RIMS Kôkyûroku Bessatsu
Bx (201x), 000–000

Annihilators of Laurent coefficients of the complex power for normal crossing singularity

By

Toshinori OAKU *

Abstract

Let f be a real-valued real analytic function defined on an open set of $\mathbb{R}^n$. Then the complex power f_+^λ is defined as a distribution with a holomorphic parameter λ. We determine the annihilator (in the ring of differential operators) of each coefficient of the principal part of the Laurent expansion of f_+^λ about $\lambda = -1$ in case $f = 0$ has a normal crossing singularity.

§ 1. Introduction

Let $\mathcal{D}_X$ be the sheaf of linear differential operators with holomorphic coefficients on the n-dimensional complex affine space $X = \mathbb{C}^n$. We denote by $\mathcal{D}_M$ the sheaf theoretic restriction of $\mathcal{D}_X$ to the n-dimensional real affine space $M = \mathbb{R}^n$, which is the sheaf of linear differential operators whose coefficients are complex-valued real analytic functions. Let us denote by $\mathcal{D}_0 = (\mathcal{D}_M)_0$, for the sake of brevity, the stalk of $\mathcal{D}_M$ (or of $\mathcal{D}_X$) at the origin $0 \in M$, which is a (left and right) Noetherian ring.

Let $\mathcal{D}'_M$ be the sheaf on M of the distributions (generalized functions) in the sense of L. Schwartz. In general, for a sheaf $\mathcal{F}$ on M and an open subset U of M, we denote by $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ the set of the sections of $\mathcal{F}$ on U. Let $C_0^\infty(U)$ be the set of the complex-valued C^∞ functions defined on U whose support is a compact set contained in U. Then $\Gamma(U, \mathcal{D}'_M)$ consists of the $\mathbb{C}$-linear maps

$$u : C_0^\infty(U) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

Received April 20, 201x. Revised September 11, 201x.
2010 Mathematics Subject Classification(s): 32W50, 46F20
Key Words: *complex power, annihilator, distribution*
Supported by JSPS Grant-in-Aid for Scientific Research (C) 26400123
*Department of Mathematics, Tokyo Woman's Christian University, Tokyo, 167-8585, Japan.
e-mail: `oaku@lab.twcu.ac.jp`

which are continuous in the sense that $\lim_{j\to\infty}\langle u, \varphi_j\rangle = 0$ holds for any sequence $\{\varphi_j\}$ of $C_0^\infty(U)$ if there is a compact set $K \subset U$ such that $\varphi_j = 0$ on $U \setminus K$ and

$$\lim_{j\to\infty} \sup_{x\in U} |\partial^\alpha \varphi_j(x)| = 0 \quad \text{for any } \alpha \in \mathbb{N}^n,$$

where we use the notation $x = (x_1, \dots, x_n)$, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$.

For a distribution u defined on an open set U of M, its annihilator $\mathrm{Ann}_{\mathcal{D}_M} u$ in $\mathcal{D}_M$ is defined to be the sheaf of left ideals of sections P of $\mathcal{D}_M$ which annihilate u. That is, for each open subset V of U, we have by definition

$$\Gamma(V, \mathrm{Ann}_{\mathcal{D}_M} u) = \{P \in \mathcal{D}_M(V) \mid Pu = 0 \text{ on } V\}.$$

Its stalk $\mathrm{Ann}_{\mathcal{D}_0} u$ at $0 \in M$ is a left ideal of $\mathcal{D}_0$.

Now let f be a real-valued real analytic function defined on an open set U of M. Then for a complex number λ with non-negative real part ($\mathrm{Re}\,\lambda \geq 0$), the distribution f_+^λ is defined to be the locally integrable function

$$f_+^\lambda(x) := \begin{cases} f(x)^\lambda = \exp(\lambda \log f(x)) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

on U and is holomorphic with respect to λ for $\mathrm{Re}\,\lambda > 0$.

For each $x_0 \in U$, there exist a nonzero polynomial $b_{f,x_0}(s)$ in an indeterminate s and some $P(s) \in (\mathcal{D}_M)_{x_0}[s]$ such that

$$b_{f,x_0}(\lambda) f_+^\lambda = P(\lambda) f_+^{\lambda+1}$$

holds in a neighborhood of x_0 for $\mathrm{Re}\,\lambda > 0$. It follows that f_+^λ is a distribution-valued meromorphic function on the whole complex plane $\mathbb{C}$ with respect to λ. This is called the complex power, and for a compactly supported C^∞-function φ on U, the meromorphic function $\langle f_+^\lambda, \varphi\rangle$ in λ is called the local zeta function (see, e.g., [1]).

By virtue of Kashiwara's theorem on the rationality of b-functions ([2]), the poles of f_+^λ are negative rational numbers. Let λ_0 be a pole of f_+^λ and x_0 be a point of U. Then there exist a positive integer m, an open neighborhood V of x_0, an open neighborhood W of λ_0 in $\mathbb{C}$, and distributions u_k defined on V such that

$$f_+^\lambda = u_{-m}(\lambda - \lambda_0)^{-m} + \cdots + u_{-1}(\lambda - \lambda_0)^{-1} + u_0 + u_1(\lambda - \lambda_0) + \cdots$$

holds as distribution on V for any $\lambda \in W \setminus \{\lambda_0\}$. To determine the poles of f_+^λ, and its Laurent expansion at each pole is an interesting problem and has been investigated by many authors.

From the viewpoint of D-module theory, it would be interesting if we can compute the annihilator of each Laurent coefficient as above explicitly. For example, we compared the annihilator of the residue of f_+^λ at $\lambda = -1$ with that of local cohomology group supported on $f = 0$ in [3].

In this paper, we treat the case where $f = 0$ has a normal crossing singularity at the origin and determine the annihilators of the coefficients of the negative degree part of the Laurent expansion about $\lambda = -1$. The two dimensional case was treated in [3].

§ 2. Main results

Let $x = (x_1, \ldots, x_n)$ be the coordinate of $M = \mathbb{R}^n$.

Proposition 2.1. *The distribution $(x_1 \cdots x_n)_+^\lambda$ has a pole of order n at $\lambda = -1$. Let*

$$(x_1 \cdots x_n)_+^\lambda = \sum_{j=-n}^{\infty} (\lambda + 1)^j u_j$$

be the Laurent expansion of the distribution $(x_1 \cdots x_n)_+^\lambda$ with respect to the holomorphic parameter λ about $\lambda = -1$, with $u_j \in \mathcal{D}'_M(M)$ for $j \geq -n$. Then for $k = 0, 1, \ldots, n-1$, the left ideal $\mathrm{Ann}_{\mathcal{D}_0} u_{-n+k}$ of $\mathcal{D}_0$ is generated by

$$x_{j_1} \cdots x_{j_{k+1}} \quad (1 \leq j_1 < \cdots < j_{k+1} \leq n), \quad x_1 \partial_1 - x_i \partial_i \quad (2 \leq i \leq n).$$

Proof. In one variable t, we have

$$\begin{aligned} t_+^\lambda &= (\lambda + 1)^{-1} \partial_t t_+^{\lambda+1} \\ &= (\lambda + 1)^{-1} \partial_t \left\{ Y(t) + \sum_{j=1}^{\infty} \frac{1}{j!} (\lambda + 1)^j (\log t_+)^j \right\} \\ &= (\lambda + 1)^{-1} \delta(t) + \sum_{j=1}^{\infty} \frac{1}{j!} (\lambda + 1)^{j-1} \partial_t (\log t_+)^j, \end{aligned}$$

where $(\log t_+)^j$ is the distribution defined by the pairing

$$\langle (\log t_+)^j, \varphi \rangle = \int_0^\infty (\log t)^j \varphi(t)\, dt$$

for $\varphi \in C_0^\infty(\mathbb{R})$.

Let us introduce the following notation:

- For a nonnegative integer j, we set

$$h_j(t) = \begin{cases} \delta(t) & (j = 0), \\ \frac{1}{j!} \partial_t (\log t_+)^j & (j \geq 1) \end{cases}$$

with $\partial_t = \partial/\partial_t$ and

$$h_\alpha(x) = h_{\alpha_1}(x_1)\cdots h_{\alpha_n}(x_n)$$

for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

- For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad [\alpha] = \max\{\alpha_i \mid 1 \le i \le n\}.$$

- Set $S(n) = \{\sigma = (\sigma_1, \dots, \sigma_n) \in \{1, -1\}^n \mid \sigma_1 \cdots \sigma_n = 1\}$.

Since

$$(x_1 \cdots x_n)_+^\lambda = \sum_{\sigma \in S(n)} (\sigma_1 x_1)_+^\lambda \cdots (\sigma_n x_n)_+^\lambda,$$

we have

$$u_{-n+k}(x) = \sum_{\sigma \in S(n)} \sum_{|\alpha|=k} h_\alpha(\sigma x).$$

In particular, we have

$$u_{-n}(x) = \sum_{\sigma \in S(n)} \delta(\sigma_1 x_1) \cdots \delta(\sigma_n x_n) = 2^{n-1}\delta(x_1)\cdots\delta(x_n).$$

It follows that $\mathrm{Ann}_{\mathcal{D}_0} u_{-n}$ is generated by $x_1, \dots, x_n$. This proves the assertion for $k = 0$ since $x_1\partial_1 - x_i\partial_i = \partial_1 x_1 - \partial_i x_i$ belongs to the left ideal of $\mathcal{D}_0$ generated by $x_1, \dots, x_n$.

We shall prove the assertion by induction on k. Assume $k \ge 1$ and $P \in \mathcal{D}_0$ annihilates u_{-n+k}, that is, $Pu_{-n+k} = 0$ holds on a neighborhood of $0 \in M$. By division, there exist $Q_1, \dots, Q_r, R \in \mathcal{D}_0$ such that

$$\begin{aligned} P &= Q_1\partial_1 x_1 + \cdots + Q_n \partial_n x_n + R, \\ R &= \sum_{\alpha_1\beta_1 = \cdots = \alpha_n\beta_n = 0} a_{\alpha,\beta} x^\alpha \partial^\beta \qquad (a_{\alpha,\beta} \in \mathbb{C}). \end{aligned} \tag{2.1}$$

Since

$$u_{-n+k}(x) = \sum_{\sigma \in S(n)} \sum_{|\alpha|=k,\, [\alpha]=1} h_\alpha(\sigma x) + \sum_{\sigma \in S(n)} \sum_{|\alpha|=k,\, [\alpha]\ge 2} h_\alpha(\sigma x), \tag{2.2}$$

we have

$$\begin{aligned} u_{-n+k}(x) &= 2^{n-k-1}\delta(x_1)\cdots\delta(x_{n-k}) h_1(x_{n-k+1})\cdots h_1(x_n) \\ &= 2^{n-k-1}\delta(x_1)\cdots\delta(x_{n-k}) \frac{1}{x_{n-k+1}} \cdots \frac{1}{x_n} \end{aligned}$$

on the domain $x_{n-k+1} > 0, \dots, x_n > 0$. Note that $\partial_i x_i$ annihilates both $\delta(x_i)$ and x_i^{-1}. Hence

$$\begin{aligned} 0 &= Pu_{-n+k} = Ru_{-n+k} \\ &= \sum_{\alpha_1 = \cdots = \alpha_{n-k} = 0, \alpha_{n-k+1}\beta_{n-k+1} = \cdots = \alpha_n \beta_n = 0} (-1)^{\beta_{n-k+1} + \cdots + \beta_n} \beta_{n-k+1}! \cdots \beta_n! a_{\alpha,\beta} \\ &\delta^{(\beta_1)}(x_1) \cdots \delta^{(\beta_{n-k})}(x_{n-k}) x_{n-k+1}^{\alpha_{n-k+1} - \beta_{n-k+1} - 1} \cdots x_n^{\alpha_n - \beta_n - 1} \end{aligned}$$

holds on $\{x \in M \mid x_{n-k+1} > 0, \dots, x_n > 0\} \cap V$ with an open neighborhood V of the origin. Hence $a_{\alpha,\beta} = 0$ holds if $\alpha_1 = \cdots = \alpha_{n-k} = 0$.

In the same way, we conclude that $a_{\alpha,\beta} = 0$ if the components of α are zero except at most k components. This implies that R is contained in the left ideal generated by $x_{j_1} \cdots x_{j_{k+1}}$ with $1 \le j_1 < \cdots < j_{k+1} \le n$.

In the right-hand-side of (2.2), each term contains the product of at least $n - k$ delta functions. Hence $x_{j_1} \cdots x_{j_{k+1}}$ with $1 \le j_1 < \cdots < j_{k+1} \le n$, and consequently R also, annihilates $u_{-n+k}(x)$. Hence we have

$$0 = Pu_{-n+k} = \sum_{i=1}^{n} Q_i \partial_i x_i u_{-n+k}.$$

On the other hand, since

$$\partial_i x_i (x_1 \cdots x_n)_+^{\lambda} = (x_i \partial_i + 1)(x_1 \cdots x_n)_+^{\lambda} = (\lambda + 1)(x_1 \cdots x_n)_+^{\lambda},$$

we have

$$\partial_i x_i u_{-k} = u_{-k-1} \qquad (k \le n - 1, \ 1 \le i \le n)$$

and consequently

$$0 = \sum_{i=1}^{n} Q_i \partial_i x_i u_{-n+k} = \sum_{i=1}^{n} Q_i u_{-n+k-1}.$$

By the induction hypothesis, $\sum_{i=1}^{n} Q_i$ belongs to the left ideal of $\mathcal{D}_0$ generated by

$$x_{j_1} \cdots x_{j_k} \quad (1 \le j_1 < \cdots < j_k \le n), \quad x_1 \partial_1 - x_i \partial_i \quad (2 \le i \le n).$$

Now rewrite (2.1) in the form

$$P = \sum_{i=1}^{n} Q_i \partial_1 x_1 + \sum_{i=2}^{n} Q_i (\partial_i x_i - \partial_1 x_1) + R.$$

If $j_1 > 1$, we have

$$x_{j_1} \cdots x_{j_k} \partial_1 x_1 = \partial_1 x_1 x_{j_1} \cdots x_{j_k}.$$

If $j_1 = 1$, let l be an integer with $2 \le l \le n$ such that $l \neq j_2, \dots, l \neq j_k$. Then we have

$$x_{j_1} \cdots x_{j_k} \partial_1 x_1 = x_{j_2} \cdots x_{j_k} x_1 \partial_1 x_1 = x_{j_2} \cdots x_{j_k} x_1 (\partial_1 x_1 - \partial_l x_l) + \partial_l x_{j_2} \cdots x_{j_k} x_1 x_l.$$

We conclude that P belongs to the left ideal generated by

$$x_{j_1} \cdots x_{j_{k+1}} \quad (1 \le j_1 < \cdots < j_{k+1} \le n), \quad x_1 \partial_1 - x_i \partial_i \quad (2 \le i \le n).$$

Conversely it is easy to see that these generators annihilate u_{-n+k} since

$$x_1 \partial_1 (x_1 \cdots x_n)_+^\lambda = x_i \partial_i (x_1 \cdots x_n)_+^\lambda = \lambda (x_1 \cdots x_n)_+^\lambda$$

and each term of (2.2) contains the product of at least $n - k$ delta functions. □

Theorem 2.2. *Let $f_1, \dots, f_m$ be real-valued real analytic functions defined on a neighborhood of the origin of $M = \mathbb{R}^n$ such that $df_1 \wedge \cdots \wedge df_m \neq 0$. Let*

$$(f_1 \cdots f_m)_+^\lambda = \sum_{j=-m}^{\infty} (\lambda + 1)^j u_j$$

be the Laurent expansion about $\lambda = -1$, with each u_j being a distribution defined on a common neighborhood of the origin. Let $v_1, \dots, v_n$ be real analytic vector fields defined on a neighborhood of the origin which are linearly independent and satisfy

$$v_i(f_j) = \begin{cases} 1 & (\text{if } i = j \le m) \\ 0 & (\text{otherwise}) \end{cases}$$

Then for $k = 0, 1, \dots, m-1$, the annihilator $\mathrm{Ann}_{\mathcal{D}_0} u_{-m+k}$ is generated by

$$\begin{aligned} &f_{j_1} \cdots f_{j_{k+1}} \quad (1 \le j_1 < \cdots < j_{k+1} \le m), \\ &f_1 v_1 - f_i v_i \quad (2 \le i \le m), \quad v_j \quad (m+1 \le j \le n). \end{aligned}$$

Proof. By a local coordinate transformation, we may assume that $f_j = x_j$ for $j = 1, \dots, m$, and $v_j = \partial / \partial x_j$ for $j = 1, \dots, n$. Then the distribution u_j does not depend on $x_{m+1}, \dots, x_n$. Hence we have only to apply Proposition 2.1 in $\mathbb{R}^m$. □

References

[1] Igusa, J., *An Introduction to the Theory of Local Zeta Functions*, American Mathematical Society, 2000.

[2] Kashiwara, M., B-functions and holonomic systems—Rationality of roots of B-functions, *Invent. Math.*, **38** (1976), 33–53.

[3] Oaku, T., Annihilators of distributions associated with algebraic local cohomology of a hypersurface, *Complex Variables and Elliptic Equations*, **59** (2014), 1533-1546.