Algorithms for D-modules applied to generalized functions

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Holonomic distributions

Definition

Let $C_0^\infty(U)$ be the set of the C^∞ functions on an open set U of $\mathbb{R}^n$ with compact support. A distribution u on U is a linear mapping

$$u : C_0^\infty(U) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

such that $\lim_{j\to\infty} \langle u, \varphi_j \rangle = 0$ holds for a sequence $\{\varphi_j\}$ of $C_0^\infty(U)$ if there is a compact set $K \subset U$ such that $\varphi_j = 0$ on $U \setminus K$ and

$$\lim_{j\to\infty} \sup_{x\in U} |\partial^\alpha \varphi_j(x)| = 0 \quad \text{for any } \alpha \in \mathbb{N}^n,$$

where $x = (x_1, \dots, x_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$. The set of the distributions on U is denoted by $\mathcal{D}'(U)$.

The derivative $\partial_k u$ of u with respect to x_k is defined by

$$\langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

For a C^∞ function a on U, the product au is defined by

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

In particular, by these actions of the derivations and the polynomial multiplications, $\mathcal{D}'(U)$ has a natural structure of left D_n-module, where

$$D_n = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

is the ring of differential operators with polynomial coefficients.

Definition

Let u be a C^∞ function or a distribution defined on an open subset U of $\mathbb{R}^n$. Then we call u a *holonomic function* or a *holonomic distribution* on U if u satisfies a holonomic system. In other words, u is holonomic if and only if its *annihilator*

$$\mathrm{Ann}_{D_n} u := \{P \in D_n \mid Pu = 0 \text{ on } U\}$$

is a holonomic ideal.

Example: Dirac's delta function $\delta(x)$ is the distribution defined by

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0) \quad (\forall \varphi \in C_0^\infty(\mathbb{R})).$$

$\delta(x)$ is holonomic since $x\delta(x) = 0$. (In fact, $\mathrm{Ann}_{D_1}\delta(x) = D_1 x$.)

Power product of polynomials as distribution

Let $f_1, \dots, f_p$ be polynomials with real coefficients. We assume that the set $\{x \in \mathbb{R}^n \mid f_i(x) > 0\,(i = 1, \dots, p)\}$ is not empty. Then the distribution $v = (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$ on $\mathbb{R}^n$ is defined to be

$$\langle v, \varphi \rangle = \int_{f_1 \geq 0, \dots, f_p \geq 0} f_1(x)^{\lambda_1} \cdots f_p(x)^{\lambda_p} \varphi(x)\, dx$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$ if $\mathrm{Re}\ \lambda_i \geq 0$ for each i. Moreover, v, that is, $\langle v, \varphi \rangle$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, is holomorphic in $(\lambda_1, \dots, \lambda_p)$ on the domain

$$\Omega_+ := \{(\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p \mid \mathrm{Re}\ \lambda_i > 0 \quad (i = 1, \dots, p)\}$$

and is continuous in $(\lambda_1, \dots, \lambda_p)$ on the closure of Ω_+.

Application to definite integrals

In particular,

$$(f_1)_+^0 \cdots (f_p)_+^0 = Y(f_1) \cdots Y(f_p),$$

where $Y(t)$ is the Heaviside function; i.e., $Y(t) = 1$ for $t > 0$ and $Y(t) = 0$ for $t \leq 0$. Then for a holonomic function $u(x)$, we have

$$\begin{aligned} v(x_1, \ldots, x_{n-d}) &= \int_{f_1 \geq 0, \ldots, f_p \geq 0} u(x)\, dx_{n-d+1} \cdots dx_n \\ &= \int_{\mathbb{R}^d} Y(f_1) \cdots Y(f_p) u(x)\, dx_{n-d+1} \cdots dx_n \end{aligned}$$

if this integral is well-defined. A holonomic system for this v is computable by the D-module theoretic integration algorithm if a holonomic system for $Y(f_1) \cdots Y(f_p) u(x)$ is obtained.

Bernstein-Sato ideals

It is known that there exist a non-zero polynomial $b(s)$ in $s = (s_1, \ldots, s_m)$ and an operator $P(s) \in D_n[s]$ such that

$$P(s)(f_1)_+^{s_1+1} \cdots (f_m)_+^{s_m+1} = b(s)(f_1)_+^{s_1} \cdots (f_m)_+^{s_m},$$

$$b(s) = \prod_{i=1}^{\nu} (c_{i1}s_1 + \cdots + c_{im}s_m + c_i)$$

with positive integers c_{ij} and positive rational numbers c_i. (Sabbah, 1987) There are several algorithms to compute the ideal consisting of such $b(s)$, which is called the Bernstein-Sato ideal.

By using this functional equation, we can extend $v(x, s)$ to a distribution in x which is meromorphic in $(\lambda_1, \ldots, \lambda_p)$ on the whole $\mathbb{C}^p$; $v(x, s)$ is holomorphic in $(\lambda_1, \ldots, \lambda_p)$ on

$$\Omega(f_1, \ldots, f_p) := \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \mid \\ b(\lambda_1 + k, \ldots, \lambda_p + k) \neq 0 \text{ for any } k \in \mathbb{N}\}.$$

Example 1 The Bernstein-Sato ideal for yz, zx, xy is generated by

$$(s_2+s_3+1)(s_2+s_3+2)(s_1+s_3+1)(s_1+s_3+2)(s_1+s_2+1)(s_1+s_2+2)$$

Example 2 The Bernstein-Sato ideal for $x^3 - y^2, y^3 - z^2$ is

$$\langle(s_2+1)(6s_2+5)(6s_2+7)(s_1+1)(6s_1+5)(6s_1+7)(12s_1+18s_2+19)$$
$$(12s_1+18s_2+23)(12s_1+18s_2+25)(12s_1+18s_2+29)$$
$$(12s_1+18s_2+31)(12s_1+18s_2+35)(12s_1+18s_2+37)$$
$$(12s_1+18s_2+41)(12s_1+18s_2+43)(12s_1+18s_2+47)\rangle$$

Difference-differential equations

The distribution $v(x,s) := (f_1)_+^{s_1} \cdots (f_p)_+^{s_p}$ satisfies a system of difference-differential equations

$$\begin{aligned} &(E_{s_j} - f_j(x))v(x,s) = 0 \quad (j = 1, \dots, p), \\ &\left(\partial_{x_i} - \sum_{j=1}^{p} s_j E_{s_j}^{-1} \frac{\partial f_j}{\partial x_i}\right) v(x,s) = 0 \quad (i = 1, \dots, n) \\ &\text{with} \quad E_{s_j} v(x, s_1, \dots, s_p) = v(x, s_1, \dots, s_j + 1, \dots, s_p) \end{aligned}$$

Note that the inverse shift $E_{s_j}^{-1}$ 'operates' on $v(x,s)$ but it reduces the domain of $v(x,s)$ w.r.t. s.

Mellin transform

$v(x, s) = (f_1)_+^{s_1} \cdots (f_p)_+^{s_p}$ can be expressed (at least formally) as the Mellin transform

$$v(x, s) = \int_{\mathbb{R}^p} (t_1)_+^{s_1} \cdots (t_p)_+^{s_p} w(t, x)\, dt_1 \cdots dt_p.$$

of

$$w(t, x) := \delta(t_1 - f_1(x)) \cdots \delta(t_p - f_p(x)).$$

We have (at least formally)

$$E_{s_j} v(x, s) = \int_{\mathbb{R}^p} (t_1)_+^{s_1} \cdots (t_p)_+^{s_p} t_j w(t, x)\, dt_1 \cdots dt_p$$

$$s_j v(x, s) = -\int_{\mathbb{R}^p} (t_1)_+^{s_1} \cdots (t_p)_+^{s_p} \partial_{t_j} t_j w(t, x)\, dt_1 \cdots dt_p$$

Let $D_n\langle s, E_s\rangle$ be the ring of difference-differential operators generated by $(x, \partial, s, E_s) = (x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}, s_1, \dots, s_p, E_{s_1}, \dots, E_{s_p})$. Let $D_{n+p} = D_n\langle t, \partial_t\rangle$ be the Weyl algebra over the variables $(x, t) = (x_1, \dots, x_n, t_1, \dots, t_p)$. We define a ring homomorphism

$$\mu : D_n\langle s, E_s\rangle \longrightarrow D_{n+p}$$

by $\mu(s_j) = -\partial_{t_j} t_j$, $\mu(E_{s_j}) = t_j$. Since μ is injective, we regard $D_n\langle s, E_s\rangle$ as a subring of D_{n+p}. Moreover, we can regard

$$D_n\langle s, E_s\rangle \subset D_{n+p} \subset D_n\langle s, E_s, E_s^{-1}\rangle$$

with identifications $t_j = E_{s_j}$, $\partial_{t_j} = -s_j E_{s_j}^{-1}$.

A holonomic system for $v(x, s)$

In terms of the ring extension $E\langle s, E_s\rangle \subset D_{n+p}$,
$v(x,s) = (f_1)_+^{s_1} \cdots (f_p)_+^{s_p}$ satisfies a holonomic system

$$(t_j - f_j(x))v(x,s) = 0 \quad (j = 1, \dots, p),$$
$$\left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j}\right) v(x,s) = 0 \quad (i = 1, \dots, n)$$

The Bernstein-Sato polynomial for the variety $f_1 = \cdots = f_p = 0$ (Budur-Mustata-Saito 2006) can be computed as the b-function (or the indicial polynomial) (with a shift) of this holonomic system along the submanifold $t_1 = \cdots = t_p = 0$.

Example 1: The Bernstein-Sato polynomial of the variety $yz = zx = xy = 0$:

```
bfct_variety([y*z,z*x,x*y],[x,y,z]);
[[-1,1],[s+2,2],[2*s+3,1]]
```

Example 2: The Bernstein-Sato polynomial of the variety $x^3 - y^2 = y^3 - z^2 = 0$:

```
bfct_variety([x^3-y^2,y^3-z^2],[x,y,z]);
[[-1,1],[s+2,1],[3*s+4,1],[3*s+5,1],[6*s+11,1],
[6*s+13,1],[18*s+25,1],[18*s+29,1],[18*s+31,1],
[18*s+35,1],[18*s+37,1],[18*s+41,1]]
```

Computation was done by using Risa/Asir.

A holonomic system for the product

Let $u(x)$ be a holonomic function and suppose that the product $u(x)v(x,s) = u(x)(f_1)_+^{s_1} \cdots (f_p)_+^{s_p}$ is well-defined as distribution. Then a holonomic system for $u(x)v(x,s)$ is obtained as follows:

Input: A set G_0 of generators of a holonomic ideal I of D_n annihilating a distribution $u(x)$, and polynomials $f_1, \dots, f_p \in \mathbb{R}[x]$.
Output: A set G of generators of a holonomic ideal J of D_{n+p} annihilating $u(f_1)_+^{s_1} \cdots (f_p)_+^{s_p}$.
1. For each $P = P(x, \partial_{x_1}, \dots, \partial_{x_n}) \in G_0$, set

$$\tau(P) := P\left(x, \partial_{x_1} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\partial_{t_j}, \dots, \partial_{x_n} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_n}\partial_{t_j}\right).$$

2. Set $G := \{\tau(P) \mid P \in G_0\} \cup \{t_j - f_j(x) \mid j = 1, \dots, p\}$.

Difference-differential equations for integrals

Consider the integral

$$\tilde{u}(x', s) := \int_{\mathbb{R}^d} u(x)v(x, s)\, dx_{n-d+1} \cdots dx_n$$

with $x' = (x_1, \ldots, x_{n-d})$. A 'holonomic system' of difference-differential equations for $\tilde{u}(x', s)$ can be computed as follows:

1. Compute the D-module theoretic integration ideal

$$\tilde{J} := (\partial_{x_{n-d+1}} D_{n+p} + \cdots + \partial_{x_n} D_{n+p} + J) \cap D_{n-d+p}$$

with $D_{n-d+p} = \mathbb{C}\langle x', \partial_{x'}, t, \partial_t \rangle$ and the holonomic ideal J annihilating $u(x)v(x, s)$.

2. Compute the intersection $J \cap D_{n-d}\langle s, E_s \rangle$, which annihilates $\tilde{u}(x', s)$.

An example of the local ζ-function

$$\tilde{u}(s) := \int_{\mathbb{R}^2} e^{-x^2-y^2}(x^3-y^2)_+^s \, dxdy$$

satisfies a difference equation

$$\begin{aligned}&(-32E_s^4 + 16(4s+13)E_s^3 + 4(s+3)(27s^2+154s+211)E_s^2 \\ &-6(s+2)(s+3)(36s^2+162s+173)E_s \\ &+3(s+1)(s+2)(s+3)(6s+5)(6s+13))\tilde{u}(s) = 0.\end{aligned}$$

It follows that $v(s)$ is holomorphic (at least) on

$$\Omega' := \{s \in \mathbb{C} \mid x \neq -\nu,\ -\frac{5}{6}-\nu,\ -\frac{13}{6}-\nu \quad (\nu = 1, 2, 3, \dots)\}.$$

2nd Part: An algorithm for Laurent coefficients of f_+^λ

For a polynomial $f(x) = f(x_1, \dots, x_n)$ with real coefficients and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\ \lambda \geq 0$,
$f_+^\lambda \in \mathcal{S}'(\mathbb{R}^n)$ (the set of tempered distributions on $\mathbb{R}^n$) is defined by

$$\langle f_+^\lambda, \varphi \rangle = \int_{f(x)>0} f(x)^\lambda \varphi(x)\, dx$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$: the set of rapidly decreasing functions, i.e.,
$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \quad (\forall \alpha, \beta \in \mathbb{N}^n)$.
In particular, $Y(f) := f_+^0$ is the Heaviside function w.r.t. f.

Functional equations

- D_n: the ring of differential operators with polynomial coefficients:

$D_n = \{P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial^\alpha$ (finite sum) $\mid a_\alpha(x) \in \mathbb{C}[x] =$
$C[x_1, \ldots, x_n]\}$ with $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_i = \partial/\partial x_i$

Theorem (J. Bernstein)

There exist $P(s) \in D_n[s]$ and $b_f(s) \in \mathbb{C}[s]$ such that

$$P(s)f^{s+1} = b_f(s)f^s$$

holds formally and $b_f(s) \neq 0$ is of minimum degree (the **Bernstein-Sato polynomial**, or the b-**function** of f).

Theorem (Kashiwara)

The roots of $b_f(s)$ are negative rational numbers.

Comparison between f^s and f_+^λ

Proposition

Assume that $\lambda = \lambda_0 \in \mathbb{C}$ is not a pole of f_+^λ and the set $\{x \in \mathbb{R}^n \mid f(x) > 0\}$ is non-empty. Consider the following three conditions on $P \in D_n$:

(1) There exists $Q(s) \in D_n[s]$ such that $Q(s)f^s = 0$ and $P = Q(\lambda_0)$.

(2) $Pf_+^{\lambda_0} = 0$ as distribution on $\mathbb{R}^n$.

(3) $Pf^{\lambda_0} = 0$ as multi-valued analytic function on $\{z \in \mathbb{C}^n \mid f(z) \neq 0\}$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds. Moerover, if $b_f(\lambda_0 - \nu) \neq 0$ for $\forall \nu = 1, 2, 3, \ldots$, then (3) $\Rightarrow$ (1) holds.

Comparison between f^s and f_+^λ — examples

(3) $\Rightarrow$ (2) does not hold in general.

Example 1 Consider $f = x = x_1$ with $n = 1$. Then $\partial f^0 = \partial 1 = 0$ as analytic function but $\partial f_+^0 = \delta(x)$.

Example 2 (cf. Björk: 'Rings of Differential Operators') Consider $f = x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $n = 4$. The b-function is $b_f(s) = (s+1)(s+2)$. The distribution f_+^λ is holomorphic in $\mathbb{C} \setminus \{-2, -3, -4, \dots\}$ with respect to λ. Set $P = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2$. Then $Pf^{-1} = 0$ as analytic function but Pf_+^{-1} is not zero.

Analytic continuation and Laurent expansion

Let $\lambda_1, \ldots, \lambda_N$ be the distinct roots of $b_f(s) = 0$. Then by using the functional equation $b(\lambda)f_+^\lambda = P(\lambda)f_+^{\lambda+1}$, the distribution f_+^λ is extended to a $\mathcal{S}'(\mathbb{R}^n)$-valued holomorphic function on

$$\Omega_f := \{\lambda \in \mathbb{C} \mid \lambda \neq \lambda_j - \nu \quad (1 \leq j \leq N,\ \nu = 0, 1, 2, \ldots)\}$$

with each $\lambda_j - \nu$ being at most a pole. Choose $\lambda_0 \in \mathbb{C} \setminus \Omega_f$. Then f_+^λ can be expressed as a Laurent series

$$f_+^\lambda = \sum_{k=-l}^{\infty} (\lambda - \lambda_0)^j u_k$$

with $u_k \in \mathcal{S}'(\mathbb{R}^n)$ and $l \in \mathbb{N}$. In particular, u_{-1} is called the *residue* of f_+^λ at λ_0, which we denote by $\mathrm{Res}_{\lambda=\lambda_0} f_+^\lambda$.

Non-singular case

• If $f = 0$ is non-singular, then f_+^λ has only simple poles at negative integers with

$$\mathrm{Res}_{\lambda=-k-1}\, f_+^\lambda = \frac{(-1)^k}{k!}\delta^{(k)}(f) \quad (k = 0, 1, 2, \dots).$$

$\delta(f)$ represents the layer (the Dirac delta function) associated with the hypersurface $f = 0$,
$\delta^{(1)}(f) = \delta'(f)$ represents the double layer (dipole),...
Cf. Gelfand-Shilov: 'Generalized Functions, Vol. 1'

Singular case

Definition

For a non-negative integer k, set

$$\delta_+^{(k)}(f) := (-1)^k k! \operatorname{Res}_{\lambda=-k-1} f_+^\lambda,$$
$$\delta_-^{(k)}(f) := k! \operatorname{Res}_{\lambda=-k-1} f_-^\lambda = k! \operatorname{Res}_{\lambda=-k-1} (-f)_+^\lambda = (-1)^k \delta_+^{(k)}(-f).$$

Then we have

Proposition

(1) $f^{k+1}\delta_\pm^{(k)}(f) = 0 \quad (k \geq 0)$.

(2) $\dfrac{\partial}{\partial x_i} Y(\pm f) = \dfrac{\partial f}{\partial x_i}\delta_\pm(f)$ for $i = 1, \dots, n$.

(3) $f\delta_\pm^{(k)}(f) = -k\delta_\pm^{(k-1)}(f) \quad (k \geq 1)$.

Algorithm

Aim

Compute a holonomic system for the Laurent coefficient u_k $(k \in \mathbb{Z})$ for f_+^λ about λ_0. (i.e. to find a left ideal $I \subset \mathrm{Ann}_{D_n} u_k$ such that D_n/I is holonomic.)

Step 1

(1) Take $m \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that $\mathrm{Re}\,\lambda_0 + m \geq 0$.

(2) Find a functional equation $b_f(s) f^s = P(s) f^{s+1}$.

(3) $Q(s) := P(s)P(s+1)\cdots P(s+m-1)$,
$b(s) := b_f(s)b_f(s+1)\cdots b_f(s+m-1)$.
Then we have $b(\lambda) f_+^\lambda = Q(\lambda) f_+^{\lambda+m}$.

Step 2

Factorize $b(s)$ as $b(s) = c(s)(s-\lambda_0)^l$ with $c(\lambda_0) \neq 0$ and $l \in \mathbb{N}$. Then we have

$$f_+^\lambda = (\lambda-\lambda_0)^{-l} c(\lambda)^{-1} Q(\lambda) f_+^{\lambda+m} = \sum_{k=-l}^{\infty} (\lambda-\lambda_0)^k u_k(x),$$

where $u_k(x) \in \mathcal{S}'(\mathbb{R}^n)$ are given by

$$\begin{aligned} u_k(x) &= \frac{1}{(l+k)!} \left[\left(\frac{\partial}{\partial\lambda}\right)^{l+k} (c(\lambda)^{-1} Q(\lambda) f_+^{\lambda+m}) \right]_{\lambda=\lambda_0} \\ &= \sum_{j=0}^{l+k} Q_j (f_+^{\lambda_0+m} (\log f)^j) \end{aligned}$$

$$\text{with} \quad Q_j := \frac{1}{j!(l+k-j)!} \left[\left(\frac{\partial}{\partial\lambda}\right)^{l+k-j} (c(\lambda)^{-1} Q(\lambda)) \right]_{\lambda=\lambda_0}.$$

Algorithm (continued)

Step 3

Compute a holonomic system for $(f_+^\lambda, \dots, f_+^\lambda(\log f)^{k+l})$ as follows:

(1) Compute a set G_0 of generators of the annihilator $\mathrm{Ann}_{D_n[s]} f^s$.

(2) Let $e_1 = (1, 0, \dots, 0)$, $\cdots$, $e_{k+l} = (0, \dots, 0, 1)$ be the canonical basis of $\mathbb{Z}^{k+l+1}$. For each $P(s) \in G_0$ and an integer j with $0 \le j \le k + l$, set

$$P^{(j)}(s) := \sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} P(s)}{\partial s^{j-i}} e_{i+1} \in (D_n[s])^{k+l+1}.$$

(3) Set $G_1 := \{P^{(j)}(\lambda_0 + m) \mid P(s) \in G_0,\ 0 \le j \le k + l\}$.

The output G_1 of Step 3 generates a left D_n-module N such that $(D_n)^{k+l+1}/N$ is holonomic and

$$P_0 f_+^{\lambda_0+m} + P_1(f_+^{\lambda_0+m}\log f) + \cdots + P_{k+l}(f_+^{\lambda_0+m}(\log f)^{k+l}) = 0$$

holds for any $P = (P_0, \ldots, P_{k+l}) \in G_1$.

Remark Step 3 is essentially differentiation of the equations

$$P(s)f_+^s = 0 \quad (P(s) \in \mathrm{Ann}_{D_n[s]} f^s)$$

with respect to s.

Algorithm (the final step)

Step 4

Let N be the left D_n-submodule of $(D_n)^{l+k+1}$ generated by the output G_1 of Step 3 and let $Q_0, Q_1, \ldots, Q_{l+k}$ be the operators computed in Step 2. Compute a set G_2 of generators of the left ideal

$$I := \{P \in D_n \mid (PQ_0, PQ_1, \ldots, PQ_{l+k}) \in N\}$$

by using quotient or syzygy computation.

Output

The ideal I annihilates the distribution u_k and D_n/I is holonomic.

Holonomicity of the output

Theorem

Let I be the left ideal of D_n computed by the preceding algorithm. Then D_n/I is holonomic.

Sketch of the proof:
(1) The left D_n-module $(D_n)^{k+l+1}/N$ is holonomic. In fact, set

$$N_j := \{(P_0, \dots, P_j, 0, \dots, 0) \in N\}.$$

Then $N_j/N_{j-1} \simeq \mathrm{Ann}_{D_n[s]} f^s/(s - \lambda_0 - m)\mathrm{Ann}_{D_n[s]} f^s$ is holonomic.

(2) D_n/I with $I := \{P \in D_n \mid (PQ_0, PQ_1, \dots, PQ_{l+k}) \in N\}$ is holonomic since the map $h : D_n/I \to (D_n)^{k+l+1}/N$ defined by $h([P]) = (PQ_0, \dots, PQ_{k+l+1})$ is an injective homomorphism of left D_n-modules.

An example: $f = x_1^2 - x_2^2$

- The functional equation is $(\lambda+1)^2 f_+^\lambda = \frac{1}{4}(\partial_1^2 - \partial_2^2) f_+^{\lambda+1}$
$\Rightarrow$ f_+^λ has poles (of order at most 2) only at $\lambda = -1, -2, -3, \dots$.
- The Laurent expansion around $\lambda = -1$ is

$$f_+^\lambda = (\lambda+1)^{-2} u_{-2}(x) + (\lambda+1)^{-1} u_{-1}(x) + u_0(x) + (\lambda+1) u_1(x) + \cdots$$

with

$$u_{-2}(x) = \frac{1}{4}(\partial_1^2 - \partial_2^2) f_+^0 = \frac{1}{4}(\partial_1^2 - \partial_2^2) Y(f),$$
$$u_{-1}(x) = \frac{1}{4}(\partial_1^2 - \partial_2^2)(Y(f)\log f).$$

Differentiating

$$(x_2\partial_1 + x_1\partial_2)f_+^s = (x_1\partial_1 + x_2\partial_2 - 2s)f_+^s = 0$$

with respect to s, we get

$$\begin{aligned}
&(x_2\partial_1 + x_1\partial_2)f_+^s = 0, \quad (x_2\partial_1 + x_1\partial_2)(f_+^s \log f) = 0,\\
&2f^s + (x_1\partial_1 + x_2\partial_2 - 2s)(f_+^s \log f) = 0,\\
&(x_1\partial_1 + x_2\partial_2 - 2s)f_+^s = 0.
\end{aligned}$$

Hence $(Y(f),\ Y(f)\log f)$ satisfies a holonomic system

$$\begin{aligned}
&(x_2\partial_1 + x_1\partial_2)Y(f) = 0, \quad (x_2\partial_1 + x_1\partial_2)(Y(f)\log f) = 0,\\
&2Y(f) + (x_1\partial_1 + x_2\partial_2)(Y(f)\log f) = 0,\\
&(x_1\partial_1 + x_2\partial_2)Y(f) = 0.
\end{aligned}$$

Let N be the left D_2-submodule of D_2^2 genererated by these vectors of differential operators. Then

$$P \cdot (\partial_1^2 - \partial_2^2, 0) \in N \quad \Rightarrow \quad Pu_{-2} = 0,$$
$$P \cdot (0, \partial_1^2 - \partial_2^2) \in N \quad \Rightarrow \quad Pu_{-1} = 0.$$

By module quotient (via intersection or syzygy computation in D_2)

- u_{-2} satisfies

  $$x_1 u_{-2}(x) = x_2 u_{-2}(x) = 0$$

  Hence $u_{-2}(x) = c\delta(x)$ $(\exists c \in \mathbb{C})$.

- u_{-1} satisfies

  $$(x_2\partial_1 + x_1\partial_2)u_{-1}(x) = (x_1^2 - x_2^2)u_{-1}(x) = 0.$$

  (This coincides with $\mathrm{Ann}_{D_2} u_{-1}$.)

More concrete computation for isolated singularity

Assume that the origin $0 \in \mathbb{C}^n$ is an isolated singularity of the hypersurface $\{z \in \mathbb{C}^n \mid f(z) = 0\}$ and $\lambda_0 = \lambda - \nu$ with $b_f(\lambda) = 0$ and $\nu = 0, 1, 2, \dots$. Assume moreover (1) $\lambda \neq -1, -2, \dots$ and $k \leq -1$, or (2) $\lambda = -1, -2, \dots$ and $k \leq -2$.
Then u_k can be expressed, in a neighborhood of 0, as a finite sum

$$u_k(x) = \sum_{\alpha} c_\alpha \partial_x^\alpha \delta(x) \quad (c_k \in \mathbb{C}).$$

Aim: Compute this expression as explicitly as possible.

Algorithm

Step 1. Let I be a left ideal of D_n annihilating u_k such that D_n/I is holonomic.
Step 2. Compute the b-function $b(s)$ of I w.r.t. the weight vector $(-1,\ldots,-1;1,\ldots,1)$. That is, there exists $Q \in D_n$ with weight $\mathrm{ord}_{(-1,1)} Q < 0$ such that $b(x_1\partial_1 + \cdots + x_n\partial_n) + Q \in I$.
Step 3. Let $s = m$ be the greatest integer root of $b(s) = 0$ if any. If $m > -n$, or there is no integer root, then $u_k(x) = 0$.
Step 4. $u_k(x)$ is written in the form

$$u_k(x) = \sum_{|\alpha| \leq -m-n} c_\alpha \partial^\alpha \delta(x).$$

Algorithm (continued)

Step 5. Let $\{P_1, \dots, P_l\}$ be a set of generators of the left ideal I. Then the differential equations

$$x^\beta P_j u_k = \sum_{|\alpha| \le -m-n} c_\alpha x^\beta P_j \partial^\alpha \delta(x) = 0$$

$$(j = 1, \dots, l,\ |\beta| \le m + \mathrm{ord}_{(-1,1)} P_j)$$

yield homogeneous linear equations for c_α.

Example: $f = x^3 - y^2$

The b-function of f is $(s+1)(s+\frac{5}{6})(s+\frac{7}{6})$. $u := \mathrm{Res}_{\lambda=-\frac{7}{6}} f_+^\lambda$ satisfies

$$yu = (x\partial_x + 2)u = x^2 u = 0.$$

The b-function with respect to $(-1,1)$ is $s+3$. Hence u is written in the form

$$u = c_0\delta(x,y) + c_1\partial_x\delta(x,y) + c_2\partial_y\delta(x,y).$$

We get $u = c_1\partial_x\delta(x,y)$.
Hence

$$\tilde{u}(s) := \int_{\mathbb{R}^2} e^{-x^2-y^2}(x^3-y^2)_+^s \, dxdy$$

is analytic at $s = -\dfrac{7}{6}$ (since $\partial_x e^{-x^2-y^2}$ vanishes at $(0,0)$).