On various b-functions of specializable D-modules

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Indicial polynomial along a submanifold

- X : an open set of $\mathbb{C}^n$ (or an n-dim. complex manifold).
- Y : a non-singular complex analytic submaifold of X.
- $\mathcal{D}_X$: the sheaf of linear differential operators with holomorphic coefficients on X.
- $\mathcal{M}$: a coherent left $\mathcal{D}_X$-module on X.
- $\{F_Y^i(\mathcal{D}_X)\}_{i\in\mathbb{Z}}$ the V-filtration of $\mathcal{D}_X$ along Y.
- θ : a vector field on a neighborhood of Y in X which induces the identity map on $\mathcal{I}_Y/\mathcal{I}_Y^2$, where $\mathcal{I}_Y$ is the defining ideal of Y. In a local coordinate $x = (x_1, \dots, x_d, x_{d+1}, \dots, x_n)$ such that $Y = \{x_1 = \cdots = x_d = 0\}$, we may take

$$\theta = x_1 \frac{\partial}{\partial x_1} + \cdots + x_d \frac{\partial}{\partial x_d}.$$

Definition

Let u be a section of $\mathcal{M}$ defined on a neighborhood of $x_0 \in Y$. The indicial polynomial of u along Y at x_0 is the monic polynomial $b(s)$, if any, in an indeterminate s of the least degree such that

$$(b(\theta) + P)u = 0 \quad (\exists P \in F_Y^{-1}(\mathcal{D}_X)_{x_0}).$$

If we impose the condition $\mathrm{ord}\, P \leq \deg b(s)$, then $b(s)$ is called a regular indicial polynomial of u along Y at x_0.
$\mathcal{M}$ is called (regular) specializable along Y if each section u of $\mathcal{M}$ has a (regular) indicial polynomial along Y.

- $\mathcal{M}$ is specializable if $\mathcal{M}$ is holonomic (Kashiwara-Kawai).
- $\mathcal{M}$ is regular specializable if $\mathcal{M}$ is regular holonomic (KK).
- If $\mathcal{M}$ is defined over the Weyl algebra, there are algorithms to detect (regular) specializability and to compute the (regular) indicial polynomial(s) (Oaku 2009 (JPAA) for regular case).

Non-uniqueness of regular indicial polynomial

The indicial polynomial is unique but a regular indicial polynomial is not necessarily unique. For example, for u such that

$$x^2\partial_x^2 u = x(\partial_x + \partial_y^2)u = 0$$

in two variables (x, y), the indicial polynomial of u along $x = 0$ is s, while $s(s-c)$ is a regular indicial polynomial of u along $x = 0$ for any c, of the least degree.

$\because)\ x\partial_x u = -x\partial_y^2 u,\ (x^2\partial_x^2 + cx\partial_x)u = -cx\partial_y^2 u.$

Note that $\mathcal{D}_X u$ is holonomic since its characteristic variety is

$$\{(x, y, \xi dx + \eta dy) \mid x = \eta = 0\} \cup \{(x, y, \xi dx + \eta dy) \mid \xi = \eta = 0\}.$$

Hence we mean by 'the regular indicial polynomial' the set of the regular indicial polynomials, which is not necessarily an ideal of $\mathbb{C}[s]$.

Examples

• Appell's F_1 is defined by $P_1 u = P_2 u = 0$ with

$$P_1 = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + (c-(a+b_1+1)x)\partial_x - b_1 y\partial_y - ab$$
$$P_2 = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + (c-(a+b_2+1)y)\partial_y - b_2 x\partial_x - ab$$

and parameters a, b_1, b_2, c. Both the indicial and the regular indicial polynomials along the origin $(0,0)$ are $s(s+c-1)$ for arbitrary values of the parameters although $\mathcal{D}_X u$ is holonomic if and only if $c \neq a+1$.

• Let $\mathcal{M}_A(\beta)$ be the A-hypergeometric (GKZ) system for an arbitrary $d \times n$ integer matrix A s.t. $\operatorname{rank} A = d$ with parameters $\beta = (\beta_1, \dots, \beta_d)$. Then $\mathcal{M}_A(\beta)$ is regular specializable along the origin for any β (Oaku 2009). In particular we have an isomorphism

$$\operatorname{Ext}^k_{(\mathcal{D}_X)_0}(\mathcal{M}_A(\beta), \mathbb{C}\{x\}) \simeq \operatorname{Ext}^k_{(\mathcal{D}_X)_0}(\mathcal{M}_A(\beta), \mathbb{C}[[x]]) \qquad (\forall k \in \mathbb{Z}).$$

Kashiwara's *b*-function

- $\mathcal{M}$: a coherent left $\mathcal{D}_X$-module.
- $\mathcal{O}_X$: the sheaf of holomorphic functions on X.
- f : a section of $\mathcal{O}_X$ on X.
- s : an inderminate (a parameter).
- $\mathcal{L}_f := \mathcal{O}_X[f^{-1}, s]f^s$, the free $\mathcal{O}_X[f^{-1}, s]$-module generated by the symbol f^s.

$\Rightarrow$ $\mathcal{L}_f$ has a natural structure of left $\mathcal{D}_X[s]$-module.

$\Rightarrow$ $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_f$ has a structure of left $\mathcal{D}_X[s]$-module.

Definition (Kashiwara)

Kashiwara's b-function of a germ u of $\mathcal{M}$ at $x_0 \in X$ w.r.t. f is the monic polynomial $b(s)$, if any, in an indeterminate s of the least degree such that

$$P(s)(u \otimes f^{s+1}) = b(s)u \otimes f^s \quad (\exists P(s) \in \mathcal{D}_X[s]_{x_0})$$

holds in $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_f$. (If $\mathrm{ord}_{\partial_x, s} P(s) \leq \deg b(s)$, we call $b(s)$ regular.)

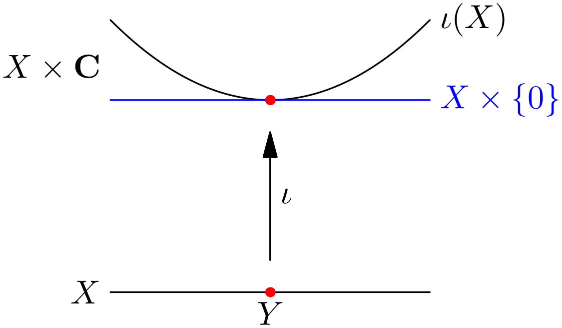
Theorem (Kashiwara 1976)

Kashiwara's b-function exists if $\mathcal{D}_X u$ is holonomic.

Kashiwara's b-function coincides with the Bernstein-Sato polynomial of f if $M = \mathcal{O}_X$ and $u = 1$.

Indicial polynomial w.r.t. a graph embedding

- f : a holomorphic function on X.
- $\iota : X \ni x \mapsto (x, f(x)) \in X \times \mathbb{C}$.
- $Y = \{x \in X \mid f(x) = 0\}$.
- $Z = \iota(X) = \{(x,t) \in X \times \mathbb{C} \mid t = f(x)\}$.
- $\mathcal{B}_{Z|X\times\mathbb{C}} = \mathcal{H}^1_Z(\mathcal{O}_{X\times\mathbb{C}}) \ni \delta(t-f) = [(t-f)^{-1}]$.
- $\mathcal{M}$: a coherent left $\mathcal{D}_X$-module defined on X.
- u : a section of $\mathcal{M}$ defined on a neighborhood of $x_0 \in Y$.
- $\iota_*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_{Z|X\times\mathbb{C}}$: direct image by ι, which is a coherent left $\mathcal{D}_{X\times\mathbb{C}}$-module with $\operatorname{supp}\iota_*\mathcal{M} \subset Z$ (Kashiwara equiv.).



Definition

The (regular) indicial polynomial of u w.r.t. the graph embedding by f at $x_0 \in Y$ is the (regular) indicial polynomial of $\iota_*(u) := u \otimes \delta(t - f) \in \iota_*\mathcal{M}$ along $X \times \{0\}$ at $(x_0, 0)$.

An example

Let $\mathcal{M}_A(\beta) = \mathcal{D}_X u$ be the A-hypergeometric system for $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ with parameters $\beta = (\beta_1, \beta_2)$; i.e.,

$$(x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - \beta_1)u = (x_2\partial_2 + 2x_3\partial_3 - \beta_2)u$$
$$= (\partial_1\partial_3 - \partial_2^2)u = 0.$$

The singular locus of $\mathcal{M}_A(\beta)$ is

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1x_3(4x_1x_3 - x_2^2) = 0\}.$$

The indicial polynomial of u w.r.t. the graph embedding by $f := 4x_1x_3 - x_2^2$ at any point $p \in \mathbb{C}^3$ such that $f(p) = 0$ is $s(s - \beta_1 - 1/2)$.

Comparison between Kashiwara's b-function and indicial polynomial w.r.t. graph embedding

- $b_0(s)$: Kashiwara's b-function of u w.r.t. f.
- $b_1(s)$: the indicial polynomial of u w.r.t. the graph embedding by f.

Proposition

$b_0(s)$ is a factor of $b_1(-s-1)$. Moreover, $b_0(s)$ and $b_1(-s-1)$ coincide if and only if $f : \mathcal{D}_X u \to \mathcal{D}_X u$ is injective at x_0.

Example: If $n = 1$, $\mathcal{M} = \mathcal{D}_X / x\mathcal{D}_X$, $u = \overline{1} \in \mathcal{M}$, $f = x$, then $b_0(s) = 1$ and $b_1(s) = s + 1$.
$\because$) $u \otimes x^s = xu \otimes x^{s-1} = 0$ in $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_f$, but $u \otimes \delta(t-x) \neq 0$ and $(t\partial_t + 1)(u \otimes \delta(t-x)) = \partial_t t(u \otimes \delta(t-x)) = \partial_t(xu \otimes \delta(t-x)) = 0$.

Comparison between the indicial polynomials along a hypersurface and w.r.t. graph embedding — Non-singular case

Theorem

Assume that f is non-singular at $x_0 \in Y$; i.e., $df \neq 0$ at x_0. Then $\mathcal{M}$ is (regular) specializable along Y at x_0 if and only if $\iota_*\mathcal{M}$ is (regular) specializable along $X \times \{0\}$ at $(x_0, 0)$. Moreover, for any section u of $\mathcal{M}$ near x_0, the (regular) indicial polynomial of u along Y at x_0 coincides with the (regular) indicial polynomial of u w.r.t. the graph embedding by f at x_0 (i.e., the (regular) indicial polynomial of $\iota_*(u)$ along $t = 0$ at $(x_0, 0)$).

Proof

We may assume that X is an open set of $\mathbb{C}^n$ containing $x_0 = 0 \in \mathbb{C}^n$, and that $f = x_1$. We use the notation $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$ and $\partial_i = \partial/\partial x_i$, $\partial' = (\partial_2, \dots, \partial_n)$. Let $b(s) \in \mathbb{C}[s]$ be the (regular) indicial polynomial of u along Y at 0. Then there exists a differential operator Q of the form $Q = Q(x, x_1\partial_1, \partial')$ such that

$$(b(x_1\partial_1) + x_1 Q(x, x_1\partial_1, \partial'))u = 0$$

(and $\mathrm{ord}\, Q \leq \deg b(s)$).

It follows that

$$
\begin{aligned}
(b(t(\partial_t + \partial_1)) + tQ(x, t(\partial_t + \partial_1), \partial'))(u \otimes \delta(t - x_1)) \\
= ((b(x_1\partial_1) + x_1 Q(x, x_1\partial_1, \partial'))u) \otimes \delta(t - x_1) = 0
\end{aligned}
$$

since one has, for any $v \in \mathcal{M}$,

$$
(t - x_1)(v \otimes \delta(t - x_1)) = 0, \quad (\partial_t + \partial_1)(v \otimes \delta(t - x_1)) = (\partial_1 v) \otimes \delta(t - x_1).
$$

There exists a differential operator R of the form $R = R(t\partial_t, \partial_1)$ such that $\mathrm{ord}\, R = \deg b(s)$ and

$$
b(t(\partial_t + \partial_1)) = b(t\partial_t) + tR(t\partial_t, \partial_1),
$$

Hence we have

$$
(b(t\partial_t) + t(R(t\partial_t, \partial_1) + Q(x, t(\partial_t + \partial_1), \partial'))(u \otimes \delta(t - x_1)) = 0.
$$

Conversely, let $b(s)$ be the (regular) indicial polynomial of $u \otimes \delta(t - x_1)$ along $t = 0$ at 0. Then there eixsts a differential operator Q of the form $Q = Q(t, x, t\partial_t, \partial_1, \partial')$ such that

$$(b(t\partial_t) + tQ(t, x, t\partial_t, \partial_1, \partial'))(u \otimes \delta(t - x_1)) = 0$$

(and $\mathrm{ord}\, Q \leq \deg b(s)$). By using $t\partial_t = t(\partial_t + \partial_1) - t\partial_1$, we rewrite the operator as

$$\begin{aligned} P &:= b(t\partial_t) + tQ(t, x, t\partial_t, \partial_1, \partial') \\ &= b(t(\partial_t + \partial_1)) + t\tilde{Q}(t, x, t(\partial_t + \partial_1), \partial_1, \partial') \\ &= b(t(\partial_t + \partial_1)) + t\sum_{i=0}^{m} Q_i(t, x, t(\partial_t + \partial_1), \partial')\partial_1^i. \end{aligned}$$

We have

$$[t - x_1, P](u \otimes \delta(t - x_1)) = 0$$

with

$$[t - x_1, P] = t\sum_{i=1}^{m} iQ_i(t, x, t(\partial_t + \partial_1), \partial')\partial_1^{i-1}.$$

It follows that

$$\tilde{P} := b(t(\partial_t + \partial_1)) + t\sum_{i=0}^{m}\Big(Q_i(t, x, t(\partial_t + \partial_1), \partial')\partial_1^i - \frac{i}{m}\partial_1 Q_i(t, x, t(\partial_t + \partial_1), \partial')\partial_1^{i-1}\Big),$$

which is of order at most $m-1$ with repect to ∂_1, also annihilates $u \otimes \delta(t - x_1)$. By induction, we get

$$((b(x_1\partial_1) + Q_0(x_1, x, x_1\partial_1, \partial'))u) \otimes \delta(t - x_1) = (b(t(\partial_t + \partial_1)) + Q_0(t, x, t(\partial_t + \partial_1), \partial'))(u \otimes \delta(t - x_1)) = 0.$$

This implies

$$(b(x_1\partial_1) + Q_0(x_1, x, x_1\partial_1, \partial'))u = 0.$$

Higher codimensional case — after Budur-Mustata-Saito

- Y : an arbitrary closed analytic subset of X.
- $\mathcal{J}$: a coherent ideal of $\mathcal{O}_X$ such that $\sqrt{\mathcal{J}} = \mathcal{I}_Y$.
- $f_1, \dots, f_d$: a set of local generators of $\mathcal{J}$ on an neighborhood U of $x_0 \in Y$.
- $\iota : U \ni x \longmapsto (x, f_1(x), \dots, f_d(x)) \in U \times \mathbb{C}^d$.
- $Z = \{(x, t_1, \dots, t_d) \in U \times \mathbb{C}^d \mid t_1 - f_1(x) = \cdots = t_d - f_d(x) = 0\}$.
- $\mathcal{B}_{Z|U\times\mathbb{C}^d} = \mathcal{H}^d_{[Z]}(\mathcal{O}_{U\times\mathbb{C}^d})$: the d-th local cohomology group.
- $\mathcal{M}$: a coherent left $\mathcal{D}_X$-module defined on X.
- u : a section of $\mathcal{M}$ defined on a neighborhood of x_0.
- $\iota_*(u) = u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d) \in \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_{Z|U\times\mathbb{C}^d}$.

Definition-Theorem

Let $b(s)$ be the (regular) indicial polynomial of $\iota_*(u)$ along $U \times \{0\}$. Then $b(s-d)$ does not depend on the choice of local generators $f_1, \dots, f_d$ of $\mathcal{J}$. We call $b(s-d)$ the (regular) b-function of u w.r.t. $\mathcal{J}$.

- A regular b-function of u w.r.t. $\mathcal{J}$ is not necessarily unique. The above statement refers to the set of the regular b-functions.
- If $\mathcal{M} = \mathcal{O}_X$ and $u = 1$, then the b-function in the above sense conicides with $b(-s)$, where $b(s)$ is the Bernstein-Sato polynomial of the variety (w.r.t. the ideal $\mathcal{J}$) defined by Budur-Mustata-Saito (2006).

The following lemma describes the behavior of the indicial polynomial via Kashiwara equivalence (direct image) w.r.t. an embedding:

Lemma (Budur-Mustata-Saito)

Let Y be a non-singular complex submanifold of X and let $\iota : X \to X \times \mathbb{C}$ be an embedding. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module and u a section of $\mathcal{M}$ near $x_0 \in Y$. Let $b(s)$ be the (regular) indicial polynomial of u along Y at $x_0 \in Y$ and $\tilde{b}(s)$ be that of $u \otimes \delta(t)$ along $\iota(Y)$ at $(x_0, 0)$. Then one has $\tilde{b}(s-1) = b(s)$.

Proof of Lemma

We may assume $Y = \{x \in X \mid x_1 = \cdots = x_d = 0\}$ and $\iota(x) = (x, 0)$. Then

$$\iota(Y) = \{(x, 0) \mid x_1 = \cdots = x_d = t = 0\}.$$

There exists $Q \in V_Y^{-1}(\mathcal{D}_X)$ such that

$$(b(x_1\partial_1 + \cdots + x_d\partial_d) + Q)u = 0.$$

Then we have

$$(b(x_1\partial_1 + \cdots + x_d\partial_d + \partial_t t) + Q)(u \otimes \delta(t)) = 0$$

and Q belongs to $V_{\iota(Y)}(\mathcal{D}_{X\times\mathbb{C}})$. Thus $\tilde{b}(s)$ is a factor of $b(s+1)$. On the other hand, there exists $Q \in V_{\iota(Y)}^{-1}(\mathcal{D}_{X\times\mathbb{C}})$ such that

$$(\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + t\partial_t) + Q)(u \otimes \delta(t)) = 0.$$

Writing Q in the form $Q = \sum_{i,j\geq 0} Q_{ij}(x,\partial)\partial_t^i t^j$, we have

$$
\begin{aligned}
0 &= (\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + t\partial_t) + Q)(u \otimes \delta(t)) \\
&= \tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d + \partial_t t - 1)(u \otimes \delta(t)) \\
&\quad + \sum_{i,j\geq 0} Q_{ij}(x,\partial)u \otimes \partial_t^i t^j \delta(t) \\
&= \tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d - 1)u \otimes \delta(t) + \sum_{i\geq 0} Q_{i0}(x,\partial)u \otimes \delta^{(i)}(t).
\end{aligned}
$$

This implies, in particular,

$$(\tilde{b}(x_1\partial_1 + \cdots + x_d\partial_d - 1) + Q_{00})u = 0.$$

Since Q_{00} belongs to $V_Y^{-1}(\mathcal{D}_X)$, we know that $b(s)$ divides $\tilde{b}(s-1)$. In conclusion, we get $b(s) = \tilde{b}(s-1)$.

Proof of Definition-Theorem (following BMS)

Suppose that there exist sections $a_1, \dots, a_d$ of $\mathcal{O}_X$ at x_0 such that $f_{d+1} = a_1 f_1 + \cdots + a_d f_d$. (i.e., assume f_{d+1} is redundant.) Define an embedding

$$\iota : X \times \mathbb{C}^d \longrightarrow X \times \mathbb{C}^{d+1} \quad \text{by}$$

$$\iota(x, t_1, \dots, t_d) = (x, t_1, \dots, t_d, a_1(x)t_1 + \cdots + a_d(x)t_d).$$

Set $Z = \{(x, t_1, \dots, t_d) \mid t_1 = \cdots = t_d = 0\}$. Then we have

$$\iota(Z) = \{(x, t_1, \dots, t_d, t_{d+1}) \mid t_1 = \cdots = t_d = t_{d+1} = 0\}$$

and

$$\begin{aligned}
&\iota_*(u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - a_1(x)t_1 - \cdots - a_d(x)t_d) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - a_1(x)f_1 - \cdots - a_d(x)f_d) \\
&= u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - f_{d+1}).
\end{aligned}$$

Let $b(s)$ be the indicial poynomial of $u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)$ along Z, and $\tilde{b}(s)$ be that of
$u \otimes \delta(t_1 - f_1) \cdots \delta(t_d - f_d)\delta(t_{d+1} - f_{d+1})$ along $\iota(Z)$.

Then $b(s - d) = \tilde{b}(s - d - 1)$ holds in view of Lemma.

Set $b_{f_1,\dots,f_d}(s) = b(s - d)$. Thus we have shown

$$b_{f_1,\dots,f_d}(s) = b_{f_1,\dots,f_d,f_{d+1}}(s)$$

if

$$\mathcal{J} = \mathcal{O}_X f_1 + \cdots + \mathcal{O}_X f_d = \mathcal{O}_X f_1 + \cdots + \mathcal{O}_X f_d + \mathcal{O}_X f_{d+1}.$$

The general case can be reduced to this situation step by step.

An example

With $X = \mathbb{C}^3 \ni (x, y, z)$, set $\mathcal{J} = \mathcal{O}_X(x^3 - y^2) + \mathcal{O}_X(x^2 - z)$, which is the defining ideal of a monomial curve $x^3 - y^2 = x^2 - z = 0$.

- The b-function $b(s)$ of $1 \in \mathcal{O}_X$ w.r.t. $\mathcal{J}$ at 0 is $(s-2)(6s-11)(6s-13)$.
- The b-function of u such that $\partial_x u = \partial_y u = (z\partial_z - a)u = 0$ w.r.t. $\mathcal{J}$ at 0 is

$$b(s,a) = (s-2)(s-a-2)(2s-2a-5)(6s-4a-11) \times (6s-4a-13)(6s-4a-15)$$

if $a \neq 0, -1, -2$.

• If $a = 0$, then $b(s) = (s-2)(2s-5)(6s-11)(6s-13)$ whereas $b(s,0) = (s-2)^2(2s-5)^2(6s-11)(6s-13)$.

• If $a = -1$, then $b(s) = (s-1)(s-2)(2s-3)(6s-7)(6s-11)$ whereas $b(s,-1) = (s-1)(s-2)(2s-3)^2(6s-7)(6s-11)$.

• If $a = -2$, then $b(s) = s(s-2)(2s-1)(6s-5)(6s-7)$ whereas $b(s,-2) = s(s-2)(2s-1)^2(6s-5)(6s-7)$.

Comparison between indicial polynomial and b-function at a regular point

Theorem

Assume that Y is non-singular at x_0. Let d be the codimension of Y near x_0. Let $b(s)$ be the (regular) indicial polynomial of u along Y at x_0. Then the b-function of u w.r.t. $\mathcal{I}_Y$ is $b(s-d)$.

This theorem also means that a (regular) b-function of u w.r.t. $\mathcal{I}_Y$ exists (this condidition does not depend on the choice of local generators of $\mathcal{I}_Y$) if and only if a (regular) indicial polynomial of u along Y exists. The proof is similar to the one-codimensional case.

(Regular) specializability along an arbitrary subvariety

Definition

Let us call $\mathcal{M}$ (regular) specializable along Y if a (regular) b-function of every section u of $\mathcal{M}$ w.r.t. $\mathcal{I}_Y$ (the defining ideal of Y) exists.

- If $\mathcal{M}$ is (regular) holonomic, then $\mathcal{M}$ is (regular) specializable along any subvariety Y of X.