times X}} (\mathcal{F} \hat{\otimes} \mathcal{M}) \\ &= \mathcal{O}_\Delta \overset{\mathbf{L}}{\otimes}_{\pi_1^{-1}\mathcal{O}_{\tilde{X}} \otimes_K \pi_2^{-1}\mathcal{O}_X} (\pi_1^{-1}\mathcal{F} \otimes_K \pi_2^{-1}\mathcal{M}) \\ &= \mathcal{F} \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}.\end{aligned}$$

The last equality follows from

$$\mathcal{O}_\Delta \simeq \pi_1^{-1}\mathcal{O}_{\tilde{X}} \otimes_K \pi_2^{-1}\mathcal{O}_X/\langle x_1 - y_1, \ldots, x_n - y_n\rangle.$$

This completes the proof. $\blacksquare$

LEMMA 6.3. *The i-th torsion group* $\mathcal{T}or_i^{\pi^{-1}\mathcal{O}_X}(\mathcal{B}_{[Z]}, \pi^{-1}\mathcal{M})$ *vanishes for* $i \neq 0$.

Proof. By definition, we have

$$(\partial/\partial x_i - f_i(x)\partial_t)(\delta(t-f)\otimes u) = 0 \quad \text{in } \mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M}$$

for any section u of $\mathcal{M}$. Hence Δ is noncharacteristic for $\mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M}$. This implies the assertion of the lemma (cf. [19]). ∎

THEOREM 6.4. *We have isomorphisms*

$$\mathcal{H}^j\left(\left(\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\right)_X^{\bullet}\right) \simeq \mathcal{H}_{[Y]}^{j+1}(\mathcal{M})$$

of left $\mathcal{D}_X$*-modules for* $j = -1, 0$.

Proof. We have by Lemma 6.1

$$\begin{aligned}
\left(\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\right)_X^{\bullet} &\simeq \mathcal{D}_{X\to\tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \left(\mathcal{B}_{[Z]} \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\right) \\
&\simeq \left(\mathcal{D}_{X\to\tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathcal{B}_{[Z]}\right) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M} \\
&\simeq \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[1] \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M} \\
&\simeq \mathbf{R}\Gamma_{[Y]}(\mathcal{M})[1]. \quad ∎
\end{aligned}$$

More elementary and concrete proof of this theorem is possible (cf. Remark 6.12 below). In what follows, we shall denote $\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$ by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}$ for a $\mathcal{D}_{\tilde{X}}$-module $\mathcal{F}$. In view of Theorems 5.7, 5.9, 5.11 and 6.3, we obtain an algorithm for computing the algebraic local cohomology groups $\mathcal{H}_{[Y]}^j(\mathcal{M})$ for $j = 0, 1$ if there is an algorithm for computing $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ as a left $\mathcal{D}_{\tilde{X}}$-module. In fact, this tensor product can be computed as follows:

LEMMA 6.5. *Let* $\mathcal{J}_Z$ *be as above. Then we have an isomorphism* $\mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M} \simeq (\mathcal{D}_{\tilde{X}\times X})^r/\mathcal{N}_Z$ *with* $\mathcal{N}_Z := \mathcal{J}_Z \hat{\otimes} (\mathcal{D}_X)^r + \mathcal{D}_{\tilde{X}} \hat{\otimes} \mathcal{N}$.

Proof. It suffices to show

$$\pi_1^{-1}\mathcal{B}_{[Z]} \otimes_K \pi_2^{-1}\mathcal{M} = \left(\pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes_K \pi_2^{-1}\mathcal{D}_X\right)/\mathcal{N}_Z'$$

with

$$\mathcal{N}_Z' := \pi_1^{-1}\mathcal{J}_Z \otimes_K \pi_2^{-1}(\mathcal{D}_X)^r + \pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes_k \pi_2^{-1}\mathcal{N}.$$

In fact we have

$$\begin{aligned}\pi_1^{-1}\mathcal{B}_{[Z]} \otimes_K \pi_2^{-1}\mathcal{M} &= \left(\pi_1^{-1}\mathcal{B}_{[Z]} \otimes_K \pi_2^{-1}(\mathcal{D}_X)^r\right)/\left(\pi_1^{-1}\mathcal{B}_{[Z]} \otimes_k \pi_2^{-1}\mathcal{N}\right)\\ &= \frac{\left(\pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes_K \pi_2^{-1}(\mathcal{D}_X)^r\right)/\left(\pi_1^{-1}\mathcal{J}_Z \otimes_K \pi_2^{-1}(\mathcal{D}_X)^r\right)}{\left(\pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes_K \pi_2^{-1}\mathcal{N}\right)/\left(\pi_1^{-1}\mathcal{J}_Z \otimes_K \pi_2^{-1}\mathcal{N}\right)}\\ &= \left(\pi_1^{-1}\mathcal{D}_{\tilde{X}} \otimes_K \pi_2^{-1}(\mathcal{D}_X)^r\right)/\mathcal{N}'_Z.\end{aligned}$$

This completes the proof. ∎

For $i = 1,\dots,n$, put

$$\Delta_i := \left\{(t,x,y) \in \tilde{X} \times X \mid x_j = y_j \text{ for } j = 1,\dots,i\right\}.$$

Then we have

$$\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \left(\cdots\left(\left(\mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M}\right)_{\Delta_1}\right)_{\Delta_2}\cdots\right)_{\Delta_n}$$

by virtue of Lemma 6.2. Since Δ_i is noncharacteristic for $\mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M}$ in view of the proof of Lemma 6.3, we can compute $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ by applying Theorem 5.7 repeatedly with $k_0 = k_1 = 0$.

LEMMA 6.6. *If* $\mathcal{M}$ *is holonomic, then* $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ *is specializable along* X.

Proof. First, $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is holonomic as the restriction of the holonomic system $\mathcal{B}_{[Z]} \hat{\otimes} \mathcal{M}$ to Δ (cf. [17]). Hence $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X by a theorem of Kashiwara-Kawai (cf. [21]). ∎

Thus we have obtained an algorithm for computing $\mathcal{H}^j_{[Y]}(\mathcal{M})$ $(j = 0,1)$ by applying Theorem 5.7 and Algorithm 5.10 to $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ under the condition that $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X. In particular, we have proved the following statement effectively:

COROLLARY 6.7. *If* $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ *is specializable along* X, *then* $\mathcal{H}^j_{[Z]}(\mathcal{M})$ $(j = 0,1)$ *are coherent left* $\mathcal{D}_X$*-modules.*

Let us describe $\mathcal{H}^1_{[Y]}(\mathcal{M})$ more concretely. First note that $\mathcal{H}^1_{[Y]}(\mathcal{M}) \simeq \mathcal{M}[f^{-1}]/\mathcal{M}$ with $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$. By applying Theorem 5.7 to $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$, we know that $\mathcal{M}[f^{-1}]/\mathcal{M}$ is generated by the modulo classes $v_{ik} := [f^{-k} \otimes u_i]$ in $(\mathcal{O}[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M})/\mathcal{M}$ with $k_0 \le k \le k_1$ and $1 \le i \le r$, and the relations among the generators $k!\,v_{ik}$ are given by $\mathcal{N}_X$ of Theorem 5.7. Actually, v_{ik_1} with $1 \le i \le r$ generate $\mathcal{M}[f^{-1}]/\mathcal{M}$ and the relations among these generators can be obtained by eliminating v_{ik} with $k < k_1$.

Our next aim is to give an algorithm of computing the b-function for a polynomial f and a section u of $\mathcal{M}$. Put $\mathcal{M}[s] := K[s] \otimes_K \mathcal{M}$. Then we have

$$\mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s] = \mathcal{L} \otimes_{\mathcal{O}_X[s]} \left(\mathcal{O}_X[s] \otimes_{\mathcal{O}_X} \mathcal{M}\right) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Note that an arbitrary element of $\mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s]$ can be expressed in the form $f^{s-m} \otimes u$ with some $m \in \mathbf{N}$ and $u \in \mathcal{M}[s]$.

LEMMA 6.8. *Let u be a section of $\mathcal{M}[s]$ and let m be a nonnegative integer. Then we have $f^{s-m} \otimes u = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s]$ if and only if $f^k u = 0$ holds in $\mathcal{M}[s]$ with some $k \in \mathbf{N}$.*

Proof. Since $\mathcal{L}$ is a free $\mathcal{O}_X[s, f^{-1}]$-module of rank one, we have $f^{s-m} \otimes u = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s]$ if and only if $1 \otimes u = 0$ in $\mathcal{M}[s, f^{-1}] := \mathcal{O}_X[s, f^{-1}] \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s]$. Assume $f^k u = 0$ in $\mathcal{M}[s]$. Then we have $1 \otimes u = f^{-k} \otimes (f^k u) = 0$ in $\mathcal{M}[s, f^{-1}]$. Letting σ be a commutative variable independent of s and x, define an $\mathcal{O}_X[s]$-homomorphism

$$\varphi \colon \mathcal{O}_X[s, \sigma] \to \mathcal{O}_X\left[s, f^{-1}\right]$$

by $\varphi(h(s, \sigma)) = h(s, f^{-1})$ for $h \in \mathcal{O}_X[s, \sigma]$. Let $\mathcal{K}$ be the kernel of φ. Then we have an exact sequence

$$\mathcal{K} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s] \longrightarrow \mathcal{O}[s, \sigma] \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s] \xrightarrow{\varphi \otimes 1} \mathcal{M}\left[s, f^{-1}\right] \longrightarrow 0.$$

Now assume $1 \otimes u = 0$ in $\mathcal{M}[s, f^{-1}]$. Then in view of the exact sequence above, there exist $\kappa_i(\sigma) = \sum_{j=0}^{d_i} \kappa_{ij} \sigma^j \in \mathcal{K}$ ($\kappa_{ij} \in \mathcal{O}_X[s]$) and $u_i \in \mathcal{M}[s]$ so that

$$1 \otimes u = \sum_{i=1}^{l} \kappa_i(\sigma) \otimes u_i$$

in $\mathcal{O}_X[s, \sigma] \otimes_{\mathcal{O}_X[s]} \mathcal{M}$. Since $\mathcal{O}_X[s, \sigma]$ is a free $\mathcal{O}_X[s]$-module, this implies

$$\sum_{i=1}^{l} \kappa_{ij} u_i = \begin{cases} u & (j = 0) \\ 0 & (j \neq 0) \end{cases}$$

in $\mathcal{M}[s]$. Put $k := \max\{d_1, \ldots, d_l\}$. Since $\kappa_i(\sigma) \in \mathcal{K}$, we have

$$\sum_{j=0}^{d_i} f^{k-j} \kappa_{ij} = f^k \kappa_i(f^{-1}) = 0$$

in $\mathcal{O}_X[s, f^{-1}]$. Thus we get

$$f^k u = \sum_{j=0}^{k} f^{k-j} \sum_{i=1}^{l} \kappa_{ij} u_i = \sum_{i=1}^{l} \left(\sum_{j=0}^{d_i} f^{k-j} \kappa_{ij} \right) u_i = 0$$

in $\mathcal{M}[s]$. This completes the proof. ∎

Let u be a section of $\mathcal{M}$ and P a section of $\mathcal{D}_X[s]$. Then the identity $P(f^s u) = 0$ means by definition that there exists $m \in \mathbf{N}$ so that $Q := f^{m-s} P f^s$ is contained in $\mathcal{D}_X[s]$ and that $Qu = 0$ holds in $\mathcal{M}[s]$ (cf. [17]).

Lemma 6.9. *For $u \in \mathcal{M}$ and $P \in \mathcal{D}_X[s]$, we have $P(f^s u) = 0$ if and only if $P(f^s \otimes u) = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$.*

Proof. For $i = 1, \ldots, n$, we have

$$\partial_i(f^s \otimes u) = f^{s-1} \otimes (s f_i + f \partial_i) u = f^{s-1} \otimes \left(f^{1-s} \partial_i f^s \right) u$$

with $f_i := \partial f / \partial x_i$. Thus by induction on the order of P, we can prove that

$$P(f^s \otimes u) = f^{s-m} \otimes (f^{m-s} P f^s) u,$$

where m denotes the order of P. By virtue of the preceding lemma, we have $P(f^s \otimes u) = 0$ if and only if $(f^{k+m-s} P f^s) u = 0$ in $\mathcal{M}[s]$ with some $k \in \mathbf{N}$, which is equivalent to $P(f^s u) = 0$. ∎

Lemma 6.10. *$\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ if and only if f: $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is injective.*

Proof. By Theorem 6.4, $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ if and only if t: $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is injective. For any $v \in \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$, there exists $m \in \mathbf{N}$ so that $(t-f)^m v = 0$. Hence if $tv = 0$, we get $f^m v = 0$. Conversely, $fv = 0$ implies $t^m v = 0$. Hence t: $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is injective if and only if so is f: $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$. ∎

Lemma 6.11. *Let p be a point of Y. Then any germ v of $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ at p is uniquely written in the form*

$$v = \sum_{i=0}^{k} \partial_t^i \, \delta(t-f) \otimes u_i \tag{6.2}$$

with $u_i \in \mathcal{M}_p$ and $k \in \mathbf{N}$.

Proof. By using the formula $t \partial_t^i \, \delta(t-f) = f \partial_t^i \, \delta(t-f) - i \partial_t^{i-1} \, \delta(t-f)$, we know that $\mathcal{B}_{[Z]}$ is generated by $\partial_t^i \, \delta(t-f)$ $(i \in \mathbf{N})$ over $\mathcal{O}_X$. Thus v can be written in the form (6.2). In order to prove the uniqueness, it suffices to note that $(\mathcal{B}_{[Z]})_p$ is a free $(\mathcal{O}_X)_p$-module generated by $\partial_t^i \, \delta(t-f)$ with $i \in \mathbf{N}$. This fact follows from the isomorphisms

$$\mathcal{B}_{[Z]} \simeq \mathcal{H}^1_{[Z]}(\mathcal{O}_X) \simeq \mathcal{O}_X\left[(t-f)^{-1}\right] / \mathcal{O}_X$$

since Z is a smooth hypersurface. This completes the proof. ∎

REMARK 6.12. In terms of the expression (6.2), the homomorphism $t\colon \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is given by

$$t\left(\sum_{i=0}^{\infty} \partial_t^i \,\delta(t-f) \otimes u_i\right) = \sum_{i=0}^{\infty} \partial_t^i \,\delta(t-f) \otimes \left(fu_i - (i+1)u_{i+1}\right).$$

This yields a more concrete proof of Theorem 6.4; to work it out is left to the reader.

PROPOSITION 6.13. *The homomorphism*

$$\iota \otimes 1\colon \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$$

is injective if and only if $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$.

Proof. Let v be a germ of $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ at $p \in Y$ given by (6.2). Then by using (6.1) we obtain

$$\begin{aligned}(\iota \otimes 1)(v) &= \sum_{i=0}^{k} (-1)^i s(s-1)\cdots(s-i+1) f^{s-i} \otimes u_i \\ &= f^{s-k} \otimes \left(\sum_{i=0}^{k} (-1)^i s(s-1)\cdots(s-i+1) f^{k-i} u_i\right).\end{aligned}$$

Now assume $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ and $(\iota \otimes 1)(v) = 0$. Then by Lemma 6.8 there exists $m \in \mathbf{N}$ so that

$$\sum_{i=0}^{k} (-1)^i s(s-1)\cdots(s-i+1) f^{m+k-i} u_i = 0 \quad \text{in } \mathcal{M}[s]. \qquad (6.3)$$

Since $u_i \in \mathcal{M}$, (6.3) is equivalent to $f^{m+k-i}u_i = 0$ for each $i = 0, \dots, k$. This implies $u_i = 0$ since $f\colon \mathcal{M} \to \mathcal{M}$ is injective by the assumption. Thus $\iota \otimes 1$ is injective.

Conversely, assume $\mathcal{H}^0_{[Y]}(\mathcal{M})_p \neq 0$ with some $p \in Y$. Then there exists $u \in \mathcal{M}_p$ and $k \in \mathbf{N}$ such that $u \neq 0$ and $f^k u = 0$. Then we have $\delta(t-f) \otimes u \neq 0$ in view of Lemma 6.11 while $(\iota \otimes 1)(\delta(t-f) \otimes u) = f^s \otimes u = 0$. This completes the proof. ∎

THEOREM 6.14. *Assume* $r = 1$ *and let* $u \in \mathcal{M}$ *be the residue class of* $1 \in \mathcal{D}_X$. *Let* $b_X(s)$ *be the b-function of* $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ *along* X *with respect to the filtration* $\{F_k(D_{\tilde{X}})(\delta(t-f) \otimes u)\}_{k \in \mathbf{Z}}$ *and let* $b(s)$ *be the b-function for* f *and* u *defined by* (1.1), *both at a point* p *of* Y. *Then we have the following*:

(1) $b(s)$ *divides* $b_X(-s-1)$;

(2) *If* $\mathcal{H}^0_{[Y]}(\mathcal{M})_p = 0$, *then we have* $b(s) = \pm b_X(-s-1)$;

(3) *A nonzero b-function* $b(s)$ *for* f *and* u *exists at* $p \in X$ *if and only if* $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ *is specializable along* X *at* p.

Proof. (1) By definition, there exists $P \in F_{-1}(\mathcal{D}_{\tilde{X}})_p$ so that

$$(b_X(t\partial_t) - P)(\delta(t-f) \otimes u) = 0 \quad \text{in } \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Writing P in a finite sum

$$P = a(t,x)^{-1} \sum_{k=1}^{\infty} P_k(t\partial_t) t^k$$

with $P_k(t\partial_t) \in \mathcal{D}_X[t\partial_t]$ and $a(t,x) \in K[t,x]$ such that $a(p) \neq 0$, put

$$Q := a(t,x)^{-1} \sum_{k=1}^{\infty} P_k(-s-1) f^{k-1}.$$

In view of (6.1), we get

$$\begin{aligned}(b_X(-s-1) - Qf)(f^s \otimes u)& \\ &= (\iota \otimes 1)\big((b_X(t\partial_t) - P)(\delta(t-f) \otimes u)\big) = 0.\end{aligned}$$

This implies (1).

(2) Now assume $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. There exists $Q(s) \in \mathcal{D}[s]_p$ so that $(b(s) - Q(s)f)(f^s \otimes u) = 0$. It follows

$$\begin{aligned}(\iota \otimes 1)\big((b(-\partial_t t) - Q(-\partial_t t)t)(\delta(t-f) \otimes u)\big)& \\ &= (b(s) - Q(s)f)(f^s \otimes u) = 0.\end{aligned}$$

Since $\iota \otimes 1$ is injective and $Q(-\partial_t t)t \in F_{-1}(\mathcal{D}_{\tilde{X}})$, this proves that $b_X(s)$ divides $b(-s-1)$.

(3) Assume that there exists a nonzero b-function $b_X(s)$ for f and u at p. Let $\bar{u}$ be the residue class of u in $\mathcal{M}' := \mathcal{M}/\mathcal{H}^0_{[Y]}(\mathcal{M})$. Then $\mathcal{M}'$ satisfies the condition of (2), and the b-function for f and $\bar{u}$ divides $b_X(s)$, hence is nonzero. Thus we know that $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}'$ is specializable along X at p by applying (2) to $\mathcal{M}'$. Since $\mathcal{B}_{[Z]}$ is $\mathcal{O}_X$-flat (cf. the proof of Lemma 6.11), we get an exact sequence

$$\begin{aligned}0 \to \operatorname{gr}_0\big(\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{H}^0_{[Y]}(\mathcal{M})\big) &\to \operatorname{gr}_0(\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}) \\ &\to \operatorname{gr}_0(\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}') \to 0.\end{aligned}$$

It is easy to see that there exists some $m \in \mathbf{N}$ so that $\partial_t^m t^m \operatorname{gr}_0(\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{H}^0_{[Y]}(\mathcal{M})) = 0$. It follows that $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X. This completes the proof. ∎

REMARK 6.15. In general case $r \geq 1$, let us assume that u is given by $u = P_1 u_1 + \cdots + P_r u_r$ with $P_i \in A_n$ given explicitly. Then we obtain an algorithm to compute $\mathcal{D}_X u$ by means of Lemmas 5.5 and 5.6 based on the fact that $Pu = 0$ holds if and only if $(PP_1, \ldots, PP_r) \in \mathcal{N}$.

Thus we have obtained an algorithm for computing the b-function for f and $u \in \mathcal{M}$ under the assumption $\mathcal{H}^0_{[Y]}(\mathcal{D}_X u) = 0$, which can be determined by Algorithm 5.10. Note that we do not need this assumption for deciding whether a nonzero b-function exists. This generalizes an algorithm of computing the Bernstein-Sato polynomial given in [33].

EXAMPLE 6.16. Put $\mathcal{M} := \mathcal{H}^1_{[Y]}(\mathcal{O}_X)$ and let u be the residue class of f^{-1} in $\mathcal{M} = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X$. Let p be a point of Y. Then the b-function for f and u at p is 1 since $fu = 0$ in $\mathcal{M}$. On the other hand, the b-function of $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ along X at p is $b_X(s) = s + 1$. In fact, since

$$t(\delta(t-f) \otimes u) = \delta(t-f) \otimes (fu) = 0,$$

we know that $b_X(s)$ divides $s + 1$. If $b_X(s) = 1$, then we should have

$$\mathcal{M} = \mathcal{H}^0_{[Y]}(\mathcal{M}) \simeq \mathcal{H}^{-1}\left((\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M})^{\bullet}_X\right) = 0$$

by virtue of Proposition 5.2 and Theorem 6.4, which is a contradiction.

EXAMPLE 6.17. Put $X = K^3 \ni (x, y, z)$ and write $\partial_x := \partial/\partial x$, $\partial_y := \partial/\partial y$, $\partial_z := \partial/\partial z$. The Bernstein-Sato polynomial for $f := x^3 - y^2 z^2$ (i.e. the b-function for f and $1 \in \mathcal{O}_X$) is

$$b_f(s) = (s+1)\left(s + \tfrac{5}{6}\right)^2\left(s + \tfrac{7}{6}\right)^2\left(s + \tfrac{4}{3}\right)\left(s + \tfrac{5}{3}\right)$$

at (0,0,0);

$$b_f(s) = (s+1)\left(s + \tfrac{5}{6}\right)\left(s + \tfrac{7}{6}\right)$$

on $\{(x,y,z) \mid x = yz = 0\} \setminus \{(0,0,0)\}$; $b_f(s) = s + 1$ on $\{(x,y,z) \mid x^3 - y^2 z^2 = 0, x \neq 0\}$; and $b_f(s) = 1$ on $\{(x,y,z) \mid x^3 - y^2 z^2 \neq 0\}$. This computation is based on Propositions 4.5 and 4.6. In practice, we have used a primary decomposition program of Risa/Asir which is based on the algorithm of [39] as well as Kan for the computation in the Weyl algebra. We can also find an operator $P \in \mathcal{D}_X[s]$ which satisfies $Pf^{s+1} = b_f(s) f^s$ at (0,0,0) in the form $P = (1/279936)P_0$ with

$$\begin{aligned} P_0 = {} & 72z^2(108s^2 + 252s + 145)\partial_x^3\partial_z^2 \\ & + 243z(108s^2 + 252s + 145)\partial_y^2\partial_z^3 \\ & + 72z(144s^3 + 900s^2 + 1508s + 755)\partial_x^3\partial_z \\ & - 972(s+1)(72s^2 + 144s + 65)\partial_y^2\partial_z^2 \\ & + 8(1296s^4 + 7776s^3 + 18072s^2 + 18576s + 6985)\partial_x^3 \end{aligned}$$

by Algorithm 5.4 of [33]. On the other hand, we have $\mathcal{H}^1_{[Y]}(\mathcal{O}_X) = \mathcal{D}_X/\mathcal{I}$ with $Y := \{(x,y,z) \mid x^3 - y^2z^2 = 0\}$, where $\mathcal{I}$ is a left ideal of $\mathcal{D}_X$ generated by

$$\begin{array}{lll} x^3 - z^2y^2, & 2x\partial_x + 3y\partial_y + 6, & \\ 2z^2y\partial_x + 3x^2\partial_y, & 2zy^2\partial_x + 3x^2\partial_z, & -y\partial_y + z\partial_z, \\ 2z^3\partial_z\partial_x + 3x^2\partial_y^2 + 2z^2\partial_x, & x^3\partial_y^2 - z^4\partial_z^2 - 4z^3\partial_z - 2z^2. & x^3\partial_y - z^3y\partial_z - 2z^2y, \end{array}$$

It is also possible (in generic cases) to compute $\mathcal{H}^j_{[Y]}(\mathcal{M})$ for algebraic set Y of codimension greater than one. For example, let $f_1(x), f_2(x)$ be two polynomials and put

$$Y_i := \{x \in X \mid f_i(x) = 0\} \quad (i = 1,2),$$
$$Y := Y_1 \cap Y_2.$$

Assume that $\mathcal{H}^j_{[Y_1]}(\mathcal{M}) = 0$ for $j \neq j_0$. Then we can compute

$$\mathcal{H}^j_{[Y]}(\mathcal{M}) = \mathcal{H}^{j-j_0}_{[Y_2]}\left(\mathcal{H}^{j_0}_{[Y_1]}(\mathcal{M})\right)$$

explicitly by applying the above method first to f_1 and $\mathcal{M}$, then to f_2 and $\mathcal{H}^{j_0}_{[Y_1]}(\mathcal{M})$.

EXAMPLE 6.18. Put $X = K^3$, $f_1 := x^2 - y^3$, $f_2 := y^2 - z^3$, and consider the space curve $Y := \{(x,y,z) \in X \mid f_1(x,y,z) = f_2(x,y,z) = 0\}$. Then we have $\mathcal{H}^j_{[Y]}(\mathcal{O}_X) = 0$ for $j \neq 2$ and

$$\mathcal{H}^2_{[Y]}(\mathcal{O}_X) \simeq \mathcal{D}_X/\mathcal{I},$$

where $\mathcal{I}$ is the left ideal of $\mathcal{D}_X$ generated by f_1, f_2 and

$$9x\partial_x + 6y\partial_y + 4z\partial_z + 30, \qquad 9y^2z^2\partial_x + 6xz^2\partial_y + 4xy\partial_z.$$

Let u_j be the residue class of f_j^{-1} in $\mathcal{H}^1_{[Y_j]}(\mathcal{O}_X) = \mathcal{O}_X[f_j^{-1}]/\mathcal{O}_X$ with $Y_j := \{(x,y,z) \mid f_j(x,y,z) = 0\}$. Then the b-function for f_2 and u_1 is

$$(s+1)\left(s+\tfrac{1}{12}\right)\left(s+\tfrac{5}{12}\right)\left(s+\tfrac{7}{12}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{11}{12}\right)\left(s+\tfrac{7}{6}\right)$$

at $(0,0,0)$, and $s+1$ on $Y \setminus \{(0,0,0)\}$. The b-function for f_1 and u_2 is

$$(s+1)\left(s+\tfrac{7}{18}\right)\left(s+\tfrac{11}{18}\right)\left(s+\tfrac{13}{18}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{17}{18}\right)$$
$$\times\left(s+\tfrac{19}{18}\right)\left(s+\tfrac{7}{6}\right)\left(s+\tfrac{23}{18}\right)$$

at $(0,0,0)$, and $s+1$ on $Y \setminus \{(0,0,0)\}$.

7. LOCALIZATION OF A D-MODULE

We retain the notation of the preceding section. Our primary goal in this section is to obtain an algorithm for computing the localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$ as a left $\mathcal{D}_X$-module under the assumption $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. For this purpose, we shall first compute

$$\mathcal{P} := \mathcal{D}_X[s](f^s \otimes u_1) + \cdots + \mathcal{D}_X[s](f^s \otimes u_r),$$

which is a left $\mathcal{D}_X[s]$-submodule of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$, and then specialize the parameter s.

PROPOSITION 7.1. *Assume $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. Then there is an algorithm to compute a set of generators on X of the left $\mathcal{D}_X[s]$-module*

$$\mathcal{Q} := \left\{ (Q_1, \dots, Q_r) \in (\mathcal{D}_X[s])^r \;\middle|\; \sum_{i=1}^r Q_i(s)(f^s \otimes u_i) = 0 \right\}.$$

Proof. By using Lemmas 6.2, 6.5 and Theorem 5.7 with $k_0 = k_1 = 0$, we get an algorithm of computing

$$\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{D}_{\tilde{X}}(\delta(t-f) \otimes u_1) + \cdots + \mathcal{D}_{\tilde{X}}(\delta(t-f) \otimes u_r)$$

as a left $\mathcal{D}_{\tilde{X}}$-module. More concretely we can get a finite subset $\tilde{\mathbf{G}} = \{P_1, \dots, P_d\}$ of $(A_{n+1})^r$ which generates the left $\mathcal{D}_{\tilde{X}}$-module

$$\tilde{\mathcal{Q}} := \left\{ (Q_1, \dots, Q_r) \in (\mathcal{D}_{\tilde{X}})^r \;\middle|\; \sum_{i=1}^r Q_i(\delta(t-f) \otimes u_i) = 0 \right\}.$$

By making use of the injectivity of $\iota \otimes 1$ (Proposition 6.13) and the relations (6.1), we get

$$\begin{aligned} \mathcal{Q} &= \left\{ (Q_1(s), \dots, Q_r(s)) \in (\mathcal{D}_X[s])^r \,\middle|\, (Q_1(-\partial_t t), \dots, Q_r(-\partial_t t)) \in \tilde{\mathcal{Q}} \right\} \\ &\simeq (\mathcal{D}_X[t\partial_t])^r \cap \tilde{\mathcal{Q}}. \end{aligned}$$

Now that we have a set of generators $\tilde{\mathbf{G}}$ of $\tilde{\mathcal{Q}}$, we can obtain a set of generators of $\mathcal{Q}$ as follows: Let x_0 and y_0 be new commutative variables independent of t, x and their derivations. For each $i = 1, \dots, d$, let $(P_i)^h \in (A_{n+1}[x_0])^r$ be the F-homogenization of P_i. Let $\mathbf{G}$ be a Gröbner basis (with respect to a term order for eliminating x_0 and y_0) of the left $A_{n+1}[x_0, y_0]$-submodule of $(A_{n+1}[x_0, y_0])^r$ which is generated by $(P_j)^h$ $(j = 1, \dots, d)$ and $(1 - x_0 y_0)e_i$ $(i = 1, \dots, r)$. Put $\mathbf{G}_0 := \mathbf{G} \cap (A_{n+1}[x_0, y_0])^r$. Then $\mathcal{Q} = (\mathcal{D}_X[t\partial_t])^r \cap \tilde{\mathcal{Q}}$ is generated by $\psi(\mathbf{G}_0)$ with the substitution $s = -t\partial_t - 1$. The proof is similar to that of Theorem 18 of [34], where the case with $r = 1$ is treated. ∎

Now let us fix an arbitrary element s_0 of K and consider the specialization $s = s_0$ of the parameter s. Let $\mathcal{L}(s_0) := \mathcal{O}_X[f^{-1}]f^{s_0}$, where f^{s_0} is regarded as a free generator. Let ρ: $\mathcal{L} \to \mathcal{L}(s_0)$ be the surjective homomorphism of left $\mathcal{D}_X$-modules defined by $\rho(g(s,x)f^{s-m}) = g(s_0,x)f^{s_0-m}$ for $g(s,x) \in \mathcal{O}_X[s, f^{-1}]$ and $m \in \mathbf{N}$. Then it is easy to see that ρ induces an isomorphism $\mathcal{L}(s_0) \simeq \mathcal{L}/(s - s_0)\mathcal{L}$ as left $\mathcal{D}_X$-modules.

Since the proof of Lemma 6.8 is also valid with s specialized to an element of K, we get the following:

LEMMA 7.2. *Let u be a section of $\mathcal{M}$ and let m be a nonnegative integer. Fix $s_0 \in K$. Then we have $f^{s_0-m} \otimes u = 0$ in $\mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M}$ if and only if $f^k u = 0$ holds in $\mathcal{M}$ with some $k \in \mathbf{N}$.*

Consider the homomorphism

$$\rho \otimes 1\colon\ \mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s] = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M}$$

and put $\mathcal{P}(s_0) := (\rho \otimes 1)(\mathcal{P})$. Our aim is to obtain an algorithm of computing $\mathcal{P}(s_0)$. Since $(s - s_0)\mathcal{P}$ is contained in the kernel of $\rho \otimes 1$, there exists a surjective homomorphism $\mathcal{P}/(s - s_0)\mathcal{P} \to \mathcal{P}(s_0)$ induced by $\rho \otimes 1$. A sufficient condition for this homomorphism to be an isomorphism is given as follows (cf. Proposition 6.2 of [16] for the case $\mathcal{M} = \mathcal{O}_X$).

PROPOSITION 7.3. *Assume that the b-function $b_i(s,p)$ for f and u_i at $p \in X$ exists for $i = 1,\dots,r$. Assume, moreover, that $b_i(s_0 - \nu) \neq 0$ for any $i = 1,\dots,r$, $\nu = 1,2,3,\dots,$ and $p \in Y$. Then the homomorphism $\mathcal{P}/(s - s_0)\mathcal{P} \to \mathcal{P}(s_0)$ is a left $\mathcal{D}_X$-module isomorphism. In particular, we have an isomorphism $\mathcal{P}(s_0) \simeq (\mathcal{D}_X)^r/\mathcal{Q}(s_0)$ with $\mathcal{Q}(s_0) := \{Q(s_0) \mid Q(s) \in \mathcal{Q}\}$.*

Proof. Let p be an arbitrary point of X and $\tilde{u} := \sum_{i=1}^r P_i(s)(f^s \otimes u_i)$ be an arbitrary element of $\mathcal{P}$ with $P_i(s) \in \mathcal{D}_X[s]_p$. Let m be the maximum of the order of each $P_i(s)$. Then there exists a germ $v(s) \in \mathcal{M}[s]_p$ so that $\tilde{u} = f^{s-m} \otimes v(s)$. Suppose $(\rho \otimes 1)(\tilde{u}) = 0$. Then there exist $w(s) \in \mathcal{M}[s]_p$ and $k \in \mathbf{N}$ so that $f^k v(s) = (s - s_0)w(s)$ in view of Lemma 7.2. Thus we have

$$\tilde{u} = (s - s_0)f^{s-m-k} \otimes w(s). \tag{7.1}$$

There exist $W_1(s),\dots,W_r(s) \in \mathcal{D}_X[s]_p$ so that $w(s) = \sum_{i=1}^r W_i(s)u_i$. Let l be the maximum of the order of each $W_i(s)$. Then, by induction on l, we can easily verify that there exists $W_i'(s) \in \mathcal{D}_X[s]_p$ of order $\leq l$ so that $f^{s-m-k} \otimes W_i(s)u_i = W_i'(s)(f^{s-m-k-l} \otimes u_i)$. Thus (7.1) reads

$$\tilde{u} = (s - s_0)\sum_{i=1}^{r} W_i'(s)\bigl(f^{s-m-k-l} \otimes u_i\bigr).$$

There exists $Q_i(s) \in \mathcal{D}_X[s]_p$ so that $Q_i(s)(f^{s+1} \otimes u_i) = b_i(s,p)f^s \otimes u_i$. Hence we have

$$Q_i(s-1) \cdots Q_i(s-m-k-l)(f^s \otimes u_i)$$
$$= b_i(s-1,p) \cdots b_i(s-m-k-l,p)\big(f^{s-m-k-l} \otimes u_i\big).$$

Since $b_i(s_0 - \nu, p) \neq 0$ for $\nu = 1,2,3,\dots$, there exist $Q_i'(s) \in \mathcal{D}_X[s]_p$ and $B(s) \in K[s]$ so that

$$B(s)f^{s-m-k-l} \otimes u_i = Q_i'(s)(f^s \otimes u_i) \qquad (i = 1,\dots,r)$$

with $B(s_0) \neq 0$. Summing up, we get

$$B(s)\tilde{u} = (s - s_0) \sum_{i=1}^{r} W_i'(s)Q_i'(s)(f^s \otimes u_i).$$

Since there exists $B'(s) \in K[s]$ such that $B(s) - B(s_0) = (s - s_0)B'(s)$, we get

$$B(s_0)\tilde{u} = (s - s_0)\left(\sum_{i=1}^{r} W_i'(s)Q_i'(s)(f^s \otimes u_i) - B'(s)\tilde{u}\right)$$
$$\in (s - s_0)\, \mathcal{P}.$$

This implies that the kernel of $\rho \otimes 1\colon \mathcal{P} \to \mathcal{P}(s_0)$ coincides with $(s - s_0)\,\mathcal{P}$ since $B(s_0) \neq 0$. This completes the proof. ∎

Thus we have obtained an algorithm for computing $\mathcal{P}(s_0)$ under the conditions of the above proposition. Note that it amounts to computing $\mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M}$ as follows.

PROPOSITION 7.4. *Under the same assumptions as in the preceding proposition, we have* $\mathcal{P}(s_0) = \mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M}$.

Proof. Let $f^{s_0-m} \otimes u$ be an arbitrary element of $(\mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M})_p$ with $u \in \mathcal{M}_p$ and $p \in Y$. Then by applying the proof of the preceding proposition with $k = l = 0$, we obtain $Q(s) \in \mathcal{D}_X[s]_p$ and $B(s) \in K[s]$ so that $Q(s)(f^s \otimes u) = B(s)f^{s-m} \otimes u$ in $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ and $B(s_0) \neq 0$. Thus we get

$$f^{s_0-m} \otimes u = B(s_0)^{-1}Q(s_0)(f^{s_0} \otimes u) \in \mathcal{P}(s_0).$$

This completes the proof. ∎

PROPOSITION 7.5. *Assume that* $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ *is specializable along* X. *Then there exists a positive integer* k_0 *so that* $\mathcal{M}[f^{-1}]$ *is isomorphic to* $(\mathcal{D}_X)^r/\mathcal{Q}(-k)$ *as left* $\mathcal{D}_X$*-module for any integer* $k \geq k_0$.

Proof. Let $b_i(s,p)$ be the b-function for f and u_i at p. In view of Propositions 4.5, 4.6 and Theorems 4.7 and 6.14, there exists a nonzero

$b(s) \in K[s]$ so that $b_i(s, p)$ divides $b(s)$ for any $i = 1, \ldots, r$ and $p \in X$. Let k_0 be the greatest positive integer, if any, such that $b(-k) = 0$. Otherwise, put $k_0 = 0$. Let k be an arbitrary integer with $k \geq k_0$. Then by Propositions 7.3 and 7.4, we have

$$\mathcal{L}(-k) \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{P}(-k) \simeq \mathcal{P}/(s+k)\mathcal{P} \simeq (\mathcal{D}_X)^r/\mathcal{Q}(-k).$$

On the other hand, $\mathcal{L}(-k) = \mathcal{O}_X[f^{-1}]f^{-k}$ is isomorphic to $\mathcal{O}_X[f^{-1}]$ as left $\mathcal{D}_X$-module. Hence $\mathcal{M}[f^{-1}]$ is isomorphic to $\mathcal{L}(-k) \otimes_{\mathcal{O}_X} \mathcal{M}$ as left $\mathcal{D}_X$-module. This completes the proof. ∎

Thus under the condition that $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X and that $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$, we have obtained an algorithm of computing $\mathcal{M}[f^{-1}]$ combining Propositions 7.1 and 7.5. More concretely, we have

$$\mathcal{M}[f^{-1}] = \sum_{i=1}^{r} \mathcal{D}_X(f^{-k_0} \otimes u_i),$$

and our algorithm computes a finite subset of $(A_n)^r$ which generates the left $\mathcal{D}_X$-module

$$\mathcal{Q}(-k_0) = \left\{ P \in \mathcal{D}_X^r \,\middle|\, \sum_{i=1}^{r} P_i(f^{-k_0} \otimes u_i) = 0 \right\}$$

on X. In particular, by applying the above argument to $\mathcal{M} := \mathcal{D}_X g^{s_2}$ with another polynomial $g \in K[s]$ and a constant $s_2 \in K$, we obtain an algorithm for computing $\mathcal{D}_X(f^{s_1} f^{s_2})$ for generic $s_1, s_2 \in K$ as follows: First, we can compute $\mathcal{D}_X g^{s_2}$ if the Bernstein-Sato polynomial $b_g(s)$ of g satisfies $b_g(s_2 - \nu) \neq 0$ for $\nu = 1, 2, 3, \ldots$ (cf. [34]). Then we have

$$(\mathcal{D}_X f^{s_1}) \otimes_{\mathcal{O}_X} (\mathcal{D}_X g^{s_2}) \simeq \mathcal{D}_X(f^{s_1} g^{s_2})$$

by virtue of Lemma 7.2, where $\mathcal{D}_X(f^{s_1} g^{s_2})$ is the left $\mathcal{D}_X$-submodule of $\mathcal{O}_X[f^{-1}, g^{-1}]f^{s_1} g^{s_2}$ generated by $f^{s_1} g^{s_2}$. Thus by applying the arguments in this section, we can compute $\mathcal{D}_X(f^{s_1} g^{s_2})$ if, in addition to the above condition, the b-function $b_{12}(s)$ for f and g^{s_2} satisfies $b_{12}(s_0 - \nu) \neq 0$ for $\nu = 1, 2, 3, \ldots$. Note that we always have $\mathcal{H}^0_{[Y]}(\mathcal{D}_X g^{s_2}) = 0$.

Hence by choosing positive integers k_1, k_2 so that $s_1 = -k_1$ and $s_2 = -k_2$ satisfy the above conditions, we get an algorithm to compute the localization $\mathcal{O}_X[f^{-1}, g^{-1}] = \mathcal{O}_X[f^{-k_1}, g^{-k_2}]$ as $\mathcal{D}_X$-module.

If we regard s_1, s_2 as indeterminates not as constants, then it is also interesting to consider the left $\mathcal{D}_X[s_1, s_2]$-module $\mathcal{D}_X[s_1, s_2]f^{s_1} g^{s_2}$. An algorithm for computing this module can be obtained by generalizing a method used in [34], or also by modifying the arguments in this section so as to be adapted to the case where $\mathcal{M}$ is a $\mathcal{D}_X[s_2]$-module. We shall discuss this problem elsewhere.

EXAMPLE 7.6. Put $X = K^3 \ni (x,y,z)$ and write $\partial_x := \partial/\partial x$, $\partial_y := \partial/\partial y$, $\partial_z := \partial/\partial z$. Put $f_1 := x^2 - y^3$ and $f_2 := y^2 - z^3$. Let $s_1, s_2 \in K$ be constants. The Bernstein-Sato polynomial of f_2 at the singular point (0,0,0) is $b_2(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$. We have $D_X f^{s_2} = D_X/I$ with the left ideal of D_X generated by

$$\partial_x,\ 3y\partial_y + 2z\partial_z - 6s_2,\quad 3z^2\partial_y + 2y\partial_z,\quad (y^2 - z^3)\partial_z + 3z^2 s_2$$

if $b_2(s_2 - \nu) \neq 0$ for any $\nu = 1,2,3,\dots$. Then the b-function for f_1 and $f_2^{s_2}$ is

$$b_{12}(s) = (s+1)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{19}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{23}{18}\right)$$
$$\times\left(s+\tfrac{2}{3}s_2+\tfrac{25}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{29}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{31}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{35}{18}\right)$$

at (0,0,0); while at the other points we have

$$b_{12}(s) = \begin{cases} (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right) & \text{on } \{(0,0,z) \mid z \neq 0\}, \\ s+1 & \text{on } \{(x,y,z) \mid x^2 - y^3 = 0,\ yz \neq 0\}, \\ 1 & \text{on } \{(x,y,z) \mid x^2 - y^3 \neq 0\}, \end{cases}$$

if s_1 satisfies $b_{12}(s_1 - \nu) \neq 0$ for any $\nu = 1,2,3,\dots$ in addition to the above condition on s_2. Under the same assumptions, we have $D_X(f_1^{s_1} f_2^{s_2}) = D_X/I(s_1,s_2)$ with the left ideal $I(s_1,s_2)$ of D_X generated by

$$\begin{cases} 9x\partial_x + 6y\partial_y + 4z\partial_z - 6(3s_1 + 2s_2), \\ (y^2 - z^3)\partial_z + 3z^2 s_2, \\ (x^2 - y^3)\partial_x - 2s_1 x, \\ 9y^2z^2\partial_x + 6xz^2\partial_y + 4xy\partial_z, \\ 3y(x^2 - y^3)\partial_y + 2z(x^2 - y^3)\partial_z + 3\left(-2s_2x^2 + (3s_1 + 2s_2)y^3\right), \\ 3z^2(x^2 - y^3)\partial_y + 2y(x^2 - y^3)\partial_z + 9s_1y^2z^2. \end{cases}$$

In particular the above assumptions are satisfied for $s_1 = s_2 = -1$. Hence we have $O_X[f_1^{-1}, f_2^{-1}] \simeq D_X/I(-1,-1)$. By regarding s_1, s_2 as indeterminates not as constants, we have also $D_X[s_1,s_2](f_1^{s_1}f_2^{s_2}) = D_X[s_1,s_2]/I(s_1,s_2)$. Then we can verify by elimination that the ideal $(I(s_1,s_2) + D_X[s_1,s_2]f_1f_2)_0 \cap K[s_1,s_2]$ of $K[s_1,s_2]$ is generated by a single element

$$b(s_1,s_2) := (s_1+1)(6s_1+5)(6s_1+7)(s_2+1)(6s_2+5)(6s_2+7)$$
$$\times(l+19)(l+23)(l+25)(l+29)(l+31)(l+35)$$
$$\times(l+37)(l+41)(l+43)(l+47)$$

with $l := 18s_1 + 12s_2$. This means that $b(s_1, s_2)$ is a minimum polynomial that satisfies a functional equation of the form $P(f_1^{s_1+1} f_2^{s_2+1}) = b(s_1, s_2) f_1^{s_1} f_2^{s_2}$ with some germ P of $\mathcal{D}_X[s_1, s_2]$ at 0 (cf. [35, 28]).

8. CORRECTNESS OF ALGORITHMS IN ANALYTIC CASE

Here we assume that K is the field $\mathbf{C}$ of complex numbers (or its subfield for actual computation). Then we can work in the analytic category rather than in the algebraic category as described so far. Let us denote by $\mathcal{O}_X^{\mathrm{an}}$ the sheaf of rings of holomorphic (complex analytic) functions on X, and by $\mathcal{D}_X^{\mathrm{an}}$ and $\mathcal{D}_{\tilde{X}}^{\mathrm{an}}$ the sheaves of rings of holomorphic differential operators on X and on $\tilde{X}$ respectively (cf. [19]). Replacing the algebraic objects by these analytic objects, we can verify that the theoretical parts are still valid. Our purpose is to show that if the inputs are algebraic, then the outputs of the algorithms presented so far provide us with the correct answers also in the analytic category.

Let $\mathcal{M} = (\mathcal{D}_{\tilde{X}})^r/\mathcal{N}$ be a coherent $\mathcal{D}_{\tilde{X}}$-module as in Sections 4 and 5 and put $\mathcal{M}^{\mathrm{an}} := \mathcal{D}_{\tilde{X}}^{\mathrm{an}} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{M}$ and $\mathcal{N}^{\mathrm{an}} := \mathcal{D}_{\tilde{X}}^{\mathrm{an}} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{N}$. Then the V-filtrations $F_k(\mathcal{D}_{\tilde{X}}^{\mathrm{an}})$, $F_k(\mathcal{M}^{\mathrm{an}})$ and $F_k(\mathcal{N}^{\mathrm{an}})$ are defined in the same way as in the algebraic case. The following lemma will be the key to the correctness proof in the analytic case.

LEMMA 8.1. *Under the notation above, we have an isomorphism*

$$F_{k_1}(\mathcal{M}^{\mathrm{an}})/F_{k_0}(\mathcal{M}^{\mathrm{an}}) \simeq \mathcal{D}_X^{\mathrm{an}}[t\partial_t] \otimes_{\mathcal{D}_X[t\partial_t]} \big(F_{k_1}(\mathcal{M})/F_{k_0}(\mathcal{M})\big)$$

as left $\mathcal{D}_X^{\mathrm{an}}$-modules for any integers $k_0 \le k_1$.

Proof. Since there is an exact sequence

$$0 \to F_{k_1}(\mathcal{N})/F_{k_0}(\mathcal{N}) \to F_{k_1}(\mathcal{D}_{\tilde{X}})^r/F_{k_0}(\mathcal{D}_{\tilde{X}})^r \to F_{k_1}(\mathcal{M})/F_{k_0}(\mathcal{M}) \to 0$$

and $\mathcal{D}_{\tilde{X}}^{\mathrm{an}}[t\partial_t]$ is faithfully flat over $\mathcal{D}_{\tilde{X}}[t\partial_t]$, it suffices to show the natural homomorphism

$$\mathcal{D}_X^{\mathrm{an}}[t\partial_t] \otimes_{\mathcal{D}_X[t\partial_t]} \big(F_{k_1}(\mathcal{N})/F_{k_0}(\mathcal{N})\big) \to F_{k_1}(\mathcal{N}^{\mathrm{an}})/F_{k_0}(\mathcal{N}^{\mathrm{an}}) \tag{8.1}$$

is an isomorphism. Since the injectivity follows from the faithfully flatness mentioned above, we have only to show that (8.1) is surjective.

Let P be a section of $F_{k_1}(\mathcal{N}^{\mathrm{an}})$ and let $P_1, \ldots, P_d$ be a set of F-involutory generators of $\mathcal{N}$. The proof of Theorem 3.16 of [33] (with trivial modification) guarantees the existence of $Q_1, \dots, Q_d \in \mathcal{D}_X^{\mathrm{an}}$ and $P_1, \dots, P_d \in \mathcal{N}$ so that $P = \sum_{i=1}^{d} Q_i P_i$ and $\mathrm{ord}_F(Q_i P_i) \le k_1$. Then we can take $Q_i' \in \mathcal{D}_X^{\mathrm{an}}[t, \partial_t]$ so that $(Q_i - Q_i')P_i \in F_{k_0}(\mathcal{N}^{\mathrm{an}})$. This proves that the homomorphism (8.1) is surjective. This completes the proof. ∎

PROPOSITION 8.2. *The b-function of $\mathcal{M}$ along X defined in the analytic category coincides with the b-function of $\mathcal{M}$ along X in the algebraic category.*

Proof. By putting $k_0 = k_1 = 0$ in the preceding lemma, we have an isomorphism

$$\mathrm{gr}_0(\mathcal{M}^{\mathrm{an}}) \simeq \mathcal{D}_X^{\mathrm{an}}[t\partial_t] \otimes_{\mathcal{D}_X[t\partial_t]} \mathrm{gr}_0(\mathcal{M}).$$

In view of Definition 4.1, this isomorphism and the faithful flatness assure the coincidence of the two definitions of the b-function. ∎

In particular, the specializability does not depend on the (algebraic or analytic) category which one works in. The restriction $(\mathcal{M}^{\mathrm{an}})_X^{\bullet}$ of $\mathcal{M}^{\mathrm{an}}$ along X is defined as a complex of left $\mathcal{D}_X^{\mathrm{an}}$-modules.

PROPOSITION 8.3. *Assume that $\mathcal{M}$ is specializable along X. Then we have an isomorphism*

$$\mathcal{H}^j\big((\mathcal{M}^{\mathrm{an}})_X^{\bullet}\big) \simeq \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{H}^j(\mathcal{M}_X^{\bullet})$$

of left $\mathcal{D}_X^{\mathrm{an}}$-modules for $j = 0, -1$.

Proof. By virtue of the above proposition, Proposition 5.2 holds also for $\mathcal{M}^{\mathrm{an}}$ with the same k_0, k_1. Hence the above isomorphism is an immediate consequence of Lemma 8.1. ∎

Now let $\mathcal{M} = (\mathcal{D}_X)^r/\mathcal{N}$ be a coherent $\mathcal{D}_X$-module as in Sections 6 and 7 and put $\mathcal{M}^{\mathrm{an}} := \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}$. Let $f \in K[s]$ be a nonconstant polynomial. Then the algebraic local cohomology group $\mathcal{H}_{[Y]}^j(\mathcal{M}^{\mathrm{an}})$ is defined and is a left $\mathcal{D}_X^{\mathrm{an}}$-module.

PROPOSITION 8.4. *We have an isomorphism $\mathcal{H}_{[Y]}^j(\mathcal{M}^{\mathrm{an}}) \simeq \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{H}_{[Y]}^j(\mathcal{M})$ if $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X.*

Proof. Put $\mathcal{B}_{[Z]}^{\mathrm{an}} := \mathcal{D}_{\tilde{X}}^{\mathrm{an}} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{B}_{[Z]}$. Then the arguments in Section 6 are valid with $\mathcal{B}_{[Z]}$ and $\mathcal{M}$ replaced by $\mathcal{B}_{[Z]}^{\mathrm{an}}$ and $\mathcal{M}^{\mathrm{an}}$ respectively. First, by Lemmas 6.2 and 6.5 in the both categories and Proposition 8.3 applied to Δ_i instead of X, we get

$$\mathcal{B}_{[Z]}^{\mathrm{an}} \otimes_{\mathcal{O}_X^{\mathrm{an}}} \mathcal{M}^{\mathrm{an}} \simeq \mathcal{D}_{\tilde{X}}^{\mathrm{an}} \otimes_{\mathcal{D}_{\tilde{X}}} \big(\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}\big).$$

Hence Theorem 6.4 in the both categories and Proposition 8.3 yield the isomorphism needed. ∎

Especially we have $\mathcal{H}^j_{[Y]}(\mathcal{M}^{\mathrm{an}}) = 0$ if and only if $\mathcal{H}^j_{[Y]}(\mathcal{M}) = 0$ by virtue of the faithful flatness of $\mathcal{D}_X^{\mathrm{an}}$ over $\mathcal{D}_X$. Put $\mathcal{L} := \mathcal{O}_X^{\mathrm{an}}[s, f^{-1}]f^s$.

Proposition 8.5. *Let $u_1, \ldots, u_r$ be generators of $\mathcal{M}$ on X. Put*

$$\mathcal{Q} := \left\{ (Q_1, \ldots, Q_r) \in (\mathcal{D}_X[s])^r \,\middle|\, \sum_{i=1}^r Q_i(f^s \otimes u_i) = 0 \text{ in } \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \right\},$$

$$\mathcal{Q}^{\mathrm{an}} := \left\{ (Q_1, \ldots, Q_r) \in (\mathcal{D}_X^{\mathrm{an}}[s])^r \,\middle|\, \sum_{i=1}^r Q_i(f^s \otimes u_i) = 0 \right.$$

$$\left. \text{in } \mathcal{L}^{\mathrm{an}} \otimes_{\mathcal{O}_X^{\mathrm{an}}} \mathcal{M}^{\mathrm{an}} \right\}.$$

Then we have an isomorphism $\mathcal{D}_X^{\mathrm{an}}[s] \otimes_{\mathcal{D}_X[s]} \mathcal{Q} \simeq \mathcal{Q}^{\mathrm{an}}$.

Proof. By replacing $\mathcal{M}$ by $\mathcal{M}/\mathcal{H}^0_{[Y]}(\mathcal{M})$, we may assume $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ since we have $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. Put

$$\tilde{\mathcal{Q}} := \left\{ (Q_1, \ldots, Q_r) \in (\mathcal{D}_{\tilde{X}})^r \,\middle|\, \sum_{i=1}^r Q_i(\delta(t-f) \otimes u_i) = 0 \right.$$

$$\left. \text{in } \mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \right\},$$

$$\tilde{\mathcal{Q}}^{\mathrm{an}} := \left\{ (Q_1, \ldots, Q_r) \in (\mathcal{D}_{\tilde{X}}^{\mathrm{an}})^r \,\middle|\, \sum_{i=1}^r Q_i(\delta(t-f) \otimes u_i) = 0 \right.$$

$$\left. \text{in } \mathcal{B}_{[Z]}^{\mathrm{an}} \otimes_{\mathcal{O}_X^{\mathrm{an}}} \mathcal{M}^{\mathrm{an}} \right\}.$$

Then by the proof of Proposition 7.1 and the faithful flatness, we get

$$\begin{aligned} \mathcal{Q}^{\mathrm{an}} &\simeq (\mathcal{D}_X^{\mathrm{an}}[t\partial_t])^r \cap \tilde{\mathcal{Q}}^{\mathrm{an}} \\ &\simeq \mathcal{D}_X^{\mathrm{an}}[t\partial_t] \otimes_{\mathcal{D}_X[t\partial_t]} \left((\mathcal{D}_X[t\partial_t])^r \cap \tilde{\mathcal{Q}} \right) \\ &\simeq \mathcal{D}_X^{\mathrm{an}}[t\partial_t] \otimes_{\mathcal{D}_X[t\partial_t]} \mathcal{Q}. \end{aligned}$$

This completes the proof. ∎

It follows immediately

$$\sum_{i=1}^r \mathcal{D}_X^{\mathrm{an}}[s](f^s \otimes u_i) \simeq \mathcal{D}_X^{\mathrm{an}}[s] \otimes_{\mathcal{D}_X[s]} \left(\sum_{i=1}^r \mathcal{D}_X[s](f^s \otimes u_i) \right).$$

By specializing s, we also get $\mathcal{D}_X^{\mathrm{an}} f^{s_0} \otimes_{\mathcal{O}_X^{\mathrm{an}}} \mathcal{M} \simeq \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} (\mathcal{D}_X f^{s_0} \otimes_{\mathcal{O}_X} \mathcal{M})$.

COROLLARY 8.6. *Let u be a section of $\mathcal{M}$. Then the b-functions for f and u in the algebraic and in the analytic sense coincide.*

Proof. Let $b^{an}(s)$ and $b(s)$ be the b-functions for f and u in the analytic and in the algebraic sense respectively. By using the above proposition with $r = 1$ and $u_1 = u$ and the faithful flatness, we get

$$\begin{aligned} \langle b^{an}(s) \rangle &= (\mathcal{Q}^{an} + \mathcal{D}_X^{an}[s]f) \cap K[s] \\ &= (\mathcal{Q} + \mathcal{D}_X[s]f) \cap K[s] \\ &= \langle b(s) \rangle . \end{aligned}$$

This completes the proof. ∎

Thus we have proved that the algorithms in the present paper are correct also in the analytic category if the input $\mathcal{D}_X^{an}$-module is written in the form $\mathcal{M}^{an} = \mathcal{D}_X^{an} \otimes_{\mathcal{D}_X} \mathcal{M}$ with a coherent $\mathcal{D}_X$-module $\mathcal{M}$ whose presentation is explicitly given.

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