An attempt to compute holonomic systems for Feynman integrals in two-dimensional space-time

Toshinori Oaku
Department of Mathematics, Tokyo Woman's Christian University

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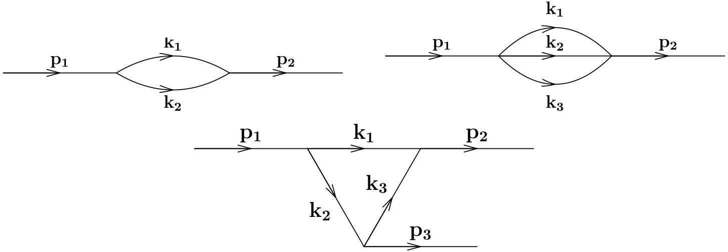
Microlocal analysis of Feynman integrals was initiated by M. Sato, T. Kawai, H.P. Stapp, M. Kashiwara, T. Oshima, et al. in the 1970's. Especially the theory of microfunctions and (holonomic systems of) microdifferential equations played a decisive role.
Recently, N. Honda and Kawai studied the geometry of Landau-Nakanishi surfaces systematically and discovered interesting phenomena in the 2-dimensional space-time in a series of papers.
Following their work, I will report on actual computation of holonomic systems for Feynman integrals associated with very simple Feynman diagrams below by computer.

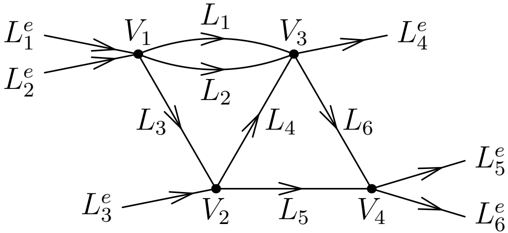


Feynman diagrams and Feynman integrals

Let G be a connected Feynman graph (diagram), i.e., G consists of

- vertices $V_1, \cdots, V_{n'}$,
- oriented line segments $L_1, \ldots, L_N$ called internal lines,
- oriented half-lines $L_1^e, \ldots, L_n^e$ called external lines.

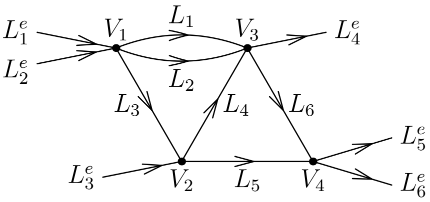
The end-points of each internal line L_l are two distinct vertices, and each external line has only one end-point, which coincides with one of the vertices.



• We associate ν-dimensional vector $\mathbf{p}_r$ to each external line L_r^e $(1 \leq r \leq n')$,
and ν-dimensional vector $\mathbf{k}_l$ and a *positive* real number m_l to each internal line L_l $(1 \leq l \leq N)$.

• For a vertex V_j and an internal or external line L_l, the incidence number $[j : l]$ is defined as follows:

$$
\begin{aligned}
&[j : l] = 1 \text{ if } L_l \text{ ends at } V_j, \\
&[j : l] = -1 \text{ if } L_l \text{ starts from } V_j, \\
&[j : l] = 0 \text{ otherwise.}
\end{aligned}
$$



The Feynman integral associated with G is defined to be

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n'} \delta^\nu\left(\sum_{r=1}^{n}[j:r]\mathbf{p}_r + \sum_{l=1}^{N}[j:l]\mathbf{k}_l\right)}{\prod_{l=1}^{N}(\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l.$$

Here δ^ν denotes the ν-dimensional delta function,

$$\mathbf{k}_l^2 := k_{l0}^2 - k_{l1}^2 - \cdots - k_{l\nu}^2$$

is the Lorentz norm of $\mathbf{k}_l = (k_{l0}, k_{l1}, \cdots, k_{l\nu})$, and $d^\nu \mathbf{k}_l$ is the ν-dimensional volume element.

The Feynman phase space integral associated with G is defined to be

$$
\begin{aligned}
& I_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) \\
&= \int_{\mathbb{R}^{\nu N}} \prod_{j=1}^{n'} \delta^\nu\Big(\sum_{r=1}^{n}[j:r]\mathbf{p}_r + \sum_{l=1}^{N}[j:l]\mathbf{k}_l\Big) \prod_{l=1}^{N} \delta_+(\mathbf{k}_l^2 - m_l^2) \prod_{l=1}^{N} d^\nu \mathbf{k}_l.
\end{aligned}
$$

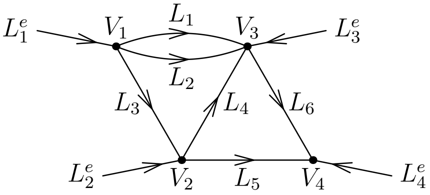
Here we denote

$$
\delta_+(\mathbf{k}_l^2 - m_l^2) = Y(k_{l0})\delta(\mathbf{k}_l^2 - m_l^2)
$$

with the Heaviside function Y.

In what follows, we assume that for each vertex V_j, there exists a unique external line, which we may assume to be L_j^e, that ends at V_j and that no external line starts from V_j. Then $n = n'$ holds and the Feynman integral is

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n} \delta^\nu\left(\mathbf{p}_j + \sum_{l=1}^{N} [j:l]\mathbf{k}_l\right)}{\prod_{l=1}^{N}(\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l$$



Well-definedness of Feynman phase space integrals

It should be well-known and is easy to prove

Proposition

If the Feynman graph G has no oriented cycles, then the Feynman phase space integral

$$I_G(\mathbf{p}_1, \dots, \mathbf{p}_n)$$
$$= \int_{\mathbb{R}^{\nu N}} \prod_{j=1}^{n} \delta^\nu\Big(\mathbf{p}_j + \sum_{l=1}^{N} [j:l]\mathbf{k}_l\Big) \prod_{l=1}^{N} \delta_+(\mathbf{k}_l^2 - m_l^2) \prod_{l=1}^{N} d^\nu \mathbf{k}_l$$

is well-defined as a hyperfunction on $\mathbb{R}^{\nu n}$.

Rewriting the Feynman integral

The delta factors of the integrand of the Feynman integral correspond to the linear equations (momentum preservation)

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \le j \le n)$$

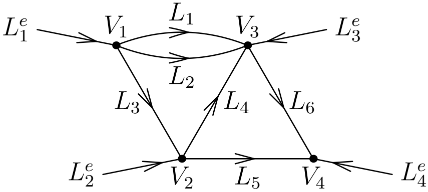
for indeterminates p_j and k_l which correspond to the vectors $\mathbf{p}_j$ and $\mathbf{k}_l$. These equations define an N-dimensional linear subspace of $\mathbb{R}^{n+N}$, which is contained in the hyperplane $p_1 + \cdots + p_n = 0$ since $\sum_{j=1}^{n} [j : l] = 0$.

Lemma

Let A be the $n \times N$ matrix whose (j, l)-element is $[j : l]$. Then the rank of A is $n - 1$.

For the example below, the matrix A is given by

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



In view of the lemma above, we can choose a set of indices

$$J = \{l_1, \ldots, l_{N-n+1}\} \subset \{1 \ldots, N\}$$

and integers a_{lr} and b_{lj} so that the system

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \leq j \leq n)$$

of linear equations is equivalent to

$$\sum_{j=1}^{n} p_j = 0, \quad k_l - \psi_l(p_1, \ldots, p_{n-1}, k_{l_1}, \ldots, k_{l_{N-n+1}}) = 0 \quad (l \in J^c).$$

Then the Feynman integral is written in the form

$$
\begin{aligned}
F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = &\int_{\mathbb{R}^{N\nu}} \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \\
&\times \prod_{l \in J^c} \delta(\mathbf{k}_l - \psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})) \\
&\times \prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l=1}^{N} d\mathbf{k}_l \\
= &\,\delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})
\end{aligned}
$$

with the amplitude function

$$
\begin{aligned}
\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) = &\int_{\mathbb{R}^{(N-n+1)\nu}} \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \\
&\times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l \in J} d\mathbf{k}_l.
\end{aligned}
$$

The Feynman phase space integral is written in the form

$$I_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n)\tilde{I}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})$$

with the amplitude

$$\tilde{I}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) = \int_{\mathbb{R}^{(N-n+1)\nu}} \prod_{l \in J} \delta_+(\mathbf{k}_l^2 - m_l^2)$$
$$\times \prod_{l \in J^c} \delta_+(\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2) \prod_{l \in J} d\mathbf{k}_l.$$

Holonomic systems for integrands

In general, since dk_l ($l \in J$) and $d\psi_l$ ($l \in J^c$) are linearly independent, the integrand Φ of $\tilde{F}_G$ is well-defined as a hyperfunction on $\mathbb{R}^N$, represented as the boundary value of the rational function

$$
\begin{aligned}
&\tilde{\Phi}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}) \\
&= \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2)^{-1} \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1}
\end{aligned}
$$

on $\mathbb{C}^{\nu N}$.

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in $\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{I_1}, \dots, \mathbf{k}_{I_{N-1}}$ and $\mathcal{B}_{\mathbb{R}^{\nu N}}$ the sheaf of hyperfunctions on $\mathbb{R}^{\nu N}$. Then the annihilator (left ideal of $D_{\nu N}$)

$$\mathrm{Ann}_{D_{\nu N}} \Phi = \{P \in D_{\nu N} \mid P\Phi = 0 \text{ in } \mathcal{B}_{\mathbb{R}^{\nu N}}(\mathbb{R}^{\nu N})\}$$

of Φ is contained in the annihilator

$$\mathrm{Ann}_{D_{\nu N}} \tilde{\Phi} = \{P \in D_{\nu N} \mid P\tilde{\Phi} = 0 \text{ as rational function}\}$$

of $\tilde{\Phi}$. There exists a general algorithm to compute the annihilator of an arbitrary rational function. However, since the denominator of $\tilde{\Phi}$ is the product of polynomials whose differentials are linearly independent at each point, the annihilator of $\tilde{\Phi}$ is generated by first order differential operators, which are much easier to compute.

The annihilator of the integrand Ψ of $\tilde{I}_G$ is contained in the annihilator of the local cohomology class $[\tilde{\Phi}]$ of the rational function $\tilde{\Phi}$ in the local cohomology group

$$H_Z^N(\mathbb{C}[\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}])$$

with the N-codimensional non-singular algebraic set

$$\begin{aligned} Z = \{ & (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}) \in \mathbb{C}^{\nu N} \mid \\ & \mathbf{k}_l^2 - m_l^2 = 0 \ (l \in J), \\ & \psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 = 0 \ (l \in J^c)\} \end{aligned}$$

and is generated by zeroth and first order operators, which are easy to compute. Note that the annhilator of the rational function $\tilde{\Phi}$ is contained in that of the local cohomology class $[\tilde{\Phi}]$.

Landau-Nakanishi varieties for amplitudes

Set

$$\begin{aligned}
\Lambda(G) = \{ & (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}; \alpha_1, \ldots, \alpha_N) \\
& \in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\
& \alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \le j \le N-n+1), \\
& \alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\
& \alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \le j \le N-n+1), \\
& \mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \le r \le n-1), \\
& \alpha_l \ge 0 \ (1 \le l \le N) \}
\end{aligned}$$

with

$$\psi_l = \sum_{r=1}^{n-1} a_{lr}\mathbf{p}_r + \sum_{j=1}^{N-n+1} b_{lj}\mathbf{k}_{l_j}.$$

and

$$
\begin{aligned}
\Lambda_+(G) = \{ & (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}; \alpha_1, \ldots, \alpha_N) \\
& \in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\
& \alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \leq j \leq N-n+1), \\
& \alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\
& \alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \leq j \leq N-n+1), \\
& \mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \leq r \leq n-1), \\
& \alpha_l > 0 \ (1 \leq l \leq N) \}.
\end{aligned}
$$

Let ϖ be the natural projection of $\Lambda(G)$ to the (purely imaginary) cotangent bundle

$$\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \\ = \{(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}; \sqrt{-1}\mathbf{u}_1 d\mathbf{p}_1 + \cdots + \sqrt{-1}\mathbf{u}_{n-1} d\mathbf{p}_{n-1})\}.$$

Then the amplitude $\tilde{F}_G$ is well-defined as a microfunction on the set

$$\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\big(\Lambda(G) \setminus \Lambda_+(G)\big)$$

and its support is contained in $\varpi(\Lambda_+(G))$.
Moreover $\tilde{F}_G$ satisfies the D-module theoretic integration (direct image) of $D_{\nu N}/\mathrm{Ann}_{D_{\nu N}}\tilde{\Phi}$ along the fibers of the projection

$$\mathbb{C}^{\nu N} \ni (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}; \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}) \longmapsto (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \in \mathbb{C}^{\nu(n-1)}.$$

Invariance under Lorentz transformations

The functions F_G, $\tilde{F}_G$, I_G, $\tilde{I}_G$ are invariant under the action of the Lorentz group: Let T be a $\nu \times \nu$ matrix such that

$$
{}^tT \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ 0 & 0 & \dots & -1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ 0 & 0 & \dots & -1 \end{pmatrix}.
$$

Then one has

$$
F_G(T\mathbf{p}_1, \dots, T\mathbf{p}_{n-1}, T\mathbf{p}_n) = F_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{p}_n),
$$

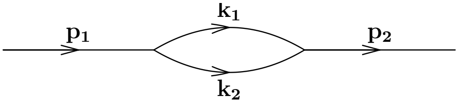
$$
\tilde{F}_G(T\mathbf{p}_1, \dots, T\mathbf{p}_{n-1}) = \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})
$$

Some examples in the two-dimensional space-time

In the sequel, we set $\nu = 2$ and consider Feynman integrals and Feynman phase space integrals associated with some simple Feynman diagrams.
In general, for a two-dimensional vector $\mathbf{p} = (p_0, p_1)$, we denote $\mathbf{p}^2 = p_0^2 - p_1^2$ for the Lorentz norm and $d\mathbf{p} = dp_0 dp_1$ for the volume element.

Example 1

Let us study the Feynman diagram G below:



Then the Feynman integral is written in the form

$$
\begin{aligned}
F_G(\mathbf{p}_1, \mathbf{p}_2) &= \int_{\mathbb{R}^4} \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) \\
&\qquad \times (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2 \\
&= \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)
\end{aligned}
$$

with the amplitude

$$
\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1.
$$

The amplitude $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a microfunction on $\sqrt{-1}T^*\mathbb{R}^2 \setminus \mathbb{R}^2$, i.e., the whole cotangent bundle with the zero section removed. In other words, $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a section of the sheaf $\mathcal{B}_{\mathbb{R}^2}/\mathcal{A}_{\mathbb{R}^2}$ on $\mathbb{R}^2$.

$\tilde{F}_G(\mathbf{p}_1)$ satisfies a holonomic system $M = D_2/I$ with the left ideal I generated by three operators

$$p_{11}\partial_{p_{10}} + p_{10}\partial_{p_{11}},$$

$$\begin{aligned}&(p_{10} - m_1 - m_2)(p_{10} - m_1 + m_2)(p_{10} + m_1 - m_2)(p_{10} + m_1 + m_2)\partial_{p_{10}}\\ &\quad + p_{11}p_{10}(2p_{10}^2 - p_{11}^2 - 2m_1^2 - 2m_2^2)\partial_{p_{11}}\\ &\quad + 2p_{10}^3 + (-2p_{11}^2 - 2m_1^2 - 2m_2^2)p_{10},\end{aligned}$$

$$\begin{aligned}&(p_{10}^2 - p_{11}^2 - (m_1 + m_2)^2)(p_{10}^2 - p_{11}^2 - (m_1 - m_2)^2)\partial_{p_{11}}\\ &\quad - 2p_{11}p_{10}^2 + 2p_{11}^3 + (2m_1^2 + 2m_2^2)p_{11}.\end{aligned}$$

The characteristic variety of M is

$$\begin{aligned}
\mathrm{Char}(M) = \{(p_{10}, p_{11}; \sqrt{-1}(u_{10}dp_{10} + u_{11}dp_{11}) \mid u_{10} = u_{11} = 0\} \\
\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
\cup \{p_{10}^2 - p_{11}^2 - (m_1 - m_2)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}
\end{aligned}$$

with each component of multiplicity one if $m_1 \neq m_2$ and

$$\begin{aligned}
\mathrm{Char}(M) = \{(p_{10}, p_{11}; \sqrt{-1}(u_{10}dp_{10} + u_{11}dp_{11}) \mid u_{10} = u_{11} = 0\} \\
\cup \{p_{10}^2 - p_{11}^2 - 4m^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
\cup \{p_{10} - p_{11} = u_{10} + u_{11} = 0\} \\
\cup \{p_{10} + p_{11} = u_{10} - u_{11} = 0\} \\
\cup \{p_{10} = p_{11} = 0\}
\end{aligned}$$

with each component of multiplicity one if $m_1 = m_2 = m$.

In view of the invariance under Lorentz transformations, let us set $\mathbf{p}_1 = (x, 0)$ with $x \neq 0$. Then $\tilde{F}_G(x, 0)$ satisfies

$$\{(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x + 2x(x^2 - m_1^2 - m_2^2)\}\tilde{F}_G(x, 0) = 0.$$

Hence the support of $\tilde{F}_G(x, 0)$ is contained in the set

$$\{(x; \sqrt{-1}udx) \mid x = \pm(m_1 + m_2), \pm(m_1 - m_2)\}$$

and one has, for example

$$\tilde{F}_G((x, 0)) = C(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2} \times (x + m_1 + m_2)^{-1/2}(x - m_1 - m_2 + \sqrt{-1}0)^{-1/2}$$

at $(m_1 + m_2; \sqrt{-1}dx)$ as a microfunction if $m_1 \neq m_2$.

If $m_1 = m_2 = m$, then the support of $\tilde{F}_G((x,0))$ is contained in $\{x = 0, \pm 2m\}$ and one has

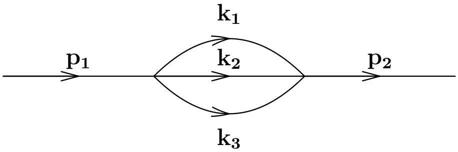
$$\tilde{F}_G((x,0)) = Cx^{-1}(x+2m)^{-1/2}(x-2m+\sqrt{-1}0)^{-1/2}$$

at $(2m, \sqrt{-1}dx)$.

$\tilde{I}_G((x,0))$ satisfies the same differential equation as $\tilde{F}_G((x,0))$.

Example 2

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)$$

with

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^4} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1}$$
$$\times ((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2.$$

We can confirm that $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a microfunction on $\sqrt{-1}T^*\mathbb{R}^2 \setminus \mathbb{R}^2$ and its support (singular spectrum) is contained in

$$
\begin{aligned}
&\{p_{10}^2 - p_{11}^2 - (-m_1 + m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 - m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2 - m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}
\end{aligned}
$$

for generic m_1, m_2, m_3.
We compute holonomic systems for $\tilde{F}_G((x, 0))$ by assigning some special values to m_1, m_2, m_3 since the computation for general m_1, m_2, m_3 (as parameters) is intractable.

First let us set $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ so that $(-m_1 + m_2 + m_3)^2$, $(m_1 - m_2 + m_3)^2$, $(m_1 + m_2 - m_3)^2$ are distinct. Then $\tilde{F}_G((x, 0))$ is annihilated by the differential operator

$$
\begin{aligned}
&30x(x-1)(x+1)(x-3)(x+3)(x-5)(x+5)(x-7)(x+7)\underline{\partial_x^3} \\
&\quad + (-2x^{12} + 191x^{10} - 5340x^8 + 35954x^6 + 273082x^4 \\
&\qquad\qquad - 2071305x^2 + 661500)\underline{\partial_x^2} \\
&\quad + (-10x^{11} + 675x^9 - 12108x^7 + 15454x^5 + 936462x^3 \\
&\qquad\qquad - 2692665x)\underline{\partial_x} \\
&\quad - 8x^{10} + 372x^8 - 3300x^6 - 36028x^4 + 457932x^2 - 356760.
\end{aligned}
$$

The singular points $x = 0, \pm 1, \pm 3, \pm 5, \pm 7$ are all regular and the indicial equations are all $s^2(s-1)$. This implies $\tilde{F}_G((x, 0)) = U \log(x + i0)$ e.g., at $(1, \sqrt{-1}dx)$ with a microdifferential operator of order zero.

Next set $m_1 = m_2 = m_3 = 1$. Then $\tilde{F}_G((x,0))$ is annihilated by

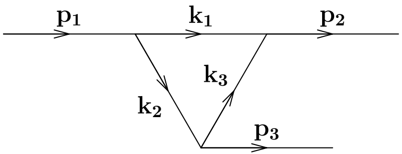
$$x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2.
This implies $\tilde{F}_G((x,0)) = U \log(x - 1 + i0)$ e.g., at $(1, \sqrt{-1}dx)$ with a microdifferential operator of order zero.

$\tilde{I}_G((x,0))$ satisfies the same differential equation as $\tilde{F}_G((x,0))$.

Example 3

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}$$
$$\times ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1})^{-1}((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1.$$

Computation for general m_1, m_2, m_3 are beyond of my (computer's) ability.
So let us set $m_1 = m_2 = m_3 = 1$ in the sequel.
In this situation, the Landau-Nakanishi variety was investigated by N. Honda and T. Kawai in detail.

The amplitude $\tilde{F}_G((x,0),(y,z))$ is well-defined on

$$\begin{aligned}&\{(x,y,z;\sqrt{-1}(udx+vdy+wdz)\mid(u,v,w)\neq(0,0,0)\}\\&\quad\setminus\Big(\{(x-y)^2-z^2-4=wx-wy+vz=u+v=0\}\\&\qquad\cup\{x-y=z=u+v=0\}\\&\qquad\cup\{y^2-z^2-4=wy-vz=u=0\}\\&\qquad\cup\{x^2-4=v=w=0\}\cup\{x=v=w=0\}\Big)\end{aligned}$$

as a microfunction and its support is contained in

$$\sqrt{-1}\,T^*_{\{f=0\}}\mathbb{R}^3\cup\sqrt{-1}\,T^*_{\{x=y=z=0\}}\mathbb{R}^3\cup\sqrt{-1}\,T^*_{\{y=z=0\}}\mathbb{R}^3$$

with

$$f=(y-z)(y+z)x^2-2(y-z)(y+z)yx+(y-z)^2(y+z)^2+4z^2,$$

where we denote by $T^*_S\mathbb{R}^3$ the closure of the conormal bundle of the regular part of a real analytic set S of $\mathbb{R}^3$.

We can compute a holonomic system $M = D_3/I$ for $\tilde{F}_G((x,0),(y,z))$, which is too complicated to show here; we get 74 generators of the left ideal I. The characteristic variety of M is

$$
\begin{aligned}
&\mathbb{C}^3 \cup T^*_{\{f=0\}}\mathbb{C}^3 \cup T^*_{\{x=f=0\}}\mathbb{C}^3 \\
&\cup T^*_{\{(x-y)^2-z^2-4=0\}}\mathbb{C}^3 \cup T^*_{\{y^2-z^2-4=0\}}\mathbb{C}^3 \\
&\cup T^*_{\{x-y-z=0\}}\mathbb{C}^3 \cup T^*_{\{x-y+z=0\}}\mathbb{C}^3 \\
&\cup T^*_{\{y-z=0\}}\mathbb{C}^3 \cup T^*_{\{y+z=0\}}\mathbb{C}^3 \cup T^*_{\{x=0\}}\mathbb{C}^3 \cup T^*_{\{x-2=0\}}\mathbb{C}^3 \cup T^*_{\{x+2=0\}}\mathbb{C}^3 \\
&\cup T^*_{\{x=y^2-z^2-4=0\}}\mathbb{C}^3 \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y+z=0\}}\mathbb{C}^3 \\
&\cup T^*_{\{x-y=z=0\}}\mathbb{C}^3 \cup T^*_{\{y=z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3,
\end{aligned}
$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$.

On the other hand, the amplitude $\tilde{I}_G((x,0),(y,z))$ of the Feynman phase space integral satisfies a holonomic system $M' = D_3/I'$, which is strictly stronger than M, i.e, $I \subsetneq I'$. The characteristic variety of M' is

$$\begin{aligned} &T^*_{\{f=0\}}\mathbb{C}^3 \cup T^*_{\{x=f=0\}}\mathbb{C}^3 \cup T^*_{\{x=0\}}\mathbb{C}^3 \\ &\quad \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y+z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3, \end{aligned}$$

which is much smaller than that of M. In particular, the support of M' as D-module is contained in the hypersurface $f = 0$.

Singularities of the surface $f = 0$

Let us investigate the singularities of the complex surface

$$Z = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\},$$
$$f = (y - z)(y + z)x^2 - 2(y - z)(y + z)yx + (y - z)^2(y + z)^2 + 4z^2.$$

Following N. Honda and T. Kawai, we rewrite f as

$$f = yzx^2 - yz(y + z)x + y^2z^2 + (y - z)^2$$

by change of coordinates $(y + z, y - z) \to (y, z)$.
Then the singular locus of Z is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$.
The projection $Z \ni (x, y, z) \mapsto (y, z)$ defines a doube covering on $\{(x, y) \mid xy \neq 0\}$ branched along the union of curves $y - z = 0$ and $yz - 4 = 0$.

The stratification of Z with respect to the (local) b-function $b_{f,p}(s)$ of f at a point p is

strata	$b_{f,p}(s)$
$\{(0,0,0)\}$	$(s+1)^3(2s+3)$
$\{(2,0,0),(-2,0,0),(2,2,2),(-2,-2,-2)\}$	$(s+1)^2(2s+3)$
$\{x=y=z\}\cup\{y=z=0\}$ $\setminus\{(0,0,0),(\pm 2,0,0),\pm(2,2,2)\}$	$(s+1)^2$
$\{f=0\}\setminus(\{x=y=z\}\cup\{y=z=0\})$	$s+1$

In comparison, that of $g := x^2 - y^2 z$ (Whitney umbrella) is

strata	$b_{g,p}(s)$
$\{(0,0,0)\}$	$(s+1)^2(2s+3)$
$\{x=y=0\}\setminus\{(0,0,0)\}$	$(s+1)^2$
$\{g=0\}\setminus\{x=y=0\}$	$s+1$

Cross section of $\tilde{F}_G$

In order to guess the multiplicity and the exponent (order) of $\tilde{F}_G$ along the conormal bundle of $f = 0$ at a non-singular point, we compute the restriction of the holonomic system M to a generic line. For example, we can take $L = \{(x, y, z) \mid y = 1, z = 2\}$. The restriction of f to L is $-3x^2 + 6x + 25 = -(3x^2 - 6x - 25)$, which have two real roots α and $2 - \alpha$. Then $F(x) := \tilde{F}_G((x, 0), (1, 2))$ is annihilated by a 5th order differential operator

$$\begin{aligned} P = {} & 147316552073926635122538062595769976812320x(x-3) \\ & \times (x-2)(x+1)(x+2)(x^2-2x-7)(3x^2-6x-25)\partial_x^5 \\ & + (287143283396437204034516799828224350871 1x^{19} + \cdots)\partial_x^4 \\ & + \cdots \end{aligned}$$

The indicial polynomial at α is $s(s-1)(s-2)(s-3)(s+1)$. Hence

$$\tilde{F}_G((x, 0), (1, 2)) = U(x - \alpha + \sqrt{-1}0)^{-1}$$

at $(\alpha, \sqrt{-1}dx)$ with a microdifferential operator U of order 0.

Landau-Nakanishi surface for general m_1, m_2, m_3

$\tilde{F}_G((x,0),(y,z))$ is well-defined on

$$
\begin{aligned}
&\{(x,y,z;\sqrt{-1}(udx+vdy+wdz) \mid (u,v,w) \neq (0,0,0)\} \\
&\quad \setminus \Big(\{(x-y)^2 - z^2 - (m_2-m_3)^2 = wx - wy + vz = u + v = 0\} \\
&\quad \{(x-y)^2 - z^2 - (m_2+m_3)^2 = wx - wy + vz = u + v = 0\} \\
&\qquad \cup \{y^2 - z^2 - (m_1-m_3)^2 = wy - vz = u = 0\} \\
&\qquad \cup \{y^2 - z^2 - (m_1+m_3)^2 = wy - vz = u = 0\} \\
&\qquad \cup \{x + m_1 + m_2 = v = w = 0\} \\
&\qquad \cup \{x + m_1 - m_2 = v = w = 0\} \\
&\qquad \cup \{x - m_1 + m_2 = v = w = 0\} \\
&\qquad \cup \{x - m_1 - m_2 = v = w = 0\} \cup \{x = v = w = 0\}
\end{aligned}
$$

as a microfunction

and the projection of its (micro-)support to the base space $\mathbb{R}^3$ is contained in the (Landau-Nakanishi) surface $f(x,y,z) = 0$ with

$$\begin{aligned} f = & (y^2 - z^2)x^4 + (-2y^3 + (2z^2 - 2m_1^2 + 2m_3^2)y)x^3 \\ & + (y^4 + (-2z^2 + 4m_1^2 - 2m_2^2 - 2m_3^2)y^2 + z^4 \\ & \quad + (2m_2^2 + 2m_3^2)z^2 + m_1^4 - 2m_3^2 m_1^2 + m_3^4)x^2 \\ & + ((-2m_1^2 + 2m_2^2)y^3 + ((2m_1^2 - 2m_2^2)z^2 - 2m_1^4 \\ & \quad + (2m_2^2 + 2m_3^2)m_1^2 - 2m_3^2 m_2^2)y)x \\ & + (m_1^4 - 2m_2^2 m_1^2 + m_2^4)y^2 + (-m_1^4 + 2m_2^2 m_1^2 - m_2^4)z^2. \end{aligned}$$

By the coordinate transformation $(y+z, y-z) \to (y, z)$, f becomes

$$\begin{aligned} f &= zy\underline{x^4} - (y+z)(zy + m_1^2 - m_3^2)\underline{x^3} \\ &+ \{(z^2+m_1^2)y^2 + 2(m_1^2 - m_2^2 - m_3^2)zy + m_1^2 z^2 + (m_1^2 - m_3^2)^2\}\underline{x^2} \\ &- (m_1 - m_2)(m_1+m_2)(y+z)(zy + m_1^2 - m_3^2)\underline{x} \\ &+ (m_1 - m_2)^2(m_1+m_2)^2 zy. \end{aligned}$$

The singular locus of $f = 0$ is given by

$$\begin{aligned} &\{f = f_x = f_y = f_z = 0\} \\ &= \{y - z = -zx^2 + (z^2 + m_1^2 - m_3^2)x + (-m_1^2 + m_2^2)z = 0\} \\ &\quad \cup \{x = m_1 - m_2 = 0\} \cup \{x = m_1 + m_2 = 0\} \\ &\quad \cup \{x^2 + (-y-z)xm_1^2 - m_2^2 = zy - m_1^2 = m_3 = 0\} \\ &\quad \cup \{(z+m_1)y + m_1 z + m_1^2 - m_3^2 = x + m_1 = m_2 = 0\} \\ &\quad \cup \{(z-m_1)y - m_1 z + m_1^2 - m_3^2 = x - m_1 = m_2 = 0\} \\ &\quad \cup \{zy - m_3^2 = x + m_2 = m_1 = 0\} \cup \{zy - m_3^2 = x - m_2 = m_1 = 0\}. \end{aligned}$$

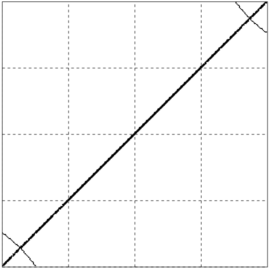
For example, if $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ (probably generic case), then the local b-function $b_{f,p}(s)$ of f at $p = (x, y, z)$ defined by the equations

$$y - z = z^4 - 20z^2 + 64 = 8x^2 - (z^3 - 12z)x - 24 = 0$$

is $(s+1)^2(2s+3)$, which is the same as that of the Whitney umbrella. There are 8 such points p, which are all in $\mathbb{R}^3$.

If $m_1 = 2$, $m_2 = m_3 = 1$, then the local b-function $b_{f,p}(s)$ of f at $p = \pm(\sqrt{3}, \sqrt{3}, \sqrt{3})$ is $(s+1)^3(2s+3)$. This implies that the singularity at p of f is not analytically equivalent to the Whitney umbrella.

The projection to the xy-space of the singular locus of $f = 0$ is as below:



If $m_1 = 1$, $m_2 = m_3 = 2$, then the local b-function $b_{f,p}(s)$ of f at $p = \pm(\sqrt{-3}, \sqrt{-3}, \sqrt{-3})$ is $(s+1)^3(2s+3)$.

Acknowledgement

I'd like to thank Professor T. Kawai for suggesting me to compute holonomic systems for Feynman integrals. However, to my regret, I can only compute very simple examples, even in the two-dimensional space-time as was shown so far.

In computation, I made use of a computer algebra system Risa/Asir developed by M. Noro et al., originally at Fujitsu Laboratories Limited. Risa/Asir programs for primary and prime decomposition have been especially useful.