An algorithmic study on the integration of holonomic distributions

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Plan of the talk

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Introduction: an example from statistics

As an example, let us consider the integral

$$F(t) = \frac{1}{2\pi} \int_{D(t)} \exp\Big(-\frac{1}{2}(x^2 + y^2)\Big)\, dxdy,$$
$$D(t) = \{(x, y) \in \mathbb{R}^2 \mid xy \leq t\}.$$

$F(t)$ can be regarded as the cumulative distribution function of a random variable XY with (X, Y) being a random vector of the two dimensional standard normal (Gaussian) distribution. By using the Heaviside function $Y(t)$ and the delta function $\delta(t)$, we have

$$F(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\Big(-\frac{1}{2}(x^2 + y^2)\Big)\, Y(t - xy)\, dxdy,$$
$$v(t) := F'(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\Big(-\frac{1}{2}(x^2 + y^2)\Big)\delta(t - xy)\, dxdy.$$

The integrand $u(x, y, t) := \exp\Big(-\frac{1}{2}(x^2 + y^2)\Big)\delta(t - xy)$ satisfies a holonomic system

$$(\partial_y + x\partial_t + y)u = (\partial_x + y\partial_t + x)u = (t - xy)u = 0$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $\partial_t = \partial/\partial t$.
We obtain (by the integration algorithm for D-modules) the relation

$$\begin{aligned} & y\partial_t(\partial_y + x\partial_t + y) - y(\partial_x + y\partial_t + x) + (\partial_t^2 - 1)(t - xy) \\ & = -\partial_x y + \partial_y y\partial_t + \underline{t\partial_t^2 + \partial_t - t}. \end{aligned}$$

Since the differential operator on the left-hand side annihilates $u(x, y, t)$, we get

$$\begin{aligned} (t\partial_t^2 + \partial_t - t)v(t) &= \int_{\mathbb{R}^2} (t\partial_t^2 + \partial_t - t)u(x, y, t)\, dxdy \\ &= \int_{\mathbb{R}^2} \partial_x(yu(x, y, t))\, dxdy - \int_{\mathbb{R}^2} \partial_y(y\partial_t u(x, y, t))\, dxdy = 0. \end{aligned}$$

The integrals on the last line vanish since $yu(x,y,t)$ and $y\partial_t u(x,y,t)$ are 'rapidly decreasing' in x,y; this reasoning shall be made precise later. It follows that $w(z) := v(-iz)$ satisfies the Bessel differential equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + z^2 w = 0.$$

Together with the property that $v(t) \to 0$ as $t \to \pm\infty$, this implies

$$v(t) = C_1 H_0^{(1)}(it) \quad (t > 0), \qquad v(t) = C_2 H_0^{(2)}(it) \quad (t < 0)$$

with some constants $C_1,\ C_2$, where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the Hankel functions. This fact was observed by Wishart and Bartlett (1932). Note that $v(t)$ is discontinuous at $t = 0$ but is locally integrable and satisfies the differential equation in the sense of generalized functions on the whole real line $\mathbb{R}$.

By the way, it follows that the characteristic function

$$\widehat{v}(\tau) := \int_{-\infty}^{\infty} e^{it\tau} v(t)\, dt = \int_{\mathbb{R}^2} \exp\Big(i\tau xy - \frac{1}{2}(x^2 + y^2)\Big)\, dxdy$$

satisfies a differential equation

$$(\tau^2 + 1)\frac{d}{d\tau}\widehat{v}(\tau) + \tau\widehat{v}(\tau) = 0.$$

Together with $\widehat{v}(0) = 1$, this implies $\widehat{v}(\tau) = \dfrac{1}{\sqrt{\tau^2 + 1}}$.
Thus we also get a representation

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{0} \frac{\exp(-i(t + i0)\tau)}{\sqrt{\tau^2 + 1}}\, d\tau + \frac{1}{2\pi}\int_{0}^{\infty} \frac{\exp(-i(t - i0)\tau)}{\sqrt{\tau^2 + 1}}\, d\tau$$

as hyperfunction.

General integrals

In general, for a holonomic function $u(x, y)$ with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$, let us consider the integral

$$v(x) = \int_{D(x)} u(x, y)\, dy_1 \cdots dy_d,$$
$$D(x) := \{y \in \mathbb{R}^d \mid f_j(x, y) \geq 0 \quad (1 \leq j \leq m)\}$$

with real polynomials $f_1, \dots, f_m$ in (x, y).

We rewrite it as

$$v(x) = \int_{\mathbb{R}^d} u(x,y) Y(f_1(x,y)) \cdots Y(f_m(x,y))\, dy_1 \cdots dy_d$$

and apply the D-module theoretic integration algorithm to obtain a holonomic system for $v(x)$, assuming that the integrand and its derivatives are 'rapidly decreasing' with respect to the integration variables y. In the process, we also need an algorithm to compute a holonomic system for the product $uY(f_1) \cdots Y(f_m)$ as a generalized function. Then the D-module theory assures us that the obtained system of differential equations for $v(x)$ is holonomic.

Distributions as generalized functions

Notations:

- $\mathbb{R}^n \ni x = (x_1, \ldots, x_n)$,
- $\partial = (\partial_1, \ldots, \partial_n)$ with $\partial_i = \partial_{x_i} = \partial/\partial x_i$,
- $D_n = \mathbb{C}\langle x, \partial \rangle$: the ring of differential operators with polynomial coefficients,
- $\mathcal{D}'(U)$: the set of Schwartz distributions on an open set U of $\mathbb{R}^n$,
- $\mathcal{S}(\mathbb{R}^n)$: the set of rapidly decreasing C^∞-functions on $\mathbb{R}^n$,
- $\mathcal{S}'(\mathbb{R}^n)$: the set of tempered distributions on $\mathbb{R}^n$.

Integration of a distribution

Let us consider distributions in variables (x, y) with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$. We regard y as the integration variables and x as parameters. Let $\pi : \mathbb{R}^{n+d} \ni (x, y) \mapsto x \in \mathbb{R}^n$ be the projection. Let U be an open set of $\mathbb{R}^n$ and let u be a distribution defined on $\pi^{-1}(U) = U \times \mathbb{R}^d$.

In order for the integral $\int_{\mathbb{R}^d} u(x, y)\, dy$ to be well-defined as a distribution on U, we need some 'tameness' of u with respect to y. Let us introduce the following two sufficient conditions:

• Let u be a distribution on $\pi^{-1}(U)$ such that $\pi : \mathrm{supp}\, u \to \mathbb{R}^n$ is proper, i.e., for any compact set K of U, $\pi^{-1}(K) \cap \mathrm{supp}\, u$ is compact. Let us denote by $\mathcal{D}'\mathcal{E}'(U \times \mathbb{R}^d)$ the set of such distributions, which constitutes a left D_{n+d}-submodule of $\mathcal{D}'(U \times \mathbb{R}^d)$. The integral of $u \in \mathcal{D}'\mathcal{E}'(U \times \mathbb{R}^d)$ in y is defined by

$$\left\langle \int_{\mathbb{R}^d} u(x,y)\, dy,\ \varphi(x) \right\rangle = \langle u(x,y), \varphi(x) 1(y) \rangle \qquad (\forall \varphi(x) \in C_0^\infty(U)),$$

where $1(y)$ denotes the constant function with value 1. This integral belongs to $\mathcal{D}'(U)$.

• Let $\mathcal{SS}'(\mathbb{R}^{n+d})$ be the subspace of $\mathcal{S}'(\mathbb{R}^{n+d})$ consisting of distributions of the form

$$u(x,y) = \sum_{j=1}^{m} u_j(y) v_j(x,y)$$

$$(m \in \mathbb{N},\ u_j(y) \in \mathcal{S}(\mathbb{R}^d),\ v_j(x,y) \in \mathcal{S}'(\mathbb{R}^{n+d})). \quad (1)$$

Then $\mathcal{SS}'(\mathbb{R}^{n+d})$ is a left D_{n+d}-submodule of $\mathcal{S}'(\mathbb{R}^{n+d})$. The integral of $u(x,y)$ in y is naturally defined as an element of $S'(\mathbb{R}^d)$ by

$$\left\langle \int_{\mathbb{R}^d} u(x,y)\,dy,\ \varphi(x) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\, \varphi(x) u_j(y) \rangle \quad (\forall \varphi(x) \in \mathcal{S}(\mathbb{R}^d)).$$

The following propositions will play a crucial role in the integration algorithm for holonomic distributions:

Proposition (differentiation under the integral sign)

Let $u(x, y)$ belong to $\mathcal{D}'\mathcal{E}'(U \times \mathbb{R}^d)$ with an open subset U of $\mathbb{R}^n$, or else to $\mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then for any $P = P(x, \partial_x) \in D_n$, we have

$$P(x, \partial_x) \int_{\mathbb{R}^d} u(x, y)\, dy = \int_{\mathbb{R}^d} P(x, \partial_x) u(x, y)\, dy.$$

Proposition

Let $u(x, y)$ belong to $\mathcal{D}'\mathcal{E}'(U)$ with an open subset U of $\mathbb{R}^n$, or else to $\mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then we have

$$\int_{\mathbb{R}^d} \partial_{y_j} u(x, y)\, dy = 0 \quad (j = 1, \ldots, d).$$

Proof: Let us assume that $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^{n+d})$. We may assume, without loss of generality, that $u(x,y) = v(y)w(x,y)$ with $v(y) \in \mathcal{S}(\mathbb{R}^d)$ and $w(x,y) \in \mathcal{S}'(\mathbb{R}^{n+d})$.
Then it follows from the definition of the integral that

$$\begin{aligned}
&\left\langle \int_{\mathbb{R}^d} \partial_{y_j}(v(y)w(x,y))\,dy,\ \varphi(x) \right\rangle \\
&= \left\langle \int_{\mathbb{R}^d} (\partial_{y_j} v(y))w(x,y)\,dy,\ \varphi(x) \right\rangle \\
&+ \left\langle \int_{\mathbb{R}^d} v(y)(\partial_{y_j} w(x,y))\,dy,\ \varphi(x) \right\rangle \\
&= \langle w(x,y),\, \varphi(x)\partial_{y_j} v(y)\rangle + \langle \partial_{y_j} w(x,y),\, \varphi(x)v(y)\rangle \\
&= \langle w(x,y),\, \varphi(x)\partial_{y_j} v(y)\rangle - \langle w(x,y),\, \partial_{y_j}(\varphi(x)v(y))\rangle = 0
\end{aligned}$$

holds for any $\varphi(x) \in \mathcal{S}(\mathbb{R}^d)$. □

Example

Set $x = (x_1, \dots, x_n)$ and let a be an arbitrary positive real number. Let $f(x)$ be a real polynomial in x. Then $\exp(-a|x|^2)Y(t - f(x))$ belongs to $\mathcal{SS}'(\mathbb{R} \times \mathbb{R}^n)$. Hence the integral

$$F(t) = \int_{\mathbb{R}^n} \exp(-a|x|^2)Y(t - f(x))\,dx$$

is well-defined as an element of $\mathcal{S}'(\mathbb{R})$. If $a = 1/2$, then $F(t)$ is the cumulative distribution function of the random variable $f(x)$ with x being the random vector of the n-dimensional normal (Gaussian) distribution. The derivative $F'(t)$ is given by the integral

$$F'(t) = \int_{\mathbb{R}^n} \exp(-a|x|^2)\delta(t - f(x))\,dx \ \in \ \mathcal{S}'(\mathbb{R}).$$

$\exp(\sqrt{-1}\,tf(x) - a|x|^2)$ also belongs to $\mathcal{SS}'(\mathbb{R} \times \mathbb{R}^n)$.

Example

Let $f(x)$ be a real polynomial in $x = (x_1, \ldots, x_n)$, and $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. Let us consider the integral

$$F(t) = \int_{\mathbb{R}^n} e^{-b_1 x_1 - \cdots - b_n x_n} Y(t - f(x))(x_1)_+^{a_1 - 1} \cdots (x_n)_+^{a_n - 1} \, dx,$$

which can be regarded, up to a constant multiple depending on a_i, b_i, as the cumulative distribution function of the random variable $f(X)$ with the random vector X of the multi-dimensional gamma distribution. Let $\chi(t)$ be a C^∞ function on $\mathbb{R}$ such that $\chi(t) = 1$ for $t \geq -1$ and $\chi(t) = 0$ for $t \leq 2$.

Then we have

$$
\begin{aligned}
& e^{-b_1x_1-\cdots-b_nx_n}(x_1)_+^{a_1-1}\cdots(x_n)_+^{a_n-1} \\
&= e^{-b_1x_1-\cdots-b_nx_n}\chi(x_1)\cdots\chi(x_n)(x_1)_+^{a_1-1}\cdots(x_n)_+^{a_n-1}
\end{aligned}
$$

and $e^{-a_1x_1-\cdots-a_nx_n}\chi(x_1)\cdots\chi(x_n)$ belongs to $\mathcal{S}(\mathbb{R}^n)$. Hence the integrand belongs to $\mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R})$ and $F(t)$ is well-defined as an element of $\mathcal{S}'(\mathbb{R})$. Its derivative is given by

$$
F'(t) = \int_{\mathbb{R}^n} e^{-b_1x_1-\cdots-b_nx_n}\delta(t-f(x))(x_1)_+^{a_1-1}\cdots(x_n)_+^{a_n-1}\,dx.
$$

Holonomic distributions

A distribution $u(x) \in \mathcal{D}(U)$, with an open set U of $\mathbb{R}^n$, is called *holonomic*, if $D_n/\mathrm{Ann}_{D_n} u$ is a holonomic D_n-module, where

$$\mathrm{Ann}_{D_n} u = \{P \in D_n \mid Pu = 0\}$$

is the annihilator (ideal) of u.

Integration as an operation on D-modules

Set (x, y) with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$. In this section we set $Y = \mathbb{C}^{n+d}$ and $X = \mathbb{C}^n$ to simplify the notation. Let $\pi : Y \ni (x, y) \longmapsto x \in X$ be the projection. We denote by $D_Y = D_{n+d}$ the ring of differential operators on the variables (x, y), and by $D_X = D_n$ that on the variables x. The module

$$D_{X \leftarrow Y} := D_Y / (\partial_{y_1} D_Y + \cdots + \partial_{y_d} D_Y).$$

has a structure of (D_X, D_Y)-bimodule. The *integral* of a left D_Y-module M along the fibers of π, or the *direct image* by π is defined to be

$$\pi_* M := D_{X \leftarrow Y} \otimes_{D_Y} M = M / (\partial_{t_1} M + \cdots + \partial_{t_d} M).$$

This is a left D_X-module since any element of D_X commutes with ∂_{y_j}.

Let us assume that M is generated by a single element $u \in M$ as left D_Y-module. Let $[u]$ be the residue class of u in $\pi_* M$. Then $\pi_* M$ is generated by $\{y^\gamma [u] \mid \gamma \in \mathbb{N}^d\}$ over D_X.

Let $\varphi \in \mathrm{Hom}_{D_Y}(M, \mathcal{S}\mathcal{S}'(\mathbb{R}^{n+d}))$. Define $\varphi' \in \mathrm{Hom}_{D_X}(M, \mathcal{S}'(\mathbb{R}^n))$ by

$$\varphi'(v) = \int_{\mathbb{R}^d} \varphi(v)\, dy \qquad (\forall v \in M).$$

Since

$$\partial_{y_1} M + \cdots + \partial_{y_d} M \subset \mathrm{Ker}\, \varphi',$$

φ' induces a D_X-homomorphism

$$\pi_*(\varphi) : \pi_* M \longrightarrow \mathcal{S}'(\mathbb{R}^n).$$

The generators $y^\gamma[u]$ of $\pi_* M$ with $\gamma' \in \mathbb{N}^d$ are sent by $\pi_*(\varphi)$ to

$$\pi_*(\varphi)(y^\gamma[u]) = \int_{\mathbb{R}^d} y^\gamma f(x, y)\, dy, \qquad f := \varphi(u) \in \mathcal{SS}'(\mathbb{R}^{n+d}).$$

In conclusion, we have defined a $\mathbb{C}$-linear map

$$\pi_* \ : \ \mathrm{Hom}_{D_Y}(M, \mathcal{SS}'(\mathbb{R}^{n+d})) \longrightarrow \mathrm{Hom}_{D_X}(\pi_* M, \mathcal{S}'(\mathbb{R}^n)).$$

Theorem(Bernstein, Kashiwara)

If M is a holonomic D_Y-module, then $\pi_* M$ is a holonomic D_X-module.

An algorithm for integration

Let M be a left D_{n+d}-module generated by $u \in M$. Let us fix the weight vector

$$w := (0, \dots, 0, 1, \dots, 1; 0, \dots, 0, -1, \dots, -1) \in \mathbb{Z}^{2(n+d)}.$$

That is, we define the weight of x_i and ∂_{x_i} to be 0, while the weight of y_j and ∂_{y_j} are 1 and -1 respectively. Set

$$F_k(M) := F_k^w(D_Y)u, \qquad \mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M) \qquad (k \in \mathbb{Z}).$$

Then $\{F_k(M)\}$ is a good w-filtration on M.

Set

$$\theta := \partial_{y_1} y_1 + \cdots + \partial_{y_d} y_d = y_1 \partial_{y_1} + \cdots + y_d \partial_{y_d} + d.$$

Theorem-Definition

If M is a holonomic D_Y-module, then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ in s such that $b(\theta)\mathrm{gr}_0(M) = 0$. Such $b(s)$ of minimum degree is called the b-function of M with respect to the weight vector w and the filtration $\{F_k(M)\}$.

Note that a non-holonomic D_Y-module can have a b-function in the above sense. The following arguments only rely on the existence of the b-function hence applies also to such non-holonomic modules.

Proposition

Suppose that a left D_Y-module $M = D_Y u = D_Y/I$ has a b-function $b(s)$ with respect to the weight vector w as above and the good w-filtration $F_k(M) := F_k^w(D_Y)u$. Let $-k_1$ be the smallest integral root, if any, of $b(s)$. Set $k_1 = -1$ if $b(s)$ has no integral root. Then the exact sequence

$$M^d \xrightarrow{(\partial_{y_1},\dots,\partial_{y_d})} M \longrightarrow \pi_* M \longrightarrow 0$$

induces an exact sequence

$$F_{k_1+1}(M)^d \xrightarrow{(\partial_{y_1},\dots,\partial_{y_d})} F_{k_1}(M) \longrightarrow \pi_* M \longrightarrow 0.$$

Let

$$(D_Y)^r \xrightarrow{\psi} D_Y \xrightarrow{\varphi} M \longrightarrow 0$$

be a presentation of M, where

$$\varphi(P) = Pu \quad (\forall P \in D_Y),$$
$$\psi((Q_1, \dots, Q_r)) = Q_1 P_1 + \cdots + Q_r P_r \quad (\forall Q_1, \dots, Q_r \in D_Y).$$

Here we assume that $P_1, \dots, P_r$ are a w-involutive basis of $I = \mathrm{Ann}_{D_Y} u$ with $\mathrm{ord}_w(P_i) = m_i$. This implies that the sequence

$$\oplus_{i=1}^r F_{k-m_i}(D_Y) \xrightarrow{\psi} F_k(D_Y) \xrightarrow{\varphi} F_k(M) \longrightarrow 0$$

is exact. Set $F_k[\mathbf{m}]((D_{X\leftarrow Y})^r) := \oplus_{i=1}^r F_{k-m_i}(D_{X\leftarrow X})$ with $\mathbf{m} = (m_1, \dots, m_r)$, and so on.

Then ψ induces homomorphisms

$$\overline{\psi} : (D_{X\leftarrow Y})^r \longrightarrow D_{X\leftarrow Y},$$
$$\overline{\psi} : F_k[\mathbf{m}]((D_{X\leftarrow Y})^r) := \oplus_{i=1}^r F_{k-m_i}(D_{X\leftarrow Y}) \longrightarrow F_k(D_{X\leftarrow Y}),$$

where $\{F_k(D_{X\leftarrow Y})\}$ denotes the filtration induced by $\{F_k^w(D_Y)\}$.
We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
F_{k_1+1}[\mathbf{m}]((D_Y)^r)^d & \longrightarrow & F_{k_1}[\mathbf{m}]((D_Y)^r) & \longrightarrow & F_{k_1}[\mathbf{m}]((D_{X\leftarrow Y})^r) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\psi,\ldots,\psi)} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\
F_{k_1+1}(D_Y)^d & \xrightarrow{(\partial_{y_1},\ldots,\partial_{y_d})} & F_{k_1}(D_Y) & \longrightarrow & F_{k_1}(D_{X\leftarrow Y}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\varphi,\ldots,\varphi)} & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
F_{k_1+1}(M)^d & \xrightarrow{(\partial_{y_1},\ldots,\partial_{y_d})} & F_{k_1}(M) & \longrightarrow & \pi_* M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

In the commutative diagram, the three horizontal sequences and the two vertical sequences except the rightmost one are exact. This implies that the rightmost vertical sequence is also exact. Note that

$$F_{k_1}(D_{X\leftarrow Y}) = \bigoplus_{|\gamma|\le k_1} y^\gamma D_X, \quad F_{k_1}[\mathbf{m}]((D_{X\leftarrow Y})^r) = \bigoplus_{i=1}^{r} \bigoplus_{|\gamma|\le k_1-m_i} y^\gamma D_X$$

as left D_X-modules. Hence $\overline{\psi}$ is a homomorphism of free left D_X-modules of finite rank, $\mathrm{coker}\,\overline{\psi}$ can be explicitly computed by linear algebra over D_X. This gives the relations among the generators $\{y^\gamma[u] \mid |\gamma| \le k_1\}$ of $\pi_* M$. By elimination, we can obtain $\mathrm{Ann}_{D_X}[u]$ so that $D_X[u] \cong D_X/\mathrm{Ann}_{D_X}[u]$ is a left D_X-submodule of $\pi_* M$.

For practical computation of integration, some computer algebra systems are available such as Kan/sm1 by N. Takayama, Risa/Asir by M. Noro et al., Macaulay2, Singular, and so on. We make use of a Risa/Asir library `nk_restriction.rr` by H. Nakayama for computing various examples in the next section.

Integrals of holonomic distributions over the whole space

Let $u(x, y) = u(x_1, \ldots, x_n, y_1, \ldots, y_d)$ be a distribution in $\mathcal{D}'\mathcal{E}'(U \times \mathbb{R}^d)$ with an open set U of $\mathbb{R}^n$, or in $\mathcal{S}\mathcal{S}'(\mathbb{R}^{n+d})$. Suppose that $u(x, y)$ is holonomic and that we have a left ideal I of D_{n+d} which annihilates $u(x, y)$ such that D_{n+d}/I is holonomic. Then the integration module $\pi_* M = M/(\partial_{y_1} M + \cdots + \partial_{y_d} M)$ gives a holonomic system of linear differential equations for

$$v(x) := \int_{\mathbb{R}^d} u(x, y)\, dy,$$

which belongs to $\mathcal{D}'(U)$ or to $\mathcal{S}'(\mathbb{R}^n)$.

Let us first consider the standard normal distribution whose density function is given by $(2\pi)^{-n/2}\exp\Big(-\dfrac{1}{2}|x|^2\Big)$. Let $f(x)$ be an arbitrary real polynomial in $x=(x_1,\dots,x_n)$. Then the cumulative function

$$F(t)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\exp\Big(-\frac{1}{2}|x|^2\Big)Y(t-f(x))\,dx$$

of the random variable $f(x)$ is well-defined as an element of $\mathcal{S}'(\mathbb{R})$ since the integrand belongs to $\mathcal{S}\mathcal{S}'(\mathbb{R}^{n+1})$. The density function $F'(t)$ is given by the integral

$$v(t):=F'(t)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\exp(-\frac{1}{2}|x|^2)\delta(t-f(x))\,dx$$

as an element of $\mathcal{S}'(\mathbb{R})$.

Moreover, $F(t)$ and $F'(t)$ are real analytic outside of the set of critical values of f:

$$C(f) := \{f(x) \mid \partial_1 f(x) = \cdots = \partial_n f(x) = 0\}.$$

By the integration algorithm, we obtain a linear ordinary differential equation which $F'(t)$ satisfies as a distribution on $\mathbb{R}$ including $C(f)$.

Remark: The Fourier transform (i.e., the characteristic function) of the density function $v(t) = F'(t)$ is expressed as an oscillatory integral

$$\widehat{v}(\tau) = \int_{-\infty}^{\infty} e^{it\tau} v(t)\,dt = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\Big(\sqrt{-1}\,\tau f(x) - \frac{1}{2}|x|^2\Big)\,dx.$$

The following well-known fact is useful for our purpose.

Proposition

Let $t_0 \in \mathbb{R}$ be a regular singular point of an ordinary differential operator

$$P = a_0(t)\partial_t^m + a_1(t)\partial_t^{m-1} + \cdots + a_m(t).$$

1. If P has no negative integer as a characteristic exponent at t_0, then P has no hyperfunction solution whose support is $\{t_0\}$ on a neighborhood of t_0.
2. If the real part of each characteristic exponent of P at t_0 is greater than -1, then any hyperfunction solution of $Pu = 0$ is locally integrable on a neighborhood of t_0.

χ^2 distribution

Set

$$u(x,t) = (2\pi)^{-n/2} \exp\Big(-\frac{1}{2}|x|^2\Big)\delta(t-|x|^2), \quad v(t) = \int_{\mathbb{R}^n} u(x,t)\,dx$$

with $|x|^2 = x_1^2 + \cdots + x_n^2$. Then $u(x,t)$ belongs to $\mathcal{SS}'(\mathbb{R}^{n+1})$ and thus $v(t)$ is well-defined as a tempered distribution on $\mathbb{R}$. Note that $v(t)$ is the density function of the χ^2 distribution. $u(x,t)$ satisfies a holonomic system

$$(t-|x|^2)u = (\partial_i + 2x_i\partial_t + x_i)u = 0 \quad (i = 1,\dots,n).$$

Since

$$\sum_{i=1}^{n} x_i(\partial_i + 2x_i\partial_t + x_i) + (1 + 2\partial_t)(t - |x|^2)$$

$$= \sum_{i=1}^{n} x_i\partial_i + 2|x|^2\partial_t + |x|^2 + (1 + 2\partial_t)(t - |x|^2)$$

$$= \sum_{i=1}^{n} x_i\partial_i + 2\partial_t t + t = \sum_{i=1}^{n} \partial_i x_i + 2t\partial_t + t - n + 2,$$

we know that $v(t)$ satisfies

$$(2t\partial_t + t - n + 2)v(t) = 0.$$

Solving this equation by quadrature and noting that $v(t) = 0$ for $t < 0$, we conclude that

$$v(t) = Ce^{-t/2}t_+^{n/2-1} \quad \text{with} \quad C = \frac{2^{-n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

quadratic forms

Set

$$v(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big) \delta(t - f(x))\, dx$$

with a quadratic form $f(x) = \sum_{i,j} a_{ij} x_i x_j$. If the absolute values of all the eigenvalues of (a_{ij}) are the same, then $v(t)$ satisfies a linear differential equation of the second order. We may assume

$$f(x) = a(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2)$$

with a constant $a > 0$. Then the integrand $u = u(x,t)$ satisfies

$$(t - f(x))u = (\partial_i + 2ax_i\partial_t + x_i)u = (\partial_j - 2ax_j\partial_t + x_j)u = 0 \qquad (1 \le i \le p < j \le n).$$

The following operators P and Q annihilate u:

$$\begin{aligned} P &= \sum_{i=1}^{p} x_i(\partial_i + 2ax_i\partial_t + x_i) + \sum_{i=p+1}^{n} x_i(\partial_i - 2ax_i\partial_t + x_i) \\ &= \sum_{i=1}^{n} \partial_i x_i + 2f\partial_t + |x|^2 - n = \sum_{i=1}^{n} \partial_i x_i + 2\partial_t t + |x|^2 - n \\ &\quad - 2\partial_t(t - f), \\ Q &= \sum_{i=1}^{p} x_i(\partial_i + 2ax_i\partial_t + x_i) - \sum_{i=p+1}^{n} x_i(\partial_i - 2ax_i\partial_t + x_i) \\ &= \sum_{i=1}^{p} \partial_i x_i - \sum_{i=p+1}^{n} \partial_i x_i + 2a|x|^2\partial_t + \frac{1}{a}f + n - 2p. \end{aligned}$$

$$2a\partial_t P - Q = 2a\sum_{i=1}^{p}\partial_i x_i\partial_t - \sum_{i=1}^{p}\partial_i x_i + \sum_{i=p+1}^{n}\partial_i x_i + (-4a\partial_t^2 + \frac{1}{a})(t-f)$$
$$+ 4a\partial_t^2 t - 2na\partial_t - \frac{1}{a}t + (2p-n)$$

implies

$$\{4a^2 t\partial_t^2 + 2a^2(4-n)\partial_t - t + (2p-n)a\}v(t) = 0.$$

The solutions of this differential equation are expressed as

$$P\left\{\begin{array}{ccc} \overbrace{\qquad\qquad\infty\qquad\qquad}^{} & & 0 \\ \frac{1}{2a} \qquad\qquad 1-\frac{p}{2} & & 0 \\ -\frac{1}{2a} \quad -\frac{1}{4}(2n-2p-4) & & \frac{n-2}{2} \end{array}\right\}$$

sum of cubes of standard normal random variables

This example was proposed by A. Takemura: Set

$$v(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1^3 - \cdots - x_n^3)\, dx.$$

If $n = 2$, $v(t)$ satisfies the ordinary differential equation $Pv(t) = 0$ with

$$P = 729t^3{\partial_t}^6 + 6561t^2{\partial_t}^5 + 12555t{\partial_t}^4 + (81t^2 + 3240){\partial_t}^3 + 243t{\partial_t}^2 + 60\partial_t + 2t.$$

The origin is a regular singular point of P with the indicial polynomial $b(s) = s(s-1)^2(s-2)(3s+1)(3s-7)$ up to a constant multiple.

Example

Let us consider

$$v(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Bigl(-\frac{|x|^2}{2}\Bigr)\delta(t - x_1^4 - x_2^4 - \cdots - x_n^4)\,dx.$$

If $n = 2$, then $v(t)$ is annihilated by

$$128t^3\partial_t{}^4 + 768t^2\partial_t{}^3 + (-24t^2 + 864t)\partial_t{}^2 + (-48t + 96)\partial_t + t - 6,$$

which has a regular singularity at 0 with the indicial polynomial $b(s) = s^2(2s-1)(2s+1)$ up to a constant multiple.

If $n = 3$, $v(t)$ is annihilated by

$$\begin{aligned}&2048t^4\partial_t{}^6 + 24576t^3\partial_t{}^5 + (-768t^3 + 77568t^2)\partial_t{}^4 \\ &\quad + (-4608t^2 + 64512t)\partial_t{}^3 + (88t^2 - 5328t + 7560)\partial_t{}^2 \\ &\quad + (176t - 720)\partial_t - 3t + 27,\end{aligned}$$

which has a regular singular point at 0 with the indicial polynomial

$$b(s) = s(s-1)(4s+1)(4s-1)(4s-3)(4s-5)$$

Example

Let us consider

$$v(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1 x_2 \cdots x_n)\, dx.$$

If $n = 2$, then $v(t)$ is annihilated by $t{\partial_t}^2 + \partial_t - t$, which has 0 as a regular singular point with the indicial equation $b(s) = s^2$.
If $n = 3$, then $v(t)$ is annihilated by

$$t^2{\partial_t}^3 + 3t{\partial_t}^2 + \partial_t + t,$$

which has 0 as a regular singular point with the indicial polynomial $b(s) = s^3$. If $n = 4$, then $v(t)$ is annihilated by

$$t^3{\partial_t}^4 + 6t^2{\partial_t}^3 + 7t{\partial_t}^2 + \partial_t - t$$

with the indicial polynomial $b(s) = s^4$ at 0.

Example

Let us consider

$$v(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t-(x_1+1)(x_2+1)\cdots(x_n+1))\,dx.$$

If $n = 2$, then $v(t)$ is annihilated by

$$t{\partial_t}^3 + (t+2){\partial_t}^2 - t\partial_t - t$$

with the indicial polynomial $s^2(s-1)$ at 0.
If $n = 3$, then $v(t)$ is annihilated by

$$t^4{\partial_t}^6 + 10t^3{\partial_t}^5 + (2t^3+22t^2){\partial_t}^4 + (2t^3+8t^2+4\partial_t){\partial_t}^3 \\ + (4t^2+4t-4){\partial_t}^2 + 2t^2\partial_t + t^2$$

with the indicial polynomial $s^3(s-1)^2(s-3)$ at 0.

Powers of polynomials times a holonomic function

Let $f_1(x), \ldots, f_p(x)$ be real polynomials in $x = (x_1, \ldots, x_n)$. Let $v(x)$ be a holonomic locally integrable function on U. Then

$$\tilde{v}(x) = (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)$$

is also locally integrable on U for complex numbers $\lambda_1, \ldots, \lambda_p$ with non-negative real parts. Especially, we have

$$\tilde{v}(x) = Y(f_1) \cdots Y(f_p) v(x)$$

if $\lambda_1 = \cdots = \lambda_p = 0$. Our purpose is to compute a holonomic system for $\tilde{v}(x)$.

Our strategy is as follows:

First we work in a purely algebraic setting and consider the D-module generated by the tensor product $f_1^{\lambda_1} \cdots f_p^{\lambda_p} \otimes u$; we show that this D-module is holonomic and introduce an algorithm to compute its structure.

Then we 'realize' these arguments and apply to the corresponding distribution $\tilde{v}(x)$, which lives in the 'real world'.

Algebraic formulation

Introducing indeterminates $s = (s_1, \ldots, s_p)$, set

$$\mathcal{L} := \mathbb{C}[x, (f_1 \cdots f_p)^{-1}, s] f_1^{s_1} \cdots f_p^{s_p},$$

which is regarded as a free $\mathbb{C}[x, (f_1 \cdots f_p)^{-1}, s]$-module generated by the 'symbol' $f_1^{s_1} \cdots f_p^{s_p}$. Then $\mathcal{L}$ is a left $D_n[s]$-module with the natural derivations

$$\partial_{x_i}(f_1^{s_1} \cdots f_p^{s_p}) = \sum_{j=1}^{p} s_j \frac{\partial f_j}{\partial x_i} f_j^{-1} f_1^{s_1} \cdots f_p^{s_p} \qquad (i = 1, \ldots, n).$$

Denote $f^s = f_1^{s_1} \cdots f_p^{s_p}$ for the sake of simplicity.

Let $M = D_n u = M/I$ be a holonomic left D_n-module generated by an element $u \in M$ with the left ideal $I = \mathrm{Ann}_{D_n} u$.

Let us consider the tensor product $M \otimes_{\mathbb{C}[x]} \mathcal{L}$, which has a natural structure of left $D_n[s]$-module with the derivations

$$\partial_{x_i}(u' \otimes v) = (\partial_{x_i} u') \otimes v + u' \otimes (\partial_{x_i} v) \quad (u' \in M,\ v \in \mathcal{L},\ i = 1, \ldots, n).$$

Our aim is to compute the annihilator (in $D_n[s]$) of $u \otimes f^s \in M \otimes_{\mathbb{C}[x]} \mathcal{L}$.

For this purpose, define shift (difference) operators E_j by

$$E_j : \mathcal{L} \ni a(x, s_1, \dots, s_p) f^s \longmapsto a(x, s_1, \dots, s_j + 1, \dots, s_p) f_j f^s \in \mathcal{L}$$

for $j = 1, \dots, p$, which are bijective with the inverse shifts $E_j^{-1} : \mathcal{L} \to \mathcal{L}$.

Let $D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra generated by $s = (s_1, \dots, s_p)$, $E = (E_1, \dots, E_p)$, and $E^{-1} = (E_1^{-1}, \dots, E_p^{-1})$. We introduce new variables $t = (t_1, \dots, t_p)$ and the associated derivations $\partial_t = (\partial_{t_1}, \dots, \partial_{t_p})$.

Let D_{n+p} be the ring of differential operators with respect to the variables $(x, t) = (x_1, \dots, x_n, t_1, \dots, t_p)$.

Let $\mu : D_{n+p} \to D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra homomorphism (Mellin transform) of D_n defined by

$$\mu(t_j) = E_j, \quad \mu(\partial_{t_j}) = -s_j E_j^{-1}.$$

Since μ is injective, we can regard $E\langle s, E, E^{-1}\rangle$ as a subring of D_{n+p} through μ. With this identification, we have

$$t_j = E_j, \qquad \partial_{t_j} = -s_j E_j^{-1}, \qquad s_j = -\partial_{t_j} t_j = -t_j \partial_{t_j} - 1.$$

Hence we have inclusions

$$D_n[s] \subset D_n\langle s, E\rangle \subset D_{n+p} \subset D_n\langle s, E, E^{-1}\rangle$$

of rings.

We are interested in the $D_n[s]$-module

$$M(f; s) = M(f_1, \dots, f_p; s_1, \dots, s_p) := D_n[s](u \otimes f^s) \subset M \otimes_{\mathbb{C}[x]} \mathcal{L}$$

and its specialization

$$M(f; \lambda_1, \dots, \lambda_p)$$
$$:= M(f; s)/((s_1 - \lambda_1)M(f; s) + \dots + (s_p - \lambda_p)M(f; s))$$

for $\lambda_1, \dots, \lambda_p \in \mathbb{C}$. For this purpose, we first compute

$$N := M \otimes_{\mathbb{C}[x]} (D_{n+p} f^s).$$

Algorithm (computing N)

Input: A set G_0 of generators of I with $M = D_n/I$ and nonzero polynomials $f_1, \ldots, f_p \in \mathbb{C}[x]$.

For $P = P(x, \partial_{x_1}, \ldots, \partial_{x_n}) \in G_0$, set

$$\tau(P) := P\left(x, \partial_{x_1} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\partial_{t_j}, \ldots, \partial_{x_n} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_n}\partial_{t_j}\right).$$

Output: $G := \{\tau(P, f_1, \ldots, f_p) \mid P \in G_0\} \cup \{t_j - f_j(x) \mid j = 1, \ldots, p\}$ generates $J := \mathrm{Ann}_{D_{n+p}}(u \otimes f^s)$.
Then $N = D_{n+p}/J$ holds and is holonomic.

Next we compute the $D_n[s]$-submodule of $N = M \otimes_{\mathbb{C}[x]} D_{n+p} f^s$:

$$M'(f;s) := D_n[s](u \otimes f^s) = D_n[s]/(J \cap D_n[s]).$$

Set

$$M'(f;\lambda) := M'(f;s)/((s_1 - \lambda_1)M'(f;s) + \cdots + (s_p - \lambda_p)M'(f;s))$$

Then there exists a natural surjective D_n-homomorphism

$$\iota : M'(f;\lambda) \longrightarrow M(f;\lambda).$$

Proposition

Set $f = f_1 \cdots f_p$. If the homomorphism $f : M \to M$ is injective, then ι is an isomorphism.

Theorem

If any λ_j is not a nonnegative integer, or else if $f : M \to M$ is injective, then $M'(f;\lambda)$ is a holonomic D_n-module.

If the assumption of this theorem may not be satisfied, then we must replace M by the homomorphic image of the localization

$$M \longrightarrow M[f^{-1}] := M \otimes_{\mathbb{C}[x]} \mathbb{C}[x, f^{-1}],$$

which is also computable. This assures that $M(f;\lambda)$ is holonomic without the assumption in the theorem above.
(We conjecture that $M'(f;\lambda)$ is also always holonomic.)

This completes the algebraic argument.

Holonomic system for $(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)$

Now let us return to the 'real world'. Assume that $f_1, \dots, f_p \in \mathbb{R}[x]$ and let $v(x)$ be a locally integrable holonomic function on an open set U of $\mathbb{R}^n$. Then

$$\tilde{v}(x) := v(x)(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$$

is well-defined as a locally integrable function on U if the real parts of $\lambda_1, \dots, \lambda_p$ are non-negative. Let I be a left ideal of D_n which annihilates $v(x)$ such that $M := D_n/I$ is holonomic.

Theorem

Suppose that $P(s) \in D_n[s]$ annihilates $u \otimes f^s$ in $M(f; s)$. Then $P(\lambda)\tilde{v}(x)$ vanishes as a distribution on U if the real parts of the components of λ are non-negative.

The preceding theorem is an immediate consequence of the folowing lemma, which was proved by Kashiwara-Kawai (1979) in case $p = 1$:

Lemma

Let $f_1, \dots, f_p$ and $v(x)$ be as above and assume $\{x \in U \mid f_1(x) > 0, \dots, f_p(x) > 0\}$ is not empty. Set $f = f_1 \cdots f_p$ and $U_f = \{x \in U \mid f(x) \neq 0\}$. Let $s_1, \dots, s_p$ be indeterminates and $\lambda_1, \dots, \lambda_p$ be complex variables. Assume that $P(s_1, \dots, s_p) \in D_n[s_1, \dots, s_p]$ satisfies

$$P(\lambda_1, \dots, \lambda_p)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)) = 0 \quad \text{in } \mathcal{D}'(U_f)$$

with $U_f := \{x \in U \mid f(x) \neq 0\}$ for $\mathrm{Re}\,\lambda_j \gg 0$ ($j = 1, \dots, p$). Then

$$P(\lambda_1, \dots, \lambda_p)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)) = 0 \quad \text{in } \mathcal{D}'(U)$$

holds for any $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}\,\lambda_j \geq 0$ ($j = 1, \dots, p$).

Integrals over the domain defined by polynomial inequalities

By the algebraic argument so far, we first obtain a holonomic system for the integrand $v(x,y)(f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}$. Then the integration algorithm gives us a holonomic system for

$$w(x) = \int_{\mathbb{R}^d} v(x,y)(f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}\,dy.$$

In particular, if $\lambda_1 = \cdots = \lambda_p = 0$, then we obtain a holonomic system for

$$w(x) = \int_{D(x)} v(x,y)\,dy, \quad D(x) = \{y \in \mathbb{R}^d \mid f_j(x,y) \geq 0 \ (1 \leq j \leq p)\}.$$

As examples, let us consider truncated multi-dimensional normal distributions: Let $f_1(x), \ldots, f_p(x)$ be real polynomials in $x = (x_1, \ldots, x_n)$ and set

$$D = \{x \in \mathbb{R}^n \mid f_j(x) \geq 0 \ (1 \leq j \leq p)\}.$$

Then $\exp\Big(-\dfrac{|x|^2}{2}\Big) Y(f_1) \ldots Y(f_p)$ is, up to a constant multiple, the probability density function of the standard normal distribution truncated by D. Let $f(x)$ be a real polynomial, which we regard as a random variable. Then the cumulative and the density functions of $f(x)$ are given by

$$F(t) = \int_D \exp\Big(-\frac{|x|^2}{2}\Big) Y(t - f(x))\, dx,$$

$$v(t) := F'(t) = \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big) \delta(t - f(x)) Y(f_1(x)) \cdots Y(f_p(x))\, dx$$

respectively up to constant multiples. The integrands belong to $\mathcal{SS}'(\mathbb{R} \times \mathbb{R}^n)$.

Example

Set $f(x) = |x|^2$,

$$D = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \ (1 \leq i \leq n), \ x_1 + \cdots + x_n \leq 1\},$$

$$v(t) = \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t-|x|^2)Y(x_1)\cdots Y(x_n)Y(1-x_1-\cdots-x_n)\,dx.$$

If $n = 2$, then $v(t)$ is annihilated by a differential operator

$$4t(t-1)(2t-1){\partial_t}^2 + 4(-2t^3 + 6t^2 - 5t + 1)\partial_t + 2t^3 - 9t^2 + 9t - 2.$$

Its indicial polynomials at 0, 1, and $1/2$ are $4s^2$, $4s(s-1)$, and $-s(2s-1)$ respectively. Here 1 is an apparent singular point, and hence every distribution solution is analytic on $\{t \in \mathbb{R} \mid t \neq 0, 1/2\}$.

Example

Set $n = 2$, $D = \{x = (x_1, x_2) \mid x_1^3 - x_2^2 \geq 0\}$ and consider

$$v(t) = F'(t) = \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - f(x))\,Y(x_1^3 - x_2^2)\,dx_1 dx_2$$

with $f(x) = x_1^2 + x_2^2$. Then $v(t)$ is annihilated by

$$\begin{aligned}
&16t^3(27t-4){\partial_t}^4 + (-864t^4 + 3368t^3 - 320t^2){\partial_t}^3 \\
&+ (648t^4 - 4956t^3 + 5724t^2 - 268t){\partial_t}^2 \\
&+ (-216t^4 + 2462t^3 - 5484t^2 + 1654t - 12)\partial_t \\
&+ 27t^4 - 409t^3 + 1351t^2 - 760t + 6.
\end{aligned}$$

The indicial polynomials at 0 at $27/4$ are $s^2(4s-1)(4s-3)$ and $s(s-1)(s-2)(s-3)$ respectively up to constant multiples. The point $27/4$ is an apparent singular point.