Some algorithmic problems for holonomic distributions

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Part I Theoretical background

Product of complex power and a locally integrable function

Let $\mathcal{D}_{\mathbb{C}^n}$ be the sheaf on $\mathbb{C}^n$ of linear partial differential operators with holomorphic coefficients and let $\mathcal{D}_{\mathbb{R}^n} := \mathcal{D}_{\mathbb{C}^n}|_{\mathbb{R}^n}$ be its sheaf theoretic restriction to $\mathbb{R}^n$. We denote by $\mathcal{D}b$ the sheaf on $\mathbb{R}^n$ of the Schwartz distributions. Assume

- f is a nonzero real-valued real analytic function defined on an open connected set U of $\mathbb{R}^n$.
- $\varphi \in L^1_{\mathrm{loc}}(U)$.

Then $f_+^\lambda \varphi$ belongs to $L^1_{\mathrm{loc}}(U)$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$, where $f_+(x) = \max\{f(x), 0\}$.

Holonomic distributions

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_{\mathbb{C}^n}$-module defined on an open set Ω of $\mathbb{C}^n$. We say that a distribution φ on an open set $U \subset \Omega \cap \mathbb{R}^n$ is a solution of $\mathcal{M}$ on U if there exist a section u of $\mathcal{M}$ and a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism $\Phi : \mathcal{D}_{\mathbb{C}^n} u \to \mathcal{D}b$ defined on U such that $\Phi(u) = \varphi$. If $\mathcal{M}$ is holonomic, then we call φ a (analytically) holonomic distribution.

Let D_n be the ring of differential operators with polynomial coefficients. Then a left $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M}$ is called algebraic if there exists a finitely generated left D_n-module M such that $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n} \otimes_{D_n} M$.

Our aim is to consider $f_+^\lambda \varphi$ for a holonomic and locally integrable function φ from theoretical as well as algorithmic viewpoints.

References

- Kashiwara, M, On the holonomic systems of linear differential equations, II, Inventiones mathematicae **49** (1978), 121-135. generalized *b*-function, tensor product of holonomic systems
- Kashiwara, M., Kawai, T., On the characteristic variety of a holonomic system with regular singularities, Advances in Mathematics **34** (1979), 163–184. complex power times a locally integrable function
- Galina, E., Laurent, Y., $\mathcal{D}$-modules and characters of semisimple Lie groups, Duke Mathematical Journal **123** (2004), 265–309. local integrability of a distribution solution of a holonomic system

Fundamental lemma

$f_+^\lambda \varphi$ is a $\mathcal{D}b(U)$-valued holomorphic function of λ on the right half plane

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda > 0\}.$$

Let s be an indeterminate corresponding to λ. Let Ω be an open set of $\mathbb{C}^n$ such that $U \subset \Omega$.

Lemma (Kashiwara-Kawai (1979))

Assume $P(s) \in \mathcal{D}_{\mathbb{C}^n}(\Omega)[s]$ and $P(\lambda)(f_+^\lambda \varphi) = 0$ holds as distribution on $U_f := \{x \in U \mid f(x) \neq 0\}$ for any $\lambda \in \mathbb{C}_+$. Then $P(\lambda)(f_+^\lambda \varphi) = 0$ holds <u>on U</u> for any $\lambda \in \mathbb{C}_+$.

Holonomicity of $f_+^\lambda \varphi$

Theorem 1 (Kashiwara-Kawai (1979))

Assume that there exists a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n} u$ defined on an open set Ω of $\mathbb{C}^n$ with $\Omega \supset U$ such that $\varphi \in L^1_{\mathrm{loc}}(U)$ is a solution of $\mathcal{M}$ on U_f. Then there exists a coherent $\mathcal{D}_{\mathbb{C}^n}[s]$-module $\mathcal{M}'$ such that $\mathcal{M}'_\lambda := \mathcal{M}'/(s-\lambda)\mathcal{M}'$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module for any $\lambda \in \mathbb{C}$ and that $f_+^\lambda \varphi$ is a solution of $\mathcal{M}'_\lambda$ for any $\lambda \in \mathbb{C}_+$

In fact, in the referenced paper, the authors assume that $\mathcal{M}$ has regular singularities on $T^*_{f^{-1}(0)}\Omega$ and that the characteristic variety of $\mathcal{M}$ is contained in $T^*_\Omega\Omega \cup \pi^{-1}(f^{-1}(0))$, and prove that $\mathcal{M}'_\lambda$ is regular holonomic and that the characteristic variety of $\mathcal{M}'_\lambda$ is contained in 'W_0'.

Sketch of the proof of Theorem 1

Let $\mathcal{L} = \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s$ where f^s is regarded as a free generator. Then $\mathcal{L}$ has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}[s]$-module.

Set

$$\mathcal{N} := \mathcal{D}_{\mathbb{C}^n}[s]f^s \subset \mathcal{L}, \qquad \mathcal{M}' := \mathcal{N} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}.$$

$\mathcal{M}'$ is a coherent $\mathcal{D}_{\mathbb{C}^n}[s]$-module. $\mathcal{N}_\lambda := \mathcal{N}/(s-\lambda)\mathcal{N}$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module for any $\lambda \in \mathbb{C}$. Hence
$\mathcal{M}'_\lambda := \mathcal{M}'/(s-\lambda)\mathcal{M}' = \mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ is holonomic for any $\lambda \in \mathbb{C}$ as the tensor product of holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules (Kashiwara (1978)).

Let f^λ be the residue class of f^s in $\mathcal{N}_\lambda$. If $\mathrm{Re}\ \lambda > 0$, then the $\mathbb{C}$-bilinear sheaf homomorphism

$$\mathcal{N}_\lambda \times \mathcal{M} \ni (Pf^\lambda, Qu) \longmapsto (Pf_+^\lambda)Q\varphi \in \mathcal{D}b,$$

which is well-defined and $\mathcal{O}_{\mathbb{C}^n}$-balanced on U_f since f_+^λ is real analytic there, induces a homomorphism

$$\Phi : \mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \longrightarrow \mathcal{D}b$$

such that $\Phi((Pf^\lambda) \otimes Qu) = (Pf_+^\lambda)Q\varphi$ on U_f. Moreover Φ is $\mathcal{D}_{\mathbb{C}^n}$-linear since

$$\partial_j(Pf^\lambda \otimes Qu) = (\partial_j Pf^\lambda) \otimes Qu + Pf^\lambda \otimes (\partial_j Qu) \quad \text{in } \mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M},$$
$$\partial_j(Pf_+^\lambda Q\varphi) = (\partial_j Pf_+^\lambda)Q\varphi + Pf_+^\lambda \partial_j Q\varphi \quad \text{on } U_f.$$

If $P \in \mathcal{D}_{\mathbb{C}^n}$ satisfies $P(f^\lambda \otimes u) = 0$ in $\mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$, then $P(f_+^\lambda \varphi) = 0$ holds on U_f. The fundamental lemma implies that $P(f_+^\lambda \varphi) = 0$ holds on U.

It follows that there exists a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism

$$\mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \ \supset \ \mathcal{D}_{\mathbb{C}^n}(f^\lambda \otimes u) \xrightarrow{\Phi'} \mathcal{D}b$$

such that $\Phi'(f^\lambda \otimes u) = f_+^\lambda \varphi$. Hence $f_+^\lambda \varphi$ is a solution of the holonomic system $\mathcal{D}_{\mathbb{C}^n}(f^\lambda \otimes u) \subset \mathcal{N}_\lambda \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$.

This completes the proof of Theorem 1.

Lemma

If $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ satisfies $P(f^s \otimes u) = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$, then $P(\lambda)(f_+^\lambda \varphi) = 0$ holds in $\mathcal{D}b(U)$ for any $\lambda \in \mathbb{C}_+$.

Proof: The inclusion $\mathcal{D}_{\mathbb{C}^n}[s]f^s \subset \mathcal{L} = \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s$ induces a homomorphism

$$\iota : \mathcal{D}_{\mathbb{C}^n}[s]f^s \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}.$$

ι is bijective on U_f since $\mathcal{D}_{\mathbb{C}^n}[s]f^s = \mathcal{L}$ there. Thus $P(s)(f^s \otimes u) = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ implies $P(\lambda)(f_+^\lambda \varphi) = 0$ on U_f and hence on U by the fundamental lemma.

Generalized b-function and analytic continuation

By Kashiwara (1978), on a neighborhood of each point of Ω, there exist nonzero $b(s) \in \mathbb{C}[s]$ and $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$P(s)(f^{s+1} \otimes u) = b(s) f^s \otimes u \quad \text{in } \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}.$$

Such $b(s)$ of the smallest degree is called the (generalized) b-function for f and u.

By the above lemma,

$$P(\lambda)(f_+^{\lambda+1}\varphi) = b(\lambda) f_+^{\lambda}\varphi$$

holds for $\mathrm{Re}\,\lambda > 0$. It follows that $f_+^{\lambda}\varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function of $\lambda \in \mathbb{C}$ if U is relatively compact in Ω. In particular, $f_+^{\lambda}\varphi$ is holomorphic on a neighborhood of $\lambda = 0$.

Corollary 1

Let φ be a locally integrable function on an open set U of $\mathbb{R}^n$ and f be a real-valued real analytic function on U. Assume that there exist a holonomic system $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n} u$ on an open set Ω of $\mathbb{C}^n$ with $U \subset \Omega$ such that φ is a solution of $\mathcal{M}$ on U_f. Then there exist a holonomic system $\mathcal{M}'_0$ of which φ is a solution on U and a surjective $\mathcal{D}_{\mathbb{C}^n}$-homomorphim $\Phi : \mathcal{M}'_0 \longrightarrow \mathcal{M}$ which is an isomorphism on U_f.

Proof: Set $\mathcal{M}'_0 = \mathcal{N}_0 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ with $\mathcal{N}_0 = \mathcal{D}_{\mathbb{C}^n} f^0$. Then $\theta(f)\varphi = f_+^0 \varphi$ and $\theta(-f)\varphi$ are solutions of $\mathcal{M}'_0$ on U, where θ denotes the Heaviside function. Hence $\varphi = \theta(f)\varphi + \theta(-f)\varphi$ is also a solution of $\mathcal{M}'_0$. The natural surjective homomorphism $\mathcal{D}_{\mathbb{C}^n} f^0 \to \mathcal{D}_{\mathbb{C}^n} 1 = \mathcal{O}_{\mathbb{C}^n}$, which is an isomorphism on U_f, induces Φ.

Corollary 2

Let φ_1 and φ_2 be locally L^p and L^q functions respectively on an open set $U \subset \mathbb{R}^n$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Assume that φ_1 and φ_2 are solutions of holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively on U. Then the product $\varphi_1\varphi_2$ is a solution of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M}$ on U.

Proof: There exist holomorphic functions f_1 and f_2 such that the singular support (the projection of the characteristic variety minus the zero section) of $\mathcal{M}_k$ is contained in $f_k = 0$ for $k = 1, 2$. Set $f(z) = f_1(z)\overline{f_1(\overline{z})}f_2(z)\overline{f_2(\overline{z})}$. Then $f(x)$ is a real-valued real analytic function and φ_1 and φ_2 are real analytic on U_f. $\varphi_1\varphi_2$ is a solution of $\mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2$ on U_f. So we can apply Corollary 1.

When is a holonomic distribution locally L^p?

Let $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n} u$ be a holonomic system on an open set Ω of $\mathbb{C}^n$ and $p \geq 1$ be a real number. Assume that $\mathcal{M}$ is p-tame; i.e., there exists a stratification $\Omega = \bigcup_\alpha X_\alpha$ such that $\mathrm{Char}(\mathcal{M}) \subset \bigcup_\alpha T^*_{X_\alpha}\Omega$ and the real parts of the roots of the b-function of $\mathcal{M}$ along each X_α are greater than $-\mathrm{codim}\, X_\alpha / p$.
Let the singular support $\mathrm{SS}(\mathcal{M})$ of $\mathcal{M}$ be $\{z \in \Omega \mid f(z) = 0\}$ with a holomorphic function $f(z)$. Then we also assume that $f(x)$ is real-valued for $x \in \Omega \cap \mathbb{R}^n$ and that at each point of $\mathrm{SS}(\mathcal{M})$, there exists a locally analytic coordinate transformation Φ so that $f \circ \Phi$ is homogeneous.

Theorem (Galina-Laurent, 2004)

Under the above assumptions, if φ is a distribution solution of $\mathcal{M}$ on $U := \Omega \cap \mathbb{R}^n$, then φ is locally L^p on U. If $\mathcal{M}$ is regular holonomic, then φ may be a hyperfunction solution.

In fact, this theorem is stated and proved with a weaker assumption (conic p-tameness with respect to a vector field).

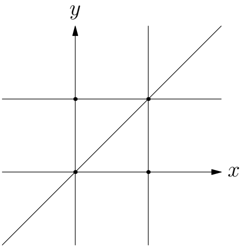
An example: Appell's F_1

Set

$$P_1 := x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + \{\gamma - (\alpha+\beta+1)x\}\partial_x \\ - \beta y \partial_y - \alpha\beta,$$

$$P_2 := y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + \{\gamma - (\alpha+\beta'+1)y\}\partial_y \\ - \beta' x \partial_x - \alpha\beta'$$

with $\alpha, \beta, \beta', \gamma \in \mathbb{C}$. Then $\mathcal{M} := \mathcal{D}_{\mathbb{C}^2}/(\mathcal{D}_{\mathbb{C}^2}P_1 + \mathcal{D}_{\mathbb{C}^2}P_2)$ is holonomic and its singular support is $\{(x,y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x-y) = 0\}$.



The b-function of $\mathcal{M}$ along each stratum is as follows:

$$
\begin{array}{ll}
x = 0 : & s(s - \beta' + \gamma - 1) \\
x = 1 : & s(s + \alpha + \beta - \gamma) \\
y = 0 : & s(s - \beta + \gamma - 1) \\
y = 1 : & s(s + \alpha + \beta' - \gamma) \\
x - y = 0 : & s(s + \beta + \beta' - 1)
\end{array}
$$

$$
\begin{array}{ll}
x = y = 0 : & s(s + \gamma - 1) \\
x - 1 = y = 0 : & s(s - \beta + \gamma - 1)(s + \alpha + \beta - \gamma) \\
x = y - 1 = 0 : & s(s - \beta' + \gamma - 1)(s + \alpha + \beta' - \gamma) \\
x - 1 = y - 1 = 0 : & s(s + \alpha + \beta + \beta' - \gamma)
\end{array}
$$

Remark: We can confirm that these b-functions are regular; there is an algorithm to find (or prove that there is none) regular b-functions (O, J. Pure and Applied Algebra (2009)).

Hence every distribution solution of $\mathcal{M}$ is locally L^p if the real parts of

$$\beta' - \gamma + 1,\ -\alpha - \beta + \gamma,\ \beta - \gamma + 1,\ -\alpha - \beta' + \gamma,\ -\beta - \beta' + 1$$

are greater than $-1/p$ and the real part of $-\alpha - \beta - \beta' + \gamma$ is greater than $-2/p$.
(This condition is satisfied, e.g., if $\alpha = \beta = \beta' = \gamma = 0$ for any $p \geq 1$.)

Laurent coefficients of $f_+^\lambda \varphi$

Let f be a real-valued real analytic function on an open connected set U of $\mathbb{R}^n$ and φ be a locally integrable function on U. Assume that φ is a solution (on U_f) of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n} u$ defined on an open set Ω of $\mathbb{C}^n$ such that U is relatively compact in Ω. Then $f_+^\lambda \varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function on $\mathbb{C}$.

Theorem 2

Under the above assumption, each coefficient of the Laurent expansion of $f_+^\lambda \varphi$ about an arbitrary $\lambda_0 \in \mathbb{C}$ is (locally at each point of U) a solution of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module.

Sketch of the proof of Theorem 2

Let λ_0 be a pole of order $l \geq 0$ of $f_+^\lambda \varphi$ and consider the Taylor expansion

$$(\lambda - \lambda_0)^l f_+^\lambda \varphi = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k \varphi_k$$

with $\varphi_k \in \mathcal{D}b(U)$ given by

$$\varphi_k = \frac{1}{k!} \lim_{\lambda \to \lambda_0} \frac{\partial^k}{\partial \lambda^k} \{(\lambda - \lambda_0)^l f_+^\lambda \varphi\}$$

but $f_+^{\lambda_0} \varphi$ is not defined in general.

Fix $m \in \mathbb{N}$ such that $\mathrm{Re}\ \lambda_0 + m > 0$. By using the functional equation involving the generalized b-function, we can find, at each point of U, a nonzero $b(s) \in \mathbb{C}[s]$ and a germ $P(s)$ of $\mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$b(\lambda) f_+^{\lambda} \varphi = P(\lambda)(f_+^{\lambda+m} \varphi).$$

Hence there exist $Q_j \in \mathcal{D}_{\mathbb{C}^n}$ such that

$$\varphi_k = \sum_{j=0}^{k} Q_j (f_+^{\lambda_0+m} (\log_+ f)^j \varphi)$$

where $\log_+ f = \log f$ if $f > 0$ and $\log_+ f = 0$ if $f \le 0$.

We have only to show that $f_+^{\lambda_0+m}(\log_+ f)^j\varphi$ with $0 \le j \le k$ satisfy a holonomic system.

Consider the left $\mathcal{D}_{\mathbb{C}^n}[s]$-module (direct sum of $\mathbb{C}$-vector spaces)

$$\mathcal{N}[k] := \mathcal{D}_{\mathbb{C}^n}[s]f^s \oplus \mathcal{D}_{\mathbb{C}^n}[s]f^s \log f \oplus \cdots \oplus \mathcal{D}_{\mathbb{C}^n}[s]f^s(\log f)^k,$$

where $\mathcal{D}_{\mathbb{C}^n}[s]$ acts on the 'symbol' $f^s(\log f)^j$ naturally. It is easy to see that $\mathcal{N}[k]/\mathcal{N}[k-1]$ is isomorphic to $\mathcal{N} = \mathcal{D}_{\mathbb{C}^n}[s]f^s$ as left $\mathcal{D}_{\mathbb{C}^n}[s]$-module. Hence $\mathcal{N}[k]/(s-\lambda)\mathcal{N}[k]$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module for any $\lambda \in \mathbb{C}$.

Now assume that $P_0(s)(f^s \otimes u) + \cdots + P_k(s)(f^s(\log f)^k \otimes u) = 0$ holds in $\mathcal{N}[k] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$. Then it is easy to see that

$$\sum_{j=0}^{k} P_j(\lambda) \frac{\partial^j}{\partial \lambda^j} f_+^\lambda \varphi = \sum_{j=0}^{k} P_j(\lambda) f_+^\lambda (\log_+ f)^j \varphi = 0$$

holds on U_f.

A generalization of the fundamental lemma implies that

$$\sum_{j=0}^{k} P_j(\lambda) f_+^\lambda (\log_+ f)^j \varphi = 0$$

holds on U. This completes the proof of Theorem 2.

Difference equations for the integral of $f_+^\lambda \varphi$

Let $f \in \mathbb{R}[x]$ be a nonconstant real polynomial in $x = (x_1, \dots, x_n)$ and φ be a locally integrable function on an open set U of $\mathbb{R}^n$. Assume that φ is a solution on U_f of a holonomic D_n-module M.

Theorem 3

Under the above assumption, if $Z(\lambda) := \int_U f_+^\lambda \varphi \, dx$ is well-defined, e.g, if the support of φ is compact in U, or if φ is rapidly decreasing with $U = \mathbb{R}^n$, then $Z(\lambda)$ satisfies a linear difference equation with polynomial coefficients in λ.

Example: $\Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x}\, dx = \int_{-\infty}^\infty x_+^\lambda e^{-x}\, dx$ satisfies $(E_\lambda - (\lambda+1))\Gamma(\lambda+1) = 0$, where $E_\lambda : \lambda \mapsto \lambda + 1$ is the shift operator.

Part II Algorithms

Let $D_n = \mathbb{C}\langle x, \partial\rangle = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the ring of differential operators with polynomial coefficients with $\partial_j = \partial/\partial x_j$. In the sequel, let f be a non-constant real polynomial of $x = (x_1, \dots, x_n)$ and φ be a locally integrable function on an open connected set U of $\mathbb{R}^n$. We assume that φ is a solution of a holonomic D_n-module M on U_f.

Our main purpose is to present an glorithm to compute a holonomic system of which $f_+^\lambda \varphi$ is a solution. As we have seen in Part I, the tensor product $D_n[s]f^s \otimes_{\mathbb{C}[x]} M$, which can be computed as the restriction to the diagonal of the exterior tensor product, provides us with such a holonomic system. But the practical computation is hard in general. So we shall present an alternative method, which is much more efficient.

Algorithms for Theorems 2 and 3 are immediate applications of this algorithm.

Mellin transform

Let us assume that φ is real analytic on U_f and set

$$\tilde{\varphi}(x,\lambda) := \int_{-\infty}^{\infty} t_+^{\lambda}\delta(t - f(x))\varphi(x)\,dt.$$

This is well-defined and coincides with $f_+^{\lambda}\varphi$ as a distribution on $U_f \times \mathbb{C}_+$. Then we have

$$\int_{-\infty}^{\infty} t_+^{\lambda} t\delta(t - f(x))\varphi(x)\,dt = \tilde{\varphi}(x,\lambda+1),$$

$$\int_{-\infty}^{\infty} t_+^{\lambda}\partial_t(\delta(t - f(x))\varphi(x))\,dt$$

$$= -\int_{-\infty}^{\infty} \partial_t(t_+^{\lambda})\delta(t - f(x)))f(x)\varphi(x)\,dt = -\lambda\tilde{\varphi}(x,\lambda-1).$$

Letting s be an indeterminate corresponding to λ, let us consider the ring $D_n\langle s, E_s\rangle$ of difference-differential operators with the shift operator $E_s : s \mapsto s+1$. In view of the preceding identities, let us define the ring homomorphism (Mellin transform of operators)

$$\mu : D_{n+1} \longrightarrow D_n\langle s, E_s, E_s^{-1}\rangle$$

by

$$\mu(t) = E_s, \quad \mu(\partial_t) = -sE_s^{-1}, \quad \mu(x_j) = x_j, \quad \mu(\partial_{x_j}) = \partial_{x_j}.$$

It is easy to see that μ is injective. Hence we may regard D_{n+1} as a subring of $D_n\langle s, E_s, E_s^{-1}\rangle$.

There are inclusions

$$D_n[s] \ \subset \ D_{n+1} \ \subset \ D_n\langle s, E_s, E_s^{-1}\rangle.$$

Let $\mathcal{F}(U)$ be the $\mathbb{C}$-vector space of the $\mathcal{D}b(U)$-valued meromorphic functions on $\mathbb{C}$. Then $\mathcal{F}(U)$ has a natural strucure of left $D_n\langle s, E_s, E_s^{-1}\rangle$-module, which is compatible with that of $D_n[s]$-module. In particular, we can regard $\mathcal{F}(U)$ as a left D_{n+1}-module.

Remark: Since we shall use only μ, we can forget the definition of the Mellin transform $\tilde{\varphi}(t, \lambda)$ as a distribution.

A holonomic system for $f_+^\lambda \varphi$

Suppose $M = D_n/I$ with a left ideal I of D_n such that $P\varphi = 0$ on U_f for any $P \in I$. Let G be a finite set of generators of I.

Since $D_n\langle s, E_s, E_s^{-1}\rangle$ acts on $\mathbb{C}[x, s, f^{-1}]f^s$, we can regard $D_{n+1}f^s \subset \mathbb{C}[x, s, f^{-1}]f^s$ as a left D_{n+1}-module.

We may regard f^s as $\delta(t - f)$ in $D_{n+1}f^s$. In fact we have

$$\mathrm{Ann}_{D_{n+1}} f^s = D_{n+1}(t - f) + \sum_{j=1}^{n} D_{n+1}\Big(\partial_{x_j} + \frac{\partial f}{\partial x_j}\partial_t\Big).$$

Step 1: a holonomic difference-differential system for $f_+^\lambda \varphi$

Introducing a new variable t, set

$$\tau(P) := P\left(x, \partial_{x_1} + \frac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_{x_n} + \frac{\partial f}{\partial x_n}\partial_t\right)$$

for $P = P(x, \partial_{x_1}, \ldots, \partial_{x_n}) \in G$. Set

$$\tilde{G} := \{\tau(P) \mid P \in G\} \cup \{t - f(x)\}$$

and let J be the left ideal of D_{n+1} generated by $\tilde{G}$. Then it is easy to see that D_{n+1}/J is holonomic.

Claim 1: $P(f_+^\lambda \varphi) = 0$ holds as an element of $\mathcal{F}(U)$ for any $P \in J$.

Proof: First note that

$$f_+^\lambda \partial_{x_j} \varphi = \Big(\partial_{x_j} + \frac{\partial f}{\partial x_j} \partial_t\Big)(f_+^\lambda \varphi)$$

holds on $U_f \times \{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda > 1\}$ for any $P \in D_n$.
Let $P \in G$. Then $P\varphi = 0$ holds on U_f. Hence

$$\tau(P)(f_+^\lambda \varphi) = f_+^\lambda P \varphi = 0$$

holds on $U_f \times \{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda > m\}$ with m being the order of P in ∂_t. Thus the fundamental lemma and the uniqueness of analytic continuation imply that $\tau(P)(f_+^\lambda \varphi) = 0$ holds in $\mathcal{F}(U)$.

Claim 2: Let u be the residue class of 1 in $M = D_n/I$. Then there exists an inclusion

$$D_{n+1}/J \ \subset \ D_{n+1} f^s \otimes_{\mathbb{C}[x]} M$$

such that $1 \mod J$ corresponds to $f^s \otimes u$.

Proof: We have only to show that for $P \in D_{n+1}$,

$$P \in J \quad \Leftrightarrow \quad P(f^s \otimes u) = 0 \quad \text{in} \quad D_{n+1} f^s \otimes_{\mathbb{C}[x]} M.$$

The right implication follows from an argument similar to (and simpler than) the one for Claim 1.

Suppose $P(f^s \otimes u) = 0$ in $D_{n+1} f^s \otimes_{\mathbb{C}[x]} M$. We can rewrite P in the form

$$P = \sum_{\alpha \in \mathbb{N}^n, \nu \in \mathbb{N}} p_{\alpha,\nu}(x) \Big(\partial_{x_1} + \frac{\partial f}{\partial x_1} \partial_t \Big)^{\alpha_1} \cdots \Big(\partial_{x_n} + \frac{\partial f}{\partial x_n} \partial_t \Big)^{\alpha_n} \partial_t^{\nu} + Q \cdot (t - f(x))$$

with $p_{\alpha,\nu}(x) \in \mathbb{C}[x]$ and $Q \in D_{n+1}$. Setting $P_\nu := \sum_{\alpha \in \mathbb{N}^n} p_{\alpha,\nu}(x) \partial_x^\alpha$, we get

$$0 = P(f^s \otimes u) = \sum_{\nu=0}^{\infty} (\partial_t^\nu f^s) \otimes P_\nu u \ \in \ D_{n+1} f^s \otimes_{\mathbb{C}[x]} M.$$

It follows that each P_ν belongs to I since $\{\partial_t^\nu f^s\}$ constitutes a free basis of $D_{n+1}f^s$ over $\mathbb{C}[x]$. Hence we have

$$P = \sum_{\nu=1}^{\infty} \partial_t^\nu \tau(P_\nu) + Q \cdot (t - f(x)) \ \in J.$$

Conclusion of Step 1: $f_+^\lambda \varphi$ is a solution of a holonomic D_{n+1}-module $D_{n+1}/J \subset D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$ if λ is regarded as a variable s subject to shift operations.

Step 2: a holonomic system for $f_+^\lambda \varphi$ with a fixed λ.

We compute the annihilator $\mathrm{Ann}_{D_n[s]}(f^s \otimes u) = J \subset D_n[s]$. This is the intersection of the left ideal J and the subring $D_n[s]$ of D_{n+1}. This can be done as follows:
Introducing new variables σ and τ. For $P \in D_{n+1}$, let $h(P) \in D_{n+1}[\tau]$ be the homogenization of P with respect to the weights

x_j	∂_{x_j}	t	∂_t	τ
0	0	-1	1	-1

Let J' be the left ideal of $D_{n+1}[\sigma, \tau]$ generated by

$$\{h(P) \mid P \in \tilde{G}\} \cup \{1 - \sigma\tau\},$$

where $\tilde{G}$ is a set of generators of J.

Set $J'' = J \cap D_{n+1}$. Since each element P of J'' is homogeneous w.r.t. the above weights, there exists $P'(s) \in D_n[s]$ such that $P = SP'(-\partial_t t)$ with $S = t^\nu$ or $S = \partial_t^\nu$ with some integer $\nu \geq 0$. We set $P'(s) = \psi(P)(s)$. Then $\{\psi(P) \mid P \in J''\}$ generates the left ideal $J \cap D_n[s]$ of $D_n[s]$. This procedure can be done by using a Gröbner basis in $D_{n+1}[\sigma, \tau]$.

Now we have a set of generators of

$$J \cap D_n[s] = \mathrm{Ann}_{D_n[s]}(f^s \otimes u)$$

with $f^s \otimes u \in D_{n+1} f^s \otimes_{\mathbb{C}[x]} M$.

Fix an arbitrary $\lambda \in \mathbb{C}$. Let f^λ be the residue class of f^s in $D_n f^\lambda := D_n[s]f^s/(s-\lambda)D_n[s]f^s$. Set $J_0 := \{P(\lambda) \mid P(s) \in J \cap D_n[s]\}$. Then we have

$$D_n/J_0 \;\cong\; D_n(f^\lambda \otimes u) \;\subset\; (D_n f^\lambda) \otimes_{\mathbb{C}[x]} M,$$

i.e, $J_0 = \mathrm{Ann}_{D_n}(f^\lambda \otimes u)$.

Claim 1: If λ is not a pole of $f_+^\lambda \varphi$, then for any $P(s) \in J \cap D_n[s]$, $P(\lambda)(f_+^\lambda \varphi) = 0$ holds as a distribution on U.

Proof: Obvious from the arguments so far.

Claim 2: D_n/J_0 is holonomic.

Proof: Let us denote by $f^s \otimes' u$ the tensor product in $D_n[s] \otimes_{\mathbb{C}[x]} M$. The natural homomorpshism $D_n[s] \otimes_{\mathbb{C}[x]} M \longrightarrow D_{n+1} \otimes_{\mathbb{C}[x]} M$ induces

$$\rho : D_n[s](f^s \otimes' u) \longrightarrow D_n[s](f^s \otimes u) \subset D_{n+1}(f^s \otimes u) = D_{n+1}/J.$$

Specializing s to λ, this induces a sujrective homomorphism

$$\rho' : D_n(f^\lambda \otimes u) \longrightarrow D_n/J_0.$$

This proves that D_n/J_0 is holonomic since $D_n(f^\lambda \otimes u)$ is holonomic as a submodule of $D_n f^\lambda \otimes_{\mathbb{C}[x]} M$.

Remark: ρ and ρ' are not injective in general. For example, set $M = D_2/(D_2 x_1 + D_2 \partial_2)$ and $f = x_1 x_2$. Then

$$\partial_2(f^s \otimes u) = \partial_2 f^s \otimes u = -x_1 \partial_t f^s \otimes u = -\partial_t f^s \otimes x_1 u = 0$$

holds in $D_{2+1} \otimes_{\mathbb{C}[x]} M$ but $\partial_2(f^s \otimes' u) \neq 0$ in $D_2[s] f^s \otimes_{\mathbb{C}[x]} M$.

Generalized b-functions

The inclusion $D_{n+1}f^s \subset \mathbb{C}[x, f^{-1}, s]f^s$ induces a homomorphism

$$D_{n+1}f^s \otimes_{\mathbb{C}[x]} M \longrightarrow \mathbb{C}[x, f^{-1}, s]f^s \otimes_{\mathbb{C}[x]} M.$$

We have computed $\mathrm{Ann}_{D_{n+1}}(f^s \otimes u) \cap D_n[s]$. Thus a generator $b(s)$ of $\mathbb{C}[s] \cap (\mathrm{Ann}_{D_{n+1}}(f^s \otimes u) \cap D_n[s]) + D_n[s]f$ is a multiple of the b-function for f and $u \in M$. If $f : M \to M$ is injective, then $b(s)$ coincides with the b-function because the above homomorphism is an isomorphism.

Difference equations for the integral

As we have seen $f_+^\lambda \varphi$ satisfies a holonomic D_{n+1}-module D_{n+1}/J.
Hence if $Z(\lambda) := \int_{\mathbb{R}^n} f_+^\lambda \varphi \, dx$ is well-defined, e.g., if φ has compact support, or rapidly decreasing, then $Z(\lambda)$ is a solution of the holonomic D_1-module

$$D_{n+1}/(J + \partial_{x_1} D_{n+1} + \cdots + \partial_{x_n} D_{n+1})$$

with $D_1 = \langle t, \partial_t \rangle$. Hence by inverse Mellin transform we obtain linear difference equations for $Z(\lambda)$.

Example 1

Set $f = x^3 - y^2 \in \mathbb{R}[x, y]$. Since the b-function of f is $b_f(s) = (s+1)(6s+5)(6s+7)$, possible poles of f_+^λ are $-1-\nu$, $-5/6-\nu$, $-6/7-\nu$ and they are at most simple poles. The residue $\mathrm{Res}_{\lambda=-1} f_+^\lambda$ is a solution of

$$D_2/(D_2(2x\partial_x + 3y\partial_y + 6) + D_2(2y\partial_x + 3x^2\partial_y) + D_2(x^3 - y^2)).$$

$\mathrm{Res}_{\lambda=-5/6} f_+^\lambda$ is a solution of $D_2/(D_2x + D_2y)$. Hence it is a constant multiple of $\delta(x, y)$.

$\mathrm{Res}_{\lambda=-7/6} f_+^\lambda$ is a solution of $D_2/(D_2x^2 + D_2(x\partial_x + 2) + D_2y)$. Hence it is a constant multiple of $\delta'(x)\delta(y)$.

Example 2

Set $f = x^3 - y^2$. $\varphi(x,y) := \exp(-x^2 - y^2)$ is a solution of $M := D_2/(D_2(\partial_x + 2x) + D_2(\partial_y + 2y))$. The generalized b-function for f and $u := [1] \in M$ is $b_f(s) = (s+1)(6s+5)(6s+7)$.

$Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi \, dxdy$ is annihilated by the difference operator

$$32E_s^4 + 16(4s+13)E_s^3 - 4(s+3)(27s^2+154s+211)E_s^2$$
$$- 6(s+2)(s+3)(36s^2+162s+173)E_s$$
$$- 3(s+1)(s+2)(s+3)(6s+5)(6s+13).$$

From this we see that $-7/6$ is not a pole of $Z(\lambda)$.

Example 3

Set $\varphi(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi(x) = 0$ for $x \le 0$. Then $\varphi(x)$ belongs to $\mathcal{S}(\mathbb{R})$ and satisfies a holonomic system

$$M := D_1/D_1(x^2\partial_x + x^2 - 1).$$

The generalized b-function for $f = x$ and $u = [1] \in M$ is $s + 1$.
$Z(\lambda) := \int_{\mathbb{R}} x_+^{\lambda}\varphi(x)\,dx$ is entire and satisfies a difference equation

$$(-E_\lambda^2 + (\lambda + 2)E_\lambda + 1)Z(\lambda) = 0.$$

This can also be deduced by integration by parts.

Example 4

Set $\varphi_1(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi_1(x) = 0$ for $x \leq 0$. Set $\varphi(x, y) = \varphi_1(x)e^{-y}$. Then φ satisfies a holonomic system

$$D_2/(D_2(x^2\partial_x + x^2 - 1) + D_2(\partial_y + 1)).$$

The generalized b-function for $f := y^3 - x^2$ and $u = [1] \in M$ is $s + 1$. $Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi(x)\, dxdy$ is well-defined since $f_+ = 0$ if $y < 0$ and satisfies difference equations: ...