Algorithms for D-modules —restriction, tensor product, localization, and local cohomology groups

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Abstract

We consider an algebraic D-module M on the affine space, i.e. a system of linear partial differential equations with polynomial coefficients. We give an algorithm for computing the cohomology groups of the restriction of M to a linear subvariety by using a free resolution of M adapted to the V-filtration. Our algorithm works, at least, if M is holonomic. As applications, we obtain algorithms for computing tensor product, localization, and algebraic local cohomology groups of holonomic systems.

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1 Introduction

The purpose of this paper is to describe algorithms for computing various functors for algebraic $\mathcal{D}$-modules, i.e. systems of linear partial differential equations with polynomial coefficients. The algorithms enable us to perform actual computations (with limitation caused by the complexity) by using e.g. a program `kan` [41] developed by the second author, as well as to establish theoretical computability of some fundamental functors in the $\mathcal{D}$-module theory.

Let K be an algebraically closed field of characteristic zero and let X be the affine space K^n with a positive integer n. We denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves on X of rings of regular functions and of algebraic linear differential operators respectively (cf. Bernstein [3], Björk [4], Borel et al. [5]). Let $\mathcal{M}$ and $\mathcal{N}$ be coherent left $\mathcal{D}_X$-modules.

Various functors are defined for (especially for holonomic) $\mathcal{D}$-modules and play the fundamental role (see [3],[4],[5] and also e.g., [15], [17], [18], [25] for their analytic counterparts). Among such functors, we are concerned with the following:

(1) The cohomology groups of the restriction $\mathcal{M}_Y^\bullet := \mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{M}$ of $\mathcal{M}$ to Y as left $\mathcal{D}_Y$-modules, where Y is a non-singular subvariety of X and $\otimes^L$ denotes the left derived functor (cf. [13]) of the tensor product.

(2) The cohomology groups $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, K[[x_1, \dots, x_n]])$ with coefficients in the formal power series solutions of $\mathcal{M}$, which equal to those with coefficients in the convergent power series solutions if $\mathcal{M}$ is regular holonomic (cf. Kashiwara-Kawai [20]).

(3) The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ and, more generally, the torsion groups $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, as left $\mathcal{D}_X$-modules.

(4) The localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$ of $\mathcal{M}$ as a left $\mathcal{D}_X$-module, where $f \in K[x_1, \dots, x_n]$ is an arbitrary non-constant polynomial.

(5) The (algebraic) local cohomology groups $\mathcal{H}^i_{[Y]}(\mathcal{M})$ with support Y as left $\mathcal{D}_X$-modules, where Y is an arbitrary algebraic set of X.

It was proved by Kashiwara [15] that these are all holonomic systems (the second one is a finite dimensional vector space) if $\mathcal{M}$ and $\mathcal{N}$ are holonomic.

Let us remark that if $K = \mathbf{C}$ and $\mathcal{M}$ is Fuchsian along Y in the sense of Laurent and Moteiro-Fernandes [23], which is the case if $\mathcal{M}$ is regular holonomic in the sense of [20], then there exists an isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^{\mathrm{an}})|_Y \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y^\bullet, \mathcal{O}_Y^{\mathrm{an}})$$

in the derived category of sheaves of $\mathbf{C}$-vector spaces; here $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{O}_Y^{\mathrm{an}}$ denote the sheaves of holomorphic functions on X and on Y respectively, and $R\mathcal{H}om$ the right derived functor of $\mathcal{H}om$. Thus roughly speaking, $\mathcal{M}_Y^\bullet$ corresponds to the system of partial differential equations which the solutions of $\mathcal{M}$ restricted to Y satisfy. Similarly, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ corresponds to the system which the product of solutions of $\mathcal{M}$ and of $\mathcal{N}$ satisfies.

As was observed by Castro [7] and Galligo [12] and was developed by many authors (e.g. [39], [40], [28], [29], [30], [1], [36]) the notion of Gröbner basis and the Buchberger algorithm

[6] are essential in the algorithmic study of $\mathcal{D}$-modules as well as in computational algebraic geometry (cf. [9], [10]). By using Gröbner bases for the Weyl algebra, we give algorithms for computing the objects listed above under some conditions on $\mathcal{M}$ and $\mathcal{N}$, which are certainly satisfied if $\mathcal{M}$ and $\mathcal{N}$ are holonomic. These algorithms also apply to the analytic counterparts of these functors as long as the input $\mathcal{D}$-module is defined algebraically.

We first give an algorithm for the restriction (Algorithm 5.4) when Y is a linear subvariety of arbitrary codimension under the condition that $\mathcal{M}$ is specializable along Y, which is the case with an arbitrary holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Here $\mathcal{M}$ is specializable along Y by definition if and only if there exists a nonzero b-function, or the indicial polynomial of $\mathcal{M}$ along Y. We also give an algorithm to compute the b-function (Algorithm 4.6).

Our method consists in computing a free resolution of $\mathcal{M}$ that is adapted to the so-called V-filtration associated with Y. Such a free resolution tensored with $\mathcal{D}_{Y\to X} := (\mathcal{D}_X)^{\bullet}_Y$ gives $\mathcal{M}^{\bullet}_Y$, but it is not a complex of coherent $\mathcal{D}_Y$-modules in general. Then we use information on the integral roots of the b-function to truncate the complex and obtain a complex of finitely generated free $\mathcal{D}_Y$-modules. The first author gave in [32] an algorithm for the case where Y is of codimension one without using free resolution.

This algorithm for the restriction also solves the other problems listed above by virtue of some isomorphisms provided by the $\mathcal{D}$-module theory, especially those described in [15],[25]. See Algorithm 6.2 for the tensor product, Algorithm 6.4 for the localization, and Algorithm 7.3 for the algebraic local cohomology groups. Finally the computation of the restriction for the general case where Y is not necessarily linear reduces to that of local cohomology through the so-called Kashiwara equivalence [15], which claims the equivalence of the category of coherent $\mathcal{D}_Y$-modules and that of coherent $\mathcal{D}_X$-modules supported by Y.

Algorithms for the local cohomology groups have been given in [32] when Y is of codimension one, and by Walther [42] under the assumption that $\mathcal{M}$ is saturated with respect to Y. See [34] for another localization algorithm, which applies to the case where M is holonomic only on $X \setminus \{f = 0\}$.

As a direct application of the restriction algorithm, we obtain an algorithm for computing the cohomology groups of the integration of a module over the Weyl algebra. This enables us, e.g. to compute the de Rham cohomology groups of some algebraic varieties. See [33] for details.

In Sections 3 and 9, we discuss how to get the free resolution mentioned above. For that purpose, we apply Schreyer's method for free resolution in the polynomial ring (see e.g. [10]) to the ring of differential operators. In doing so, we need some modification because of the non-commutativity and the fact that the term order we use is not a well-order. We have two methods to cope with this difficulty: one is the homogenization with respect to the V-filtration by the first author ([29],[30]), and the other is what we call the homogenized Weyl algebra which was introduced and implemented by the second author in the 2nd version of `kan/sm1` [41] that was released in 1994, but has not been published in the literature. We discuss the first method in Section 3 and the latter method in 9 in a more general situation than the V-filtration; i.e. for the filtration defined by a general weight vector. Methods similar to the latter one were also employed by Castro-Jimenez and others (implicitly in [1] and explicitly in [8]) and applied to the computation of the slopes of a $\mathcal{D}$-module.

In Section 10, we give a criterion for a free resolution to be adapted to the filtration defined by a general weight vector. In particular, we prove that being a standard base in the sense of Robbiano [35, Definition 3.1] and Mora [26, p.6] is equivalent to being an involutive base (Definition 10.1). We expect that this serves as a theoretical basis for obtaining as small adapted resolution as possible.

We have implemented the algorithms by using `kan/sm1` [41] for computations of Gröbner bases and free resolutions (as Schreyer resolutions) in the Weyl algebra, and Risa/Asir [27] for factorization and primary decomposition in the polynomial ring.

2 V-filtration and adapted free resolution

Let K be an algebraically closed field of characteristic zero. We fix positive integers d and n. Let X be the affine space K^{d+n} with the coordinate system $(t, x) = (t_1, ..., t_d, x_1, \ldots, x_n)$. We denote by $\partial_t = (\partial_{t_1}, ..., \partial_{t_d})$ and $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ the corresponding derivations with $\partial_{x_i} = \partial/\partial x_i$, $\partial_{t_j} = \partial/\partial t_j$. We use the notation $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial_x^\beta := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$, $t^\mu := t_1^{\mu_1} \cdots t_d^{\mu_d}$, $\partial_t^\nu := \partial_{t_1}^{\nu_1} \cdots \partial_{t_n}^{\nu_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$ and $\mu = (\mu_1, \ldots, \mu_d), \nu = (\nu_1, ..., \nu_d) \in \mathbf{N}^d$, where we put $\mathbf{N} := \{0, 1, 2, \ldots\}$. We also use the notation $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

Let Y be the d-codimensional linear subvariety of X given by $Y := \{(t, x) \in X \mid t = 0\}$. Let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the sheaves of regular functions on X and on Y respectively. We denote by $\mathcal{D}_X$ and $\mathcal{D}_Y$ the sheaves of rings of algebraic linear differential operators on X and on Y respectively.

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module on X. Then the set of the global sections $M := \Gamma(X, \mathcal{M})$ is a finitely generated left module over the Weyl algebra $D_{d+n} := \Gamma(X, \mathcal{D}_X)$. Conversely, for a finitely generated left D_{d+n}-module M, its sheafification $\mathcal{M} := \mathcal{D}_X \otimes_{D_{d+n}} M$ is a coherent $\mathcal{D}_X$-module. More precisely, this correspondence gives an equivalence between the category of finitely generated D_{d+n}-modules and that of coherent $\mathcal{D}_X$-modules (cf. [3],[5]). Hence we could work only in the first category. However, as to e.g., the restriction functor, it would be preferable to work in the latter category since a coherent $\mathcal{D}_X$-module can be specializable along some Zariski open subset of Y but not along whole Y (cf. Section 4). In any case, actual computations are done for modules over the Weyl algebra.

In the sequel, we define the notion of free resolution adapted to the V-filtration. Let $\mathcal{D}_X|_Y$ be the sheaf theoretic restriction of $\mathcal{D}_X$ to Y. Let $\mathcal{J}_Y := \mathcal{O}_X t_1 + \ldots + \mathcal{O}_X t_d$ be the defining ideal of Y. Then for each integer k we put

$$\begin{aligned} F_Y^k(\mathcal{D}_X) &:= \{P \in \mathcal{D}_X|_Y \mid P(\mathcal{J}_Y)^j \in (\mathcal{J}_Y)^{j-k} \quad \text{for any } j \geq k\} \\ &= \{P = \sum_{|\nu| \leq l} \sum_{|\beta| \leq m} a_{\nu\beta}(t, x) \partial_t^\nu \partial_x^\beta \mid l, m \in \mathbf{N},\ a_{\nu\beta}(t, x) \in \mathcal{J}_Y^{|\nu|-k}\} \end{aligned}$$

with the convention $\mathcal{J}_Y^j = \mathcal{O}_X$ for $j \leq 0$. This is called the V-filtration attached to Y (cf. [16],[24]). More generally, given an r-vector $\mathbf{m} := (m_1, ..., m_r)$ of integers, we put

$$F_Y^k[\mathbf{m}](\mathcal{D}_X^r) := \bigoplus_{i=1}^r F_Y^{k-m_i}(\mathcal{D}_X) e_i,$$

where $e_1, \ldots, e_r$ are the canonical generators of $\mathcal{D}_X^r$. We may assume that $\mathcal{M}$ has a presentation $\mathcal{M} = \mathcal{D}_X^r/\mathcal{N}$ on X, where $\mathcal{N}$ is a coherent left $\mathcal{D}_X$-submodule of $\mathcal{D}_X^r$. In fact, $M := \Gamma(X, \mathcal{M})$ can be written in the form $M = D_{d+n}^r/N$ with an integer r and an D_{d+n}-submodule N of D_{d+n}^r. Then $\mathcal{N} := \mathcal{D}_X \otimes_{D_{d+n}} N$ satisfies the above property. Let u_i be the residue class of $e_i \in \mathcal{D}_X^r$ in $\mathcal{M}$. Then for $\mathbf{m} \in \mathbf{Z}^r$, we put

$$\begin{aligned} F_Y^k[\mathbf{m}](\mathcal{N}) &:= \mathcal{N} \cap F_Y^k[\mathbf{m}](\mathcal{D}_X^r), \\ F_Y^k[\mathbf{m}](\mathcal{M}) &:= F_Y^{k-m_1}(\mathcal{D}_X)u_1 + \cdots + F_Y^{k-m_r}(\mathcal{D}_X)u_r \end{aligned}$$

for each integer $k \in \mathbf{Z}$. The graded ring and modules associated with these filtrations are defined by

$$\begin{aligned} \mathrm{gr}_Y(\mathcal{D}_X) &:= \bigoplus_{k\in\mathbf{Z}} F_Y^k(\mathcal{D}_X)/F_Y^{k-1}(\mathcal{D}_X), \\ \mathrm{gr}_Y[\mathbf{m}](\mathcal{D}_X^r) &:= \bigoplus_{k\in\mathbf{Z}} F_Y^k[\mathbf{m}](\mathcal{D}_X^r)/F_Y^{k-1}[\mathbf{m}](\mathcal{D}_X^r), \\ \mathrm{gr}_Y[\mathbf{m}](\mathcal{M}) &:= \bigoplus_{k\in\mathbf{Z}} F_Y^k[\mathbf{m}](\mathcal{M})/F_Y^{k-1}[\mathbf{m}](\mathcal{M}). \end{aligned}$$

Then $\mathrm{gr}_Y[\mathbf{m}](\mathcal{N})$ and $\mathrm{gr}_Y[\mathbf{m}](\mathcal{M})$ are coherent left $\mathrm{gr}_Y(\mathcal{D}_X)$-modules. If $\mathbf{m}$ is the zero vector, we shall omit the notation $[\mathbf{m}]$.

For a nonzero section P of $\mathcal{D}_X^r|_Y$, let $k = \mathrm{ord}_Y[\mathbf{m}](P)$ be the minimum $k \in \mathbf{Z}$ such that $P \in F_Y^k[\mathbf{m}](\mathcal{D}_X^r)$. (We put $\mathrm{ord}_Y[\mathbf{m}](0) := -\infty$.) Then let $\sigma_Y[\mathbf{m}](P)$ be the residue class of P in

$$\mathrm{gr}_Y^k[\mathbf{m}](\mathcal{D}_X^r) := F_Y^k[\mathbf{m}](\mathcal{D}_X^r)/F_Y^{k-1}[\mathbf{m}](\mathcal{D}_X^r).$$

Definition 2.1 Let $\mathcal{M} = \mathcal{D}_X^r/\mathcal{N}$ be as above. Let us consider a free resolution (i.e., an exact sequence)

$$\mathcal{D}_X^{r_l} \xrightarrow{\psi_l} \mathcal{D}_X^{r_{l-1}} \xrightarrow{\psi_{l-1}} \cdots \xrightarrow{\psi_2} \mathcal{D}_X^{r_1} \xrightarrow{\psi_1} \mathcal{D}_X^{r_0} \xrightarrow{\varphi} \mathcal{M} \longrightarrow 0 \tag{2.1}$$

of $\mathcal{M}$, where ψ_i are homomorphisms of left $\mathcal{D}_X$-modules, and φ is defined by $\varphi(e_i) = u_i$ for $i = 1, ..., r_0$ with $r_0 = r$. This free resolution is said to be *adapted* to the $F_Y[\mathbf{m}]$-filtration if and only if there exist vectors $\mathbf{m}_1 \in \mathbf{Z}^{r_1}$,...,$\mathbf{m}_l \in \mathbf{Z}^{r_l}$ such that

$$\psi_{j+1}(F_Y^k[\mathbf{m}_{j+1}](\mathcal{D}_X^{r_{j+1}})) \subset F_Y^k[\mathbf{m}_j](\mathcal{D}_X^{r_j})$$

holds for $j = 0, 1, \ldots, l-1$ with $\mathbf{m}_0 = \mathbf{m}$ and that (2.1) induces an exact sequence

$$\mathrm{gr}_Y[\mathbf{m}_l](\mathcal{D}_X^{r_l}) \xrightarrow{\overline{\psi}_l} \cdots \xrightarrow{\overline{\psi}_2} \mathrm{gr}_Y[\mathbf{m}_1](\mathcal{D}_X^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}_Y[\mathbf{m}_0](\mathcal{D}_X^{r_0}) \xrightarrow{\overline{\varphi}} \mathrm{gr}_Y[\mathbf{m}](\mathcal{M}) \to 0$$

of $\mathrm{gr}_Y(\mathcal{D}_X)$-modules. We call $\mathbf{m}_1$,...,$\mathbf{m}_l$ the *shift vectors* associated with the free resolution (2.1).

The definition above is general in the sense that it is local and also applies to the analytic case (cf. Section 8). However, from the computational viewpoint, working in the Weyl algebra would be more convenient: Let D_n and D_{d+n} be the Weyl algebras on the n variables x and on the $d+n$ variables (t,x) respectively with coefficients in K (cf. [4]). Put $L := \mathbf{N}^{2(d+n)} = \mathbf{N}^d \times \mathbf{N}^d \times \mathbf{N}^n \times \mathbf{N}^n$. An element P of D_{d+n}^r is written uniquely in a finite sum

$$P = \sum_{i=1}^{r} \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta i} t^\mu \partial_t^\nu x^\alpha \partial_x^\beta e_i \tag{2.2}$$

with $a_{\mu\nu\alpha\beta i} \in K$, $e_1 := (1,0,\dots,0), \dots, e_r := (0,\dots,0,1)$. Put

$$\begin{aligned}
F_Y^k(D_{d+n}) &:= \{P = \sum_{|\nu|-|\mu|\le k} \sum_{\beta} a_{\mu\nu\beta}(x) t^\mu \partial_t^\nu \partial_x^\beta \in D_{d+n} \mid a_{\mu\nu\beta}(x) \in K[x]\},\\
F_Y^k[\mathbf{m}](D_{d+n}^r) &:= \bigoplus_{i=1}^{r} F_Y^{k-m_i}(D_{d+n}) e_i,\\
F_Y^k[\mathbf{m}](N) &:= N \cap F_Y^k[\mathbf{m}](D_{d+n}^r),\\
F_Y^k[\mathbf{m}](M) &:= F_Y^{k-m_1}(D_{d+n}) u_1 + \cdots + F_Y^{k-m_r}(D_{d+n}) u_r,
\end{aligned}$$

where u_i is the residue class of e_i in M. The graded ring and modules $\mathrm{gr}_Y(D_{d+n})$, $\mathrm{gr}_Y[\mathbf{m}](D_{d+n}^r)$, $\mathrm{gr}_Y[\mathbf{m}](N)$ and $\mathrm{gr}_Y[\mathbf{m}](M)$ are defined in the same way as for $\mathcal{D}_X$.

The following lemma follows immediately from the definition:

Lemma 2.2 *In the notation above, we have*

$$\mathrm{gr}_Y[\mathbf{m}](\mathcal{N}) = \mathrm{gr}_Y(\mathcal{D}_X) \otimes_{\mathrm{gr}_Y(D_{d+n})} \mathrm{gr}_Y[\mathbf{m}](N).$$

By using this lemma and the flatness of $\mathcal{D}_X$ over D_{d+n}, we can easily get the following:

Proposition 2.3 *Let*

$$D_{d+n}^{r_l} \xrightarrow{\psi_l} D_{d+n}^{r_{l-1}} \xrightarrow{\psi_{l-1}} \cdots \xrightarrow{\psi_2} D_{d+n}^{r_1} \xrightarrow{\psi_1} D_{d+n}^{r_0} \xrightarrow{\varphi} M \longrightarrow 0 \tag{2.3}$$

be a free resolution of M adapted to the $F_Y[\mathbf{m}]$-filtration, i.e., ψ_i are homomorphisms of left D_{d+n}-modules, φ is defined by $\varphi(e_i) = u_i$ for $i = 1, ..., r_0$ with $r_0 = r$, and there exist $\mathbf{m}_1 \in \mathbf{Z}^{r_1}$,..., $\mathbf{m}_l \in \mathbf{Z}^{r_l}$ such that

$$\psi_{j+1}(F_Y^k[\mathbf{m}_{j+1}](D_{d+n}^{r_{j+1}})) \subset F_Y^k[\mathbf{m}_j](D_{d+n}^{r_j})$$

holds for $j = 0, 1, \dots, l-1$ with $\mathbf{m}_0 = \mathbf{m}$ and that (2.3) induces an exact sequence

$$\mathrm{gr}_Y[\mathbf{m}_l](D_{d+n}^{r_l}) \xrightarrow{\bar\psi_l} \cdots \xrightarrow{\bar\psi_2} \mathrm{gr}_Y[\mathbf{m}_1](D_{d+n}^{r_1}) \xrightarrow{\bar\psi_1} \mathrm{gr}_Y[\mathbf{m}_0](D_{d+n}^{r_0}) \xrightarrow{\bar\varphi} \mathrm{gr}_Y[\mathbf{m}](M) \to 0$$

of $\mathrm{gr}_Y(D_{d+n})$-modules. Under this assumption, the exact sequence (2.3) tensored by $\mathcal{D}_X$ from the left gives a free resolution of $\mathcal{M}$ adapted to the $F_Y[\mathbf{m}]$-filtration.

Definition 2.4 Let N be a D_{d+n}-submodule of D_{d+n}^r. Then a subset G of N is said to be an $F_Y[\mathbf{m}]$-*involutive base* of N if G generates N and $\{\sigma_Y[\mathbf{m}](P) \mid P \in G\}$ generates $\mathrm{gr}_Y[\mathbf{m}](N)$ in $\mathrm{gr}_Y[\mathbf{m}](D_{d+n}^r)$.

In Section 10, we shall prove the following equivalence in a more general situation (Theorem 10.7):

Theorem 2.5 *Let M and $\mathbf{m}$ be as in the preceding proposition. Consider an exact sequence (2.3) and vectors $\mathbf{m}_1 \in \mathbf{Z}^{r_1}$,..., $\mathbf{m}_l \in \mathbf{Z}^{r_l}$ such that*

$$\psi_{j+1}(F_Y^k[\mathbf{m}_{j+1}](D_{d+n}^{r_{j+1}})) \subset F_Y^k[\mathbf{m}_j](D_{d+n}^{r_j})$$

holds for $j = 0, 1, \ldots, l-1$ with $\mathbf{m}_0 = \mathbf{m}$. Then the following three conditions are equivalent:

(1) The resolution (2.3) is adapted to the $F_Y[\mathbf{m}]$-filtration.

(2) For each $k \in \mathbf{Z}$, the sequence

$$F_Y^k[\mathbf{m}_l](D_{d+n}^{r_l}) \xrightarrow{\psi_l} \cdots \xrightarrow{\psi_2} F_Y^k[\mathbf{m}_1](D_{d+n}^{r_1}) \xrightarrow{\psi_1} F_Y^k[\mathbf{m}_0](D_{d+n}^{r_0}) \xrightarrow{\varphi} F_Y^k[\mathbf{m}](M) \to 0$$

is exact.

(3) Let $e_1^{(i)}, \ldots, e_{r_i}^{(i)}$ be the canonical generators of $D_{d+n}^{r_i}$. Then $\{\psi_i(e_1^{(i)}), \ldots, \psi_i(e_{r_i}^{(i)})\}$ is an $F_Y[\mathbf{m}_{i-1}]$-involutive base of $\operatorname{Ker} \psi_{i-1}$ for each $i = 1, 2, \ldots, l$ with $\psi_0 := \varphi$.

Note that the implications (2) $\Rightarrow$ (1), (2) $\Rightarrow$ (3) are trivial but the others are not. However, we do not need this theorem in the following sections; it possibly concerns the problem of how to get an adapted resolution as small as possible, in connection with the Schreyer resolution discussed in Section 9.

3 Adapted free resolution by V-homogenization

The purpose of this section is to show that Gröbner bases homogenized with respect to the V-filtration provide a free resolution adapted to the V-filtration. An alternative and more efficient method will be described in Section 9.

We fix a natural number r and a vector $\mathbf{m} \in \mathbf{Z}^r$. Let $\prec$ be a well-order (i.e. a linear order) on $L \times \{1, \ldots, r\}$ which satisfies

$$\begin{aligned}&(\tilde\alpha, i) \prec (\tilde\beta, j) \text{ implies } (\tilde\alpha + \tilde\gamma, i) \prec (\tilde\beta + \tilde\gamma, j)\\ &\text{for any } \tilde\alpha, \tilde\beta, \tilde\gamma \in L \text{ and } i, j \in \{1, \ldots, r\}.\end{aligned} \qquad (3.1)$$

Then we define a total order $\prec_F$ on $L \times \{1, ..., r\}$ by

$$\begin{aligned}&(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j) \text{ if and only if}\\ &|\nu - \mu| + m_i < |\nu' - \mu'| + m_j \text{ or else}\\ &|\nu - \mu| + m_i = |\nu' - \mu'| + m_j,\ (\mu, \nu, \alpha, \beta, i) \prec (\mu', \nu', \alpha', \beta', j).\end{aligned} \qquad (3.2)$$

Let P be a nonzero element of D_{d+n}^r which is written in the form (2.2). Then the *leading exponent* $\operatorname{lexp}_F(P) \in L \times \{1, \ldots, r\}$ of P with respect to $\prec_F$ is defined as the maximum element of $\{(\mu, \nu, \alpha, \beta, i) \mid a_{\mu\nu\alpha\beta i} \neq 0\}$ in the order $\prec_F$. Moreover, for $(\mu, \nu, \alpha, \beta, i) = \operatorname{lexp}_F(P)$, the leading coefficient of P is defined by $\operatorname{lcoef}_F(P) := a_{\mu\nu\alpha\beta i}$. The set of leading exponents $E_F(N)$ of a subset N of D_{n+1}^r is defined by

$$E_F(N) := \{\operatorname{lexp}_F(P) \mid P \in N \setminus \{0\}\}.$$

Definition 3.1 A finite subset G of a left D_{d+n}-submodule N of D_{d+n}^r is called a Gröbner basis of N with respect to $\prec_F$ (or an $F[\mathbf{m}]$-Gröbner basis) if G generates N and

$$E_F(N) = \bigcup_{P \in G} (\mathrm{lexp}_F(P) + L)$$

holds with the notation

$$(\tilde{\alpha}, i) + L = \{(\tilde{\alpha} + \tilde{\beta}, i) \mid \tilde{\beta} \in L\}$$

for $\tilde{\alpha} \in L$ and $i \in \{1, \ldots, r\}$.

We define an order $\prec_H$ on $\mathbf{N} \times L \times \{1, ..., r\}$ by

$$\begin{aligned} &(\lambda, \tilde{\alpha}, i) \prec_H (\lambda', \tilde{\alpha}', i) \text{ if and only if } \lambda < \lambda' \text{ or else} \\ &\lambda = \lambda',\ (\lambda, \tilde{\alpha}, i) \prec (\lambda', \tilde{\alpha}', j), \end{aligned} \tag{3.3}$$

where $\lambda, \lambda' \in \mathbf{N}, \tilde{\alpha}, \tilde{\alpha}' \in L$ and $i, j \in \{1, ..., r\}$. It is easy to see that $\prec_H$ is a well-order and satisfies

Lemma 3.2 *If* $|\nu - \mu| + m_i - \lambda = |\nu' - \mu'| + m_j - \lambda'$*, then we have*

$$(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$$

if and only if

$$(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j).$$

We introduce an indeterminate t_0 which commutes with any element of D_{d+n} in order to define the homogenization.

Definition 3.3 An element P of $D_{d+n}[t_0]^r$ of the form

$$P = \sum_{i=1}^{r} \sum_{\lambda,\mu,\nu,\alpha,\beta} a_{\lambda\mu\nu\alpha\beta i} t_0^{\lambda} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial_x^{\beta} e_i$$

is said to be $F[\mathbf{m}]$-*homogeneous* of order k if $a_{\lambda\mu\nu\alpha\beta i} = 0$ whenever $|\nu - \mu| - \lambda + m_i \neq k$.

Definition 3.4 For an element P of D_{d+n}^r of the form (2.2), put

$$k := \min\{|\nu - \mu| + m_i \mid a_{\mu\nu\alpha\beta i} \neq 0 \text{ for some } \alpha, \beta\}.$$

Then the $F[\mathbf{m}]$-*homogenization* $h(P) \in D_{d+n}[t_0]^r$ of P is defined by

$$h(P) := \sum_{i=1}^{r} \sum_{\mu,\nu,\alpha,\beta} a_{\mu\nu\alpha\beta i} t_0^{|\nu-\mu|+m_i-k} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial_x^{\beta} e_i.$$

Then $h(P)$ is $F[\mathbf{m}]$-homogeneous of order k.

When $\mathbf{m}$ is the zero vector, we simply say F-homogeneous instead of $F[\mathbf{m}]$-homogeneous.

Lemma 3.5 *If $P \in D_{d+n}[t_0]$ is F-homogeneous and $Q \in D_{d+n}[t_0]^r$ is $F[\mathbf{m}]$-homogeneous, then PQ is $F[\mathbf{m}]$-homogeneous.*

Lemma 3.6 *For $P_1, \dots, P_k \in D_{d+n}^r$, put $P = P_1 + \cdots + P_k$. Then there exist $l, l_1, \dots, l_k \in \mathbf{N}$ so that*

$$t_0^l h(P) = t_0^{l_1} h(P_1) + \cdots + t_0^{l_k} h(P_k).$$

Let us define $\varpi : \mathbf{N} \times L \times \{1, \dots, r\} \longrightarrow L \times \{1, \dots, r\}$ by $\varpi(\lambda, \mu, \nu, \alpha, \beta, i) = (\mu, \nu, \alpha, \beta, i)$. For a nonzero element $P = P(t_0)$ of $D_{d+n}[t_0]^r$, let us denote by $\mathrm{lexp}_H(P) \in \mathbf{N} \times L \times \{1, \dots, r\}$ and $\mathrm{lcoef}_H(P) \in K$ the leading exponent and the leading coefficient of P with respect to $\prec_H$.

Lemma 3.7 *(1) If $P(t_0) \in D_{d+n}[t_0]^r$ is $F[\mathbf{m}]$-homogeneous, then we have $\mathrm{lexp}_F(P(1)) = \varpi(\mathrm{lexp}_H(P(t_0)))$.*

(2) For any $P \in D_{d+n}^r$, we have $\mathrm{lexp}_F(P) = \varpi(\mathrm{lexp}_H(h(P)))$.

Since the Buchberger algorithm preserves the $F[\mathbf{m}]$-homogeneity, we have

Proposition 3.8 *Let N be a left $D_{d+n}[t_0]$-submodule of $D_{d+n}[t_0]^r$ generated by $F[\mathbf{m}]$-homogeneous operators. Then there exists a Gröbner basis with respect to $\prec_H$ of N consisting of $F[\mathbf{m}]$-homogeneous operators. Moreover, such a Gröbner basis can be computed by the Buchberger algorithm.*

The following proposition can be easily proved in the same way as [30, Theorem 3.12]

Proposition 3.9 *Let N be a left D_{d+n}-submodule of D_{d+n}^r generated by $P_1, \dots, P_l \in D_{d+n}^r$. Let us denote by $h(N)$ the left $D_{d+n}[t_0]$-submodule of $D_{d+n}[t_0]^r$ generated by $h(P_1), \dots, h(P_l)$. Let $G = \{Q_1(t_0), \dots, Q_k(t_0)\}$ be a Gröbner basis of $h(N)$ with respect to $\prec_H$ consisting of $F[\mathbf{m}]$-homogeneous operators. Then $G(1) := \{Q_1(1), \dots, Q_k(1)\}$ is an $F[\mathbf{m}]$-Gröbner basis of N.*

Thus we have an algorithm of computing an $F[\mathbf{m}]$-Gröbner basis for an arbitrary shift vector $\mathbf{m} \in \mathbf{Z}^r$. We can prove the following in the same way as [30, Proposition 3.11]

Proposition 3.10 *Let N and $Q_j(t_0)$ be as in Proposition 3.9 and put $\mathcal{N} := \mathcal{D}_X \otimes_{D_{d+n}} N \subset \mathcal{D}_X^r$. Then for any germ P of $\mathcal{N}$ at $p \in Y$, there exist germs U_j of $\mathcal{D}_X$ at p such that $P = U_1 Q_1(1) + \cdots + U_k Q_k(1)$ and $\mathrm{ord}_Y[\mathbf{m}](U_j Q_j(1)) \le \mathrm{ord}_Y[\mathbf{m}](P)$ for $j = 1, ..., k$.*

If the leading exponent of $P \in D_{d+n}[t_0]^r$ is $\mathrm{lexp}_H(P) = (\lambda, \tilde{\alpha}, i) \in \mathbf{N} \times L \times \{1, \dots, r\}$, we define the leading position $\mathrm{lp}_H(P)$ of P by i. For $\tilde{\alpha}, \tilde{\beta} \in L$ and $i \in \{1, ..., r\}$, we put

$$\begin{aligned} (\tilde{\alpha}, i) \vee (\tilde{\beta}, i) &:= (\max\{\tilde{\alpha}_1, \tilde{\beta}_1\}, ..., \max\{\tilde{\alpha}_{2d+2n}, \tilde{\beta}_{2d+2n}\}, i), \\ (\tilde{\alpha}, i) + (\tilde{\beta}, i) &:= (\tilde{\alpha} + \tilde{\beta}, i). \end{aligned}$$

Let N and $Q_j(t_0)$ be as in Proposition 3.9 and put $\Lambda := \{(i,j) \mid 1 \le i < j \le k,\ \mathrm{lp}_H(Q_i(t_0)) = \mathrm{lp}_H(Q_j(t_0))\}$. For $(i,j) \in \Lambda$, let $S_{ij}(t_0), S_{ji}(t_0) \in D_{d+n}[t_0]$ be monomials such that

$$\mathrm{lexp}_H(S_{ji}(t_0)Q_i(t_0)) = \mathrm{lexp}_H(S_{ij}(t_0)Q_j(t_0)) = \mathrm{lexp}_H(Q_i(t_0)) \vee \mathrm{lexp}_H(Q_j(t_0)),$$
$$\mathrm{lcoef}_H(S_{ji}(t_0)Q_i(t_0)) = \mathrm{lcoef}_H(S_{ij}(t_0)Q_j(t_0)).$$

Then by the division algorithm, there exist F-homogeneous $U_{ijl}(t_0) \in D_{d+n}[t_0]$ so that we have

$$S_{ji}(t_0)Q_i(t_0) - S_{ij}(t_0)Q_j(t_0) = \sum_{l=1}^{k} U_{ijl}(t_0)Q_l(t_0)$$

and either $U_{ijl}(t_0) \neq 0$ or else

$$\mathrm{lexp}_H(U_{ijl}(t_0)Q_l(t_0)) \prec_H \mathrm{lexp}_H(Q_i(t_0)) \vee \mathrm{lexp}(Q_j(t_0))$$

for each $l = 1, ..., k$.

The proof of the following proposition is similar to that of [30, Theorem 3.13]:

Proposition 3.11 *In the same notation as in Proposition 3.10, the left D_{d+n}-module*

$$\mathrm{Syz}(Q_1(1), \dots, Q_k(1)) := \{(U_1, ..., U_k) \in D_{d+n}^k \mid U_1Q_1(1) + \cdots + U_kQ_k(1) = 0\}$$

is generated by $\{V_{ij}(1) \mid (i,j) \in \Lambda\}$ with

$$V_{ij}(t_0) := (0, \dots, \overset{(i)}{S_{ji}(t_0)}, \dots, -\overset{(j)}{S_{ij}(t_0)}, \dots, 0) - (U_{ij1}(t_0), \dots, U_{ijk}(t_0)).$$

Now let us describe an algorithm for computing a free resolution of M which is adapted to the filtration $F_Y[\mathbf{m}]$ (cf. Proposition 2.3). Let N be a left D_{d+n}-submodule of D_{d+n}^r such that $M = D_{d+n}^r/N$.

First, starting with a given $\mathbf{m} \in \mathbf{Z}^r$, let $\{P_1, ..., P_{r_1}\}$ be an $F_Y[\mathbf{m}]$-Gröbner basis of N constructed as in Proposition 3.9. Put

$$\mathbf{m}_1 := (\mathrm{ord}_Y[\mathbf{m}](P_1), \dots, \mathrm{ord}_Y[\mathbf{m}](P_{r_1}))$$

and define $\psi_1 : D_{d+n}^{r_1} \longrightarrow D_{d+n}^r$ by

$$\psi_1(Q_1, ..., Q_{r_1}) := Q_1P_1 + \cdots + Q_{r_1}P_{r_1}.$$

Then we get a set of generators of the kernel $\mathrm{Ker}\,\psi_1$ by using Proposition 3.11.

By the same procedure as above with N, r, and $\mathbf{m}$ replaced by $\mathrm{Ker}\,\psi_1$, r_1, and $\mathbf{m}_1$ respectively, we obtain a homomorphism $\psi_2 : D_{d+n}^{r_2} \to D_{d+n}^{r_1}$ so that $\mathrm{Im}\,\psi_2 = \mathrm{Ker}\,\psi_1$. In view of Propositions 3.10 and 3.11, the sequence

$$F_Y^k[\mathbf{m}_2](D_{d+n}^{r_2}) \xrightarrow{\psi_2} F_Y^k[\mathbf{m}_1](D_{d+n}^{r_1}) \xrightarrow{\psi_1} F_Y^k[\mathbf{m}](D_{d+n}^r)$$

is exact for any $k \in \mathbf{Z}$ with $\mathbf{m}_2 \in \mathbf{Z}^{r_2}$ defined by

$$\mathbf{m}_2 := (\mathrm{ord}_Y[\mathbf{m}_1](\psi_2(1, \dots, 0)), \dots, \mathrm{ord}_Y[\mathbf{m}_1](\psi_2(0, \dots, 1))).$$

Proceeding in the same way, we can obtain a free resolution (2.3) which is adapted to the $F_Y[\mathbf{m}]$-filtration for any given $l \in \mathbf{N}$.

4 The b-function of a D-module

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module on X. We assume that a left D_{d+n}-submodule N of D_{d+n}^r is given explicitly so that $\mathcal{M} = \mathcal{D}_X \otimes_{D_{d+n}} M$ holds with $M := D_{d+n}^r/N$. Set $\mathcal{N} := \mathcal{D}_X \otimes_{D_{d+n}} N \subset \mathcal{D}_X^r$. In this and the following sections, we restrict ourselves to the case $\mathbf{m} = (0,\dots,0) \in \mathbf{Z}^r$ and omit the notation $[\mathbf{m}]$. Then the F_Y-filtrations and the associated graded rings and modules are defined as in the previous sections with $u_1,\dots,u_r$ being the residue classes in M (or in $\mathcal{M}$) of the canonical generators of D_{d+n}^r (or of $\mathcal{D}_X^r$). Then, for any $k \in \mathbf{Z}$,

$$\mathrm{gr}_Y^k(\mathcal{M}) := F_Y^k(\mathcal{M})/F_Y^{k-1}(\mathcal{M})$$

is a left $\mathrm{gr}_Y^0(\mathcal{D}_X)$-module and we have

$$\mathrm{gr}_Y^k(\mathcal{M}) = \mathrm{gr}_Y^k(\mathcal{D}_X)\mathrm{gr}_Y^0(\mathcal{M}).$$

We put $\vartheta := t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$. This is the unique vector field modulo $F_Y^{-1}(\mathcal{D}_X)$ that operates on $\mathcal{J}_Y/\mathcal{J}_Y^2$ as identity. Let θ be a commutative variable corresponding to ϑ.

Definition 4.1 The *b-function* (or the *indicial polynomial*) $b(\theta,p) \in K[\theta]$ of $\mathcal{M}$ along Y with respect to the filtration F_Y at $p \in Y$ is the monic polynomial $b(\theta,p) \in K[\theta]$ of the least degree, if any, that satisfies

$$b(\vartheta,p)\mathrm{gr}_Y^0(\mathcal{M})_p = 0.$$

If such $b(\theta,p)$ exists, $\mathcal{M}$ is called *specializable* along Y at p. If $\mathcal{M}$ is not specializable at p, we put $b(\theta,p) = 0$. The *global b-function* $b(\theta)$ of $\mathcal{M}$ along Y is defined to be the least common multiple of $b(\theta,p)$ with p running through Y.

It is known that the specializability does not depend on the choice of the generators $u_1,\dots,u_r$ of $\mathcal{M}$, while the b-function itself can depend on it (cf. [24]). It is also known that if $\mathcal{M}$ is holonomic, then $\mathcal{M}$ is specializable along an arbitrary submanifold ([19],[20],[22]).

Let us describe the procedures for computing $b(\theta,p)$ and $b(\theta)$: First, we reduce to the case $r = 1$. For each $i = 1,\dots,r$, let π_i be the projection of $\mathrm{gr}_Y(\mathcal{D}_X^r)$ to the i-th component and put

$$\mathrm{gr}_Y(\mathcal{N})^{(i)} := \{P \in \mathrm{gr}_Y(\mathcal{N}) \mid \pi_j(P) = 0 \text{ for } j = i+1,\dots,r\}.$$

Note that $\mathrm{gr}_Y(\mathcal{N})^{(i)}/\mathrm{gr}_Y(\mathcal{N})^{(i-1)}$ can be regarded as a left ideal of $\mathrm{gr}_Y(\mathcal{D}_X)$ by the projection to the i-th component. Then we get the following lemma:

Lemma 4.2 *Under the above notation, $b(\vartheta,p)$ is a generator of the ideal*

$$\bigcap_{i=1}^{r} \left(K[\vartheta] \cap (\mathrm{gr}_Y(\mathcal{N})^{(i)}/\mathrm{gr}_Y(\mathcal{N})^{(i-1)})_p\right).$$

Let us now assume that the order $\prec$ satisfies

$$(\tilde\alpha, i) \prec (\tilde\alpha', j) \text{ if } i < j \text{ for } \tilde\alpha, \tilde\alpha' \in L \text{ and } i,j \in \{1,\dots,r\}. \tag{4.1}$$

Let G be a Gröbner basis of N with respect to $\prec_F$ defined by $\prec$ as in Section 3. Then $\hat{\mathcal{I}} := \mathrm{gr}_Y(\mathcal{N})^{(i)}/\mathrm{gr}_Y(\mathcal{N})^{(i-1)}$ is generated by

$$\{\pi_i(\sigma_Y(P)) \mid P \in G,\ \sigma_Y(P) \in \mathrm{gr}_Y(\mathcal{N})^{(i)}\}.$$

Our next task is to compute the intersection $\hat{\mathcal{I}} \cap \mathcal{D}_Y[t_1\partial_1, \dots, t_d\partial_d]$. For this purpose, we introduce commutative indeterminates $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d)$, and work with the ring $D_{d+n}[v, w]$. For an element P of D_{d+n} of the form

$$P = \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta} t^\mu \partial_t^\nu x^\alpha \partial_x^\beta,$$

its multi-homogenization $\mathrm{mh}(P) \in D_{d+n}[v]$ is defined by

$$\mathrm{mh}(P) := \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta} v_1^{\nu_1-\mu_1-\kappa_1} \cdots v_d^{\nu_d-\mu_d-\kappa_d} t^\mu \partial_t^\nu x^\alpha \partial_x^\beta$$

with $\kappa_j := \min\{\nu_j - \mu_j \mid a_{\mu\nu\alpha\beta} \neq 0\}$. Let $\prec_{mh}$ be an order on $\mathbf{N}^d \times \mathbf{N}^d \times L \ni (\rho, \sigma, \tilde{\alpha})$ defined by

$$\begin{aligned} &(\rho, \sigma, \tilde{\alpha}) \prec_{mh} (\rho', \sigma', \tilde{\alpha}') \text{ if and only if } |\rho + \sigma| < |\rho' + \sigma'| \\ &\text{or else } |\rho + \sigma| = |\rho' + \sigma'|,\ \tilde{\alpha} < \tilde{\alpha}' \end{aligned} \tag{4.2}$$

with an arbitrary well-order $<$ on L satisfying (3.1). Fixing an $i \in \{1, \dots, d\}$, we assign weight 1 to w_i, ∂_{t_i}, weight -1 to v_i, t_i, and weight 0 to all the other variables. An element of $D_{d+n}[v, w]$ is said to be multi-homogeneous if it is homogeneous with respect to the weight above for each $i = 1, ..., d$. Thus $\mathrm{mh}(P)$ is multi-homogeneous for any $P \in D_{d+n}$. Put $S_\kappa := S_{1\kappa_1} \cdots S_{d\kappa_d}$ for $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbf{Z}^d$ with $S_{ij} = \partial_{t_i}^j$ if $j \geq 0$ and $S_{ij} := t_i^{-j}$ otherwise. Let $s = (s_1, \dots, s_d)$ be commutative indeterminates. Assume that $P \in D_{d+n}$ is multi-homogeneous. Then we have

$$S_\kappa P = Q(t_1\partial_{t_1}, \dots, t_d\partial_{t_d}, x, \partial_x)$$

with some $Q(s_1, \dots, s_d, x, \partial_x) \in D_n[s_1, \dots, s_d]$ and $\kappa \in \mathbf{Z}^d$. We put

$$\psi(P)(s_1, \dots, s_d) := Q(s_1, \dots, s_d).$$

Proposition 4.3 *Let $\hat{\mathcal{I}}$ be a left ideal of $\mathrm{gr}_Y(\mathcal{D}_X)$. Let G_0 be a finite subset of $\mathrm{gr}_Y(D_{d+n})$ which generates $\hat{\mathcal{I}}$. Let G_1 be a Gröbner basis with respect to $\prec_{mh}$ of the ideal of $D_{d+n}[v, w]$ generated by*

$$\{\mathrm{mh}(P) \mid P \in G_0\} \cup \{1 - v_i w_i \mid i = 1, \dots, d\}.$$

We may assume that G_1 consists of multi-homogeneous elements since so does the input. Then the left ideal $\hat{\mathcal{I}} \cap \mathcal{D}_Y[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]$ of $\mathcal{D}_Y[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]$ is generated by

$$G_2 := \{\psi(P)(t_1\partial_{t_1}, \dots, t_d\partial_{t_d}) \mid P \in G_1 \cap D_{d+n}\}.$$

Proof: Let P be an element of $G_1 \cap D_{d+n}$. Since P is multi-homogeneous and free of v, w, there exists $\kappa \in \mathbf{Z}^d$ so that $Q := \psi(P)(t_1\partial_{t_1}, \dots, t_d\partial_{t_d}) = S_\kappa P$. By definition, P belongs to the ideal generated by $\mathrm{mh}(G_0)$ and $1 - v_i w_i$ $(i = 1, \dots, d)$. Setting $v_i = w_i = 1$, we know that Q belongs to $\hat{\mathcal{I}}$.

Conversely, let P be an arbitrary germ of $\hat{\mathcal{I}} \cap \mathcal{D}_Y[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]$ at $p \in Y$. Multiplying P by a polynomial in x which does not vanish at p, we may assume $P \in D_n[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]$. In view of the definition of the multi-homogenization and the fact that $\mathrm{mh}(P) = P$, there exists $\rho \in \mathbf{N}^d$ so that $v^\rho P$ belongs to the ideal generated by $\mathrm{mh}(G_0)$. This implies that P belongs to the ideal generated by $\mathrm{mh}(G_0)$ and $\{1 - v_i w_i \mid i = 1, \dots, d\}$ since

$$P = (1 - v^\rho w^\rho)P + v^\rho w^\rho P$$

and $1 - v^\rho w^\rho$ belongs to the ideal generated by $\{1 - v_i w_i \mid i = 1, \dots, d\}$. Set $G_1 \cap D_{d+n} = \{P_1, \dots, P_k\}$. Then by the definition of $\prec_{mh}$ and G_1, there exist $Q_1, \dots, Q_k \in D_{d+n}$ so that

$$P = Q_1 P_1 + \cdots + Q_k P_k.$$

Since $P_1, \dots, P_k$ are multi-homogeneous as well as P, we may assume that such is also the case with $Q_1, \dots, Q_k$. Hence there exist $\kappa^{(1)}, \dots, \kappa^{(k)} \in \mathbf{Z}^d$ and $Q'_i \in D_n$ so that

$$P = \psi(P) = Q'_1 S_{\kappa^{(1)}} P_1 + \cdots + Q'_k S_{\kappa^{(k)}} P_k = Q'_1 \psi(P_1) + \cdots + Q'_k \psi(P_k).$$

This completes the proof. []

Let $\hat{\mathcal{I}}$ be as in Proposition 4.3. Now we have obtained a set of generators G_2 of $\hat{\mathcal{I}} \cap \mathcal{D}_Y[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]$. We identify each $t_i\partial_{t_i}$ with s_i. Then from G_2, we can compute a set of generators G_3 of the ideal $\hat{\mathcal{I}} \cap \mathcal{O}_Y[s]$ of $\mathcal{O}_Y[s]$ by eliminating ∂_x by means of Gröbner basis in the Weyl algebra (see e.g. [28] for details). Then it is easy to obtain a subset G_4 of $K[x, \theta]$ which generates the sheaf of ideals

$$\mathcal{J} := \hat{\mathcal{I}} \cap \mathcal{O}_Y[\theta] = \hat{\mathcal{I}} \cap \mathcal{O}_Y[s] \cap \mathcal{O}_Y[\theta]$$

with $\theta = s_1 + \cdots + s_d$ again by Gröbner basis in the polynomial ring. Let us denote by J the ideal of $K[x, \theta]$ generated by G_4. Then $b(\theta, p)$ is a generator of

$$\mathcal{J}_p \cap K[\theta] = (\mathcal{O}_Y[\theta])_p J \cap K[\theta].$$

Our final task is to compute the b-function at each point of Y by using the input G_4. This is achieved by primary decomposition. Let us state the method in a more general setting, where we replace the variable θ by the variables $s = (s_1, \dots, s_d)$ for the sake of generality: So let J be an arbitrary ideal of $K[x, s]$ whose generators are given. For each point p of Y, put

$$B(J, p) := (\mathcal{O}_Y[s])_p J \cap K[s],$$

which is an ideal of $K[s]$. Let $J = Q_1 \cap \cdots \cap Q_l$ be a primary decomposition in $K[x, s]$. Then by the flatness of $(\mathcal{O}_Y[s])_p$ over $K[x, s]$ we have

$$B(J, p) = B(Q_1, p) \cap \cdots \cap B(Q_l, p).$$

Each ideal on the right hand side can be computed easily by the following:

Lemma 4.4 *Let Q be a primary ideal of $K[x,s]$ and put*

$$\mathbf{V}_Y(Q) := \{x \in Y = K^n \mid f(x) = 0 \textit{ for any } f \in Q \cap K[x]\}.$$

Then we have

$$B(Q,p) = \begin{cases} Q \cap K[s] & \textit{if } p \in \mathbf{V}_Y(Q) \\ K[s] & \textit{if } p \in Y \setminus \mathbf{V}_Y(Q). \end{cases}$$

Proof: First assume $p \notin \mathbf{V}_Y(Q)$. Then there exists $a(x) \in Q \cap K[x]$ such that $a(p) \neq 0$. This implies that $B(Q,p) = K[s]$. Next assume $p \in \mathbf{V}_Y(Q)$ and $b(s) \in B(Q,p)$. Then there exists $a(x) \in K[x]$ so that $a(x)b(s) \in Q$ and $a(p) \neq 0$. Suppose $b(s)$ does not belong to Q. Then we have $a(x)^j \in Q$ with some $j \in \mathbf{N}$ since Q is primary. This implies $a(p) = 0$, which is a contradiction. Thus we have $B(Q,p) \subset Q \cap K[s]$. The converse inclusion is obvious. []

Lemma 4.5 *Let J be an ideal of $K[x,s]$. Then we have*

$$\bigcap_{p \in Y} B(J,p) = J \cap K[s].$$

Proof: Let $J = Q_1 \cap \cdots \cap Q_l$ be a primary decomposition. Then by the preceding lemma, we have

$$\begin{aligned} \bigcap_{p \in Y} B(J,p) &= \bigcap_{p \in Y} \bigcap_{j=1}^{l} B(Q_j,p) \\ &= \bigcap \{Q_j \cap K[s] \mid Q_j \cap K[x] \neq K[x]\} \\ &= J \cap K[s] \end{aligned}$$

since $Q_j \cap K[s] = K[s]$ if and only if $Q_j \cap K[x] = K[x]$. []

Returning back to the ideal J of $K[s,\theta]$ generated by G_4, we have only to apply Lemma 4.4 with s replaced by θ. This gives us an algebraic stratification of Y so that $b(s,p)$ is constant on each stratum as a function of p. Moreover, Lemma 4.5 tells us that the global b-function of $\mathcal{M}$ along Y is simply a generator of $J \cap K[\theta]$. Thus the algorithm is summarized as follows:

Algorithm 4.6 (The b-function of $\mathcal{M} := \mathcal{D}_X \otimes_{D_{d+n}} M$)
Input: $M = D_{d+n}^r / N$ with an D_{d+n}-submodule N of D_{n+d}^r.

(1) Compute a Gröbner basis G of N with respect to the order $\prec_F$ that is defined through (3.2) by using an order $\prec$ satisfying (3.1) and (4.1) with $\mathbf{m} = 0$.

(2) For $i = 1$ to r do

  (a) $G_i := \{\pi_i(\sigma_Y(P)) \mid P \in G,\ \pi_j(\sigma_Y(P)) = 0 \text{ for any } j > i\}$.

  (b) Let G_{i1} be a Gröbner basis of the left ideal of $D_{d+n}[v,w]$ generated by

$$\{\mathrm{mh}(P) \mid P \in G_i\} \cup \{1 - v_i w_i \mid i = 1, \ldots, d\}$$

  with respect to an order $\prec_{mh}$ satisfying (4.2).

(c) $G_{i2} := \{\psi(P) \in D_n[s] \mid P \in G_{i1} \cap D_{d+n}\}$.

(d) Compute $J_i := \langle G_{i2} \rangle \cap K[x, \theta]$ first by eliminating ∂_x, then eliminating $s'_2, \dots, s'_d$ after substitution $\theta = s_1 + \cdots + s_d$ and $s'_j = s_j$ for $j = 2, \dots, d$; here $\langle G_{i2} \rangle$ denotes the left ideal of $D_n[s]$ generated by G_{i2}.

(3) The global b-function $b(\theta)$ of $\mathcal{M}$ is the generator of $\bigcap_{i=1}^{r} (J_i \cap K[\theta])$.

(4) For $i = 1$ to r do

(a) Compute a primary decomposition $J_i = \bigcap_{j=1}^{l_i} Q_{ij}$ in $K[x, \theta]$.

(b) Compute (generators of) $Q_{ij} \cap K[s]$ and $Q_{ij} \cap K[x]$ for $j = 1, \dots, l_i$ by elimination.

(5) For each $p \in Y$, the local b-function $b(\theta, p)$ is the generator of the ideal

$$\bigcap_{i=1}^{r} \bigcap_{j=1}^{l_i} \{Q_{ij} \cap K[\theta] \mid g(p) = 0 \text{ for any } g(x) \in Q_{ij} \cap K[x]\}.$$

Let us remark on the coefficient field: Suppose that the input is defined over a subfield K_0 of K. Then the steps 1–3 can be done over K_0 instead of K and $b(\theta, p)$ divides $b(\theta)$ for any $p \in Y$. However, the primary decomposition in the step 4 must be one in $K[x, \theta]$ not in $K_0[x, \theta]$. In fact, we need a primary decomposition over an intermediate field K_1 with $K_0 \subset K_1 \subset K$ so that $b(\theta)$ factors into linear polynomials in $K_1[\theta]$. If, e.g., K_0 is the rationals $\mathbf{Q}$, such K_1 is computable. Hence the primary decomposition in the step 4 is certainly computable if the input is defined over $\mathbf{Q}$ in view of e.g., [2], [11], [38] and gives the local b-function at any $p \in Y = K^n$.

As a special case where all the computation can be done over K_0, suppose that the ideal $J \cap K_0[\theta]$ is generated by a polynomial which is a multiple of linear factors over K_0. Then the step 4 of Algorithm 4.6 can be computed over K_0 and the step 5 is true for any $p \in K^n$; one can easily verify this by considering a projection of K to K_0. Note that this is exactly the case with the classical Bernstein-Sato polynomial (cf. [30],[31],[32] for algorithms) and $K_0 = \mathbf{Q}$ by virtue of Kashiwara's theorem on the rationality [14].

At this occasion, let us make a correction to [32]: Lemma 4.4 of [32] does not hold in general; we need field extension as explained above. This correction does not affect the rest of [32].

Example 4.7 Let K be an arbitrary field of characteristic zero and put $X := K^5 \ni (t_1, t_2, x, y, z)$, $Y := \{(t_1, t_2, x, y, z) \in X \mid t_1 = t_2 = 0\}$. Put $M := D_5/I$ with the left ideal I generated by

$$t_1 - x^3 + y^2,\ t_2 - y^3 + z^2,\ \partial_x + 3x^2\partial_{t_1},\ \partial_y - 2y\partial_{t_1} + 3y^2\partial_{t_2},\ \partial_z - 2z\partial_{t_2}.$$

Then the b-function $b(s, p)$ of $\mathcal{M} := \mathcal{D}_X \otimes_{D_5} M$ along Y at $p \in Y = K^3$ is given as follows:

$$\begin{aligned} b(s,p) = \ & s\left(s - \tfrac{5}{18}\right)\left(s - \tfrac{1}{6}\right)\left(s - \tfrac{1}{18}\right)\left(s + \tfrac{1}{18}\right)\left(s + \tfrac{1}{6}\right) \\ & \times \left(s + \tfrac{5}{18}\right)\left(s + \tfrac{1}{3}\right)\left(s + \tfrac{7}{18}\right)\left(s + \tfrac{11}{18}\right)\left(s + \tfrac{2}{3}\right) \end{aligned}$$

if $p = (0, 0, 0) \in Y$; $b(s, p) = s$ if $p \in \{(x, y, z) \in K^3 \mid x^3 - y^2 = 0,\ y^3 - z^2 = 0\} \setminus \{(0, 0, 0)\}$; and $b(s, p) = 1$ otherwise.

5 Restriction of a D-module

We retain the notation of the preceding sections. Put

$$\mathcal{D}_{Y\to X} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

Then $\mathcal{D}_{Y\to X}$ has a natural structure of $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule. Let $\mathcal{M} = \mathcal{D}_X \otimes_{D_{d+n}} M$ be a coherent $\mathcal{D}_X$-module with a finitely generated D_{d+n}-module $M = D_{d+n}^r/N$. Then the (D-module theoretic) *restriction* of $\mathcal{M}$ to Y is defined by

$$\mathcal{M}_Y^\bullet := \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X}^L \mathcal{M}$$

in the derived category of left $\mathcal{D}_X$-modules, where $\otimes^L$ denotes the left derived functor of the tensor product (see [13] for the derived category and derived functors).

In general, let $\mathcal{L}$ be a $K[t]$-module and $\mathcal{L}_j$ ($j \in \mathbf{Z}$) be additive subgroups of $\mathcal{L}$ such that $t_i\mathcal{L}_j \subset \mathcal{L}_{j-1}$ holds for $i = 1, \ldots, d$ and $j \in \mathbf{Z}$. Then for any integer k, we define the Koszul complex $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ associated with $\mathcal{L}_\bullet = \{\mathcal{L}_j\}_{j\in\mathbf{Z}}$ and $t_1, \ldots, t_d$ by

$$0 \longrightarrow \mathcal{L}_{k+d}\otimes_{\mathbf{Z}} \overset{0}{\wedge}\, \mathbf{Z}^d \xrightarrow{\delta} \mathcal{L}_{k+d-1}\otimes_{\mathbf{Z}} \overset{1}{\wedge}\, \mathbf{Z}^d \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{L}_k\otimes_{\mathbf{Z}} \overset{d}{\wedge}\, \mathbf{Z}^d \longrightarrow 0,$$

where δ is defined by

$$\delta(u \otimes e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{l=1}^{d} t_l u \otimes e_l \wedge e_{i_1} \wedge \cdots \wedge e_{i_j}$$

for a subset $\{i_1, \ldots, i_j\}$ of $\{1, \ldots, d\}$ with the unit vectors $e_1, \ldots, e_d$ of $\mathbf{Z}^d$. When $\mathcal{L}_j = \mathcal{L}$ for each j, we also denote this simply by $\mathcal{K}^\bullet(\mathcal{L}, t_1, \ldots, t_d)$. Here we regard $\mathcal{L}_{k+d-j}\otimes \overset{j}{\wedge}\, \mathbf{Z}^d$ as being placed at the degree $-j$ to be compatible with the cohomology theory.

In particular, $\mathcal{K}^\bullet(\mathcal{D}_X, t_1, \ldots, t_d)$ is quasi-isomorphic to $\mathcal{D}_{Y\to X}$ in the derived category of right $\mathcal{D}_X$-modules. Hence we can identify $\mathcal{M}_Y^\bullet$ with the complex

$$\mathcal{K}^\bullet(\mathcal{D}_X, t_1, \ldots, t_d) \otimes_{\mathcal{D}_X} \mathcal{M} = \mathcal{K}^\bullet(\mathcal{M}, t_1, \ldots, t_d).$$

Our purpose below is to describe an algorithm to compute each cohomology group $\mathcal{H}^i(\mathcal{M}_Y^\bullet)$ (for $i = 0, -1, \ldots, -d$ since it is zero for other i) under the assumption that $\mathcal{M}$ is specializable along Y. Let $b(\theta, p)$ be the b-function of $\mathcal{M}$ along Y at $p \in Y$.

Proposition 5.1 *Let p be a point of Y and k be an integer such that $b(k, p) \neq 0$. Then the Koszul complex $\mathcal{K}^\bullet(\mathrm{gr}_Y^\bullet(\mathcal{M})[k], t_1, \ldots, t_d)$ associated with $\{\mathrm{gr}_Y^j(\mathcal{M})\}_{j\in\mathbf{Z}}$ is exact at p (i.e., on a Zariski neighbourhood of p).*

We shall prove this proposition in a slightly more general situation. Let $D_d := K[t]\langle\partial_t\rangle$ be the Weyl algebra on the variables $t = (t_1, \ldots, t_d)$ and define a filtration on it and the associated graded module by

$$F_k(D_d) := \{ \sum_{|\nu-\mu|\leq k} a_{\mu\nu} t^\mu \partial_t^\nu \}, \qquad \mathrm{gr}_k(D_d) := F_k(D_d)/F_{k-1}(D_d).$$

Note that $\mathrm{gr}(D_d) := \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k(D_d)$ is isomorphic to D_d. In particular, we can regard $\mathrm{gr}_0(D_d)$ as a subring of D_d.

Proposition 5.2 *Let $\mathcal{L} = \bigoplus_{j\in\mathbf{Z}} \mathcal{L}_j$ be a graded $\mathrm{gr}(D_d)$-module; i.e., assume $\mathrm{gr}_j(D_d)\mathcal{L}_i \subset \mathcal{L}_{i+j}$ for $i, j \in \mathbf{Z}$. Assume moreover that there exists a nonzero polynomial $b(\theta) \in K[\theta]$ which satisfies $b(\vartheta + j)\mathcal{L}_j = 0$ for any $j \in \mathbf{Z}$ with $\vartheta = t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$. Let k be an integer such that $b(k) \neq 0$. Then $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ is exact.*

Proof: We argue by induction on d. First suppose $d = 1$. Then $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1)$ is the complex

$$0 \longrightarrow \mathcal{L}_{k+1} \xrightarrow{t_1} \mathcal{L}_k \longrightarrow 0.$$

Assume $u \in \mathcal{L}_{k+1}$ satisfies $t_1 u = 0$. Then we have $u = 0$ since

$$0 = b(t_1\partial_{t_1} + k + 1)u = b(\partial_{t_1}t_1 + k)u = b(k)u.$$

On the other hand, there exists $P \in D_d$ so that $b(t_1\partial_{t_1} + k) = t_1P + b(k)$. Hence for an arbitrary $v \in \mathcal{L}_k$, we conclude $v \in t_1\mathcal{L}_{k+1}$ from $b(t_1\partial_{t_1} + k)v = 0$.

Now assume the proposition is true with d replaced by $d - 1$. It is easy to see, as in the case of the usual Koszul complex (see e.g., [37, p.188]), that $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ is quasi-isomorphic to the complex associated with the double complex

$$\begin{array}{c} \mathcal{K}^\bullet(\mathcal{L}_\bullet[k+1], t_1, \ldots, t_{d-1}) \\ \downarrow t_d \\ \mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_{d-1}). \end{array} \tag{5.1}$$

Let us denote by $\mathcal{L}'_j$ and $\mathcal{L}''_j$ the kernel and the cokernel of $t_d : \mathcal{L}_{j+1} \longrightarrow \mathcal{L}_j$. Then $\mathcal{L}' := \bigoplus_{j\in\mathbf{Z}} \mathcal{L}'_j$ and $\mathcal{L}'' := \bigoplus_{j\in\mathbf{Z}} \mathcal{L}''_j$ are graded $\mathrm{gr}(D_{d-1})$-modules. For any $u \in \mathcal{L}'_j$, we have

$$\begin{aligned} 0 &= b(t_1\partial_{t_1} + \cdots + t_d\partial_{t_d} + j + 1)u \\ &= b(t_1\partial_{t_1} + \cdots + t_{d-1}\partial_{t_{d-1}} + \partial_{t_d}t_d + j)u \\ &= b(t_1\partial_{t_1} + \cdots + t_{d-1}\partial_{t_{d-1}} + j)u. \end{aligned}$$

On the other hand, for $v \in \mathcal{L}_j$, let $\overline{v}$ be its residue class in $\mathcal{L}''_j$. Then we have

$$0 = b(t_1\partial_{t_1} + \cdots + t_d\partial_{t_d} + j)\overline{v} = b(t_1\partial_{t_1} + \cdots + t_{d-1}\partial_{t_{d-1}} + j)\overline{v}.$$

Thus both $\mathcal{L}'$ and $\mathcal{L}''$ satisfy the conditions of the proposition with d replaced by $d - 1$. By the induction hypothesis, the complexes $\mathcal{K}^\bullet(\mathcal{L}'_\bullet[k]; t_1, \ldots, t_{d-1})$ and $\mathcal{K}^\bullet(\mathcal{L}''_\bullet[k]; t_1, \ldots, t_{d-1})$ are exact. Hence the vertical chain map of (5.1) is a quasi-isomorphism, which implies that $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ is exact. □

Under the assumption of Proposition 5.1, we have $b(\vartheta + j)\mathrm{gr}^j_Y(\mathcal{M}) = 0$ for any $j \in \mathbf{Z}$. In fact, for $P \in \mathrm{gr}^j_Y(\mathcal{D}_X)$, we easily get $b(\vartheta + j)P = Pb(\vartheta)$. This yields

$$\begin{aligned} b(\vartheta + j)\mathrm{gr}^j_Y(\mathcal{M}) &= b(\vartheta + j)\mathrm{gr}^j_Y(\mathcal{D}_X)\mathrm{gr}^0_Y(\mathcal{M}) \\ &= \mathrm{gr}^j_Y(\mathcal{D}_X)b(\vartheta)\mathrm{gr}^0_Y(\mathcal{M}) = 0. \end{aligned}$$

Hence Proposition 5.1 is an immediate consequence of Proposition 5.2.

In general for $\mathbf{m} \in \mathbf{Z}^r$, we define the $F_Y[\mathbf{m}]$-filtration on $\mathcal{D}^r_{Y\to X}$ by

$$\begin{aligned} F^k_Y[\mathbf{m}](\mathcal{D}^r_{Y\to X}) &:= F^k_Y[\mathbf{m}](\mathcal{D}^r_X)/(t_1F^{k+1}_Y[\mathbf{m}](\mathcal{D}^r_X) + \cdots + t_dF^{k+1}_Y[\mathbf{m}](\mathcal{D}^r_X)) \\ &\simeq \{P = \sum_{i=1}^r \sum_{\nu,\beta} a_{\nu\beta}(x)\partial_t^\nu \partial_x^\beta e_i \mid a_{\nu\beta}(x) = 0 \text{ if } |\nu| > k - m_i\} \\ &= \bigoplus_{i=1}^r \bigoplus_{|\nu|\le k-m_i} \mathcal{D}_Y. \end{aligned}$$

We also define $F^k_Y[\mathbf{m}](D_n[\partial_t]^r)$ in the same way identifying $D_n[\partial_t]$ with $D_{d+n}/(t_1D_{d+n}+\cdots+t_dD_{d+n})$.

Theorem 5.3 *Put $\mathbf{m} = \mathbf{m}_0 = 0$ and take a free resolution (2.1) of Definition 2.1 with $l = d+1$, which is adapted to the F_Y-filtration with the shift vectors $\mathbf{m}_1, \ldots, \mathbf{m}_{d+1}$. Let p be a point of Y and let k_0 and k_1 be the minimum and the maximum integral root of the equation $b(\theta, p) = 0$ in θ. Then, for $0 \le i \le d$, the cohomology group $\mathcal{H}^{-i}(\mathcal{M}^\bullet_Y)$ is isomorphic as $\mathcal{D}_Y$-module to the $-i$-th cohomology group of the complex*

$$\frac{F^{k_1}_Y[\mathbf{m}_{d+1}](\mathcal{D}^{r_{d+1}}_{Y\to X})}{F^{k_0-1}_Y[\mathbf{m}_{d+1}](\mathcal{D}^{r_{d+1}}_{Y\to X})} \xrightarrow{\overline{\psi}_{d+1}} \cdots \xrightarrow{\overline{\psi}_2} \frac{F^{k_1}_Y[\mathbf{m}_1](\mathcal{D}^{r_1}_{Y\to X})}{F^{k_0-1}_Y[\mathbf{m}_1](\mathcal{D}^{r_1}_{Y\to X})} \xrightarrow{\overline{\psi}_1} \frac{F^{k_1}_Y(\mathcal{D}^r_{Y\to X})}{F^{k_0-1}_Y(\mathcal{D}^r_{Y\to X})} \longrightarrow 0 \qquad (5.2)$$

at p, where $\overline{\psi}_j$ is a homomorphism induced by ψ_j. In particular, we have $\mathcal{H}^{-i}(\mathcal{M}^\bullet_Y) = 0$ at p for any i if $b(\theta, p) = 0$ has no integral roots.

Proof: For any $k \in \mathbf{Z}$, the complex

$$\cdots \to \mathrm{gr}^k_Y[\mathbf{m}_{d+1}](\mathcal{D}^{r_{d+1}}_X) \xrightarrow{\overline{\psi}_{d+1}} \cdots \xrightarrow{\overline{\psi}_1} \mathrm{gr}^k_Y[\mathbf{m}_0](\mathcal{D}^{r_0}_X) \to 0,$$

is quasi-isomorphic to $\mathrm{gr}^k_Y(\mathcal{M})$ since (2.1) is adapted to the F_Y-filtration. Hence we know that $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_Y(\mathcal{M})[k], t_1, \ldots, t_d)$ is quasi-isomorphic to the complex associated with the double complex

$$\begin{array}{c} 0 \\ \uparrow \\ \mathcal{K}^\bullet(\mathrm{gr}^\bullet_Y[\mathbf{m}_0](\mathcal{D}^{r_0}_X)[k], t_1, \ldots, t_d) \\ \uparrow \overline{\psi}_1 \\ \vdots \\ \uparrow \overline{\psi}_d \\ \mathcal{K}^\bullet(\mathrm{gr}^\bullet_Y[\mathbf{m}_d](\mathcal{D}^{r_d}_X)[k], t_1, \ldots, t_d) \\ \uparrow \overline{\psi}_{d+1} \\ \mathcal{K}^\bullet(\mathrm{gr}^\bullet_Y[\mathbf{m}_{d+1}](\mathcal{D}^{r_{d+1}}_X)[k], t_1, \ldots, t_d) \\ \uparrow \\ \vdots \end{array}$$

On the other hand, we have a quasi-isomorphism

$$\mathcal{K}^\bullet(\mathrm{gr}_Y^\bullet[\mathbf{m}_i](\mathcal{D}_X^{r_i})[k], t_1, \dots, t_d) \simeq \mathrm{gr}_Y^k[\mathbf{m}_i](\mathcal{D}_{Y\to X}^{r_i}).$$

Hence $\mathcal{K}^\bullet(\mathrm{gr}_Y^\bullet(\mathcal{M})[k], t_1, \dots, t_d)$ is quasi-isomorphic to the complex

$$\cdots \to \mathrm{gr}_Y^k[\mathbf{m}_{d+1}](\mathcal{D}_{Y\to X}^{r_{d+1}}) \xrightarrow{\overline{\psi}_{d+1}} \cdots \xrightarrow{\overline{\psi}_1} \mathrm{gr}_Y^k[\mathbf{m}_0](\mathcal{D}_{Y\to X}^{r_0}) \to 0. \tag{5.3}$$

Thus, by virtue of Proposition 5.1, the complex (5.3) is exact at p if $b(k,p) \neq 0$. This implies the theorem since we have

$$F_Y^k[\mathbf{m}_i](\mathcal{D}_{Y\to X}^{r_i}) = 0$$

for sufficiently small $k \in \mathbf{Z}$. ▯

Note that $F_Y^{k_1}[\mathbf{m}_i](\mathcal{D}_{Y\to X}^{r_i})/F_Y^{k_0-1}[\mathbf{m}_i](\mathcal{D}_{Y\to X}^{r_i})$ is a free $\mathcal{D}_Y$-module of rank

$$\sum_{j=1}^{r_i} \sharp\{\nu \in \mathbf{Z}^d \mid k_0 - m_{ij} \leq |\nu| \leq k_1 - m_{ij}\}.$$

Hence Theorem 5.3 gives us an algorithm to compute each cohomology group $\mathcal{H}^{-i}(\mathcal{M}_Y^\bullet)$. In fact, we have only to compute the cohomology groups of the complex (5.2) as left D_n-modules with $\mathcal{D}_{Y\to X}$ replaced by $D_n[\partial_t]$. The flatness of $\mathcal{D}_Y$ over D_n assures us that the generators of the cohomology group over D_n also give the ones over $\mathcal{D}_Y$. The algorithm is summarized as follows:

Algorithm 5.4 (The cohomology groups of the restriction of $\mathcal{M}$ to Y)
Input: $M = D_{d+n}^r/N$ with a D_{d+n}-submodule N of D_{d+n}^r.
Output: $\mathcal{H}^{-i}(\mathcal{M}_Y^\bullet) = \mathcal{D}_Y \otimes_{D_n} (D_n^{l_i}/I_i)$ for $0 \leq i \leq d$.

(1) Compute the global b-function $b(\theta)$ of $\mathcal{M}$ along Y by the steps 1–3 of Algorithm 4.6 with M as input.

(2) If $b(\theta) = 0$, then $\mathcal{M}$ is not globally specializable along Y; quit.

(3) Let k_0 and k_1 be the minimum and the maximum integral roots of $b(\theta) = 0$. If there is no integral root, then we have $\mathcal{H}^{-i}(\mathcal{M}_Y^\bullet) = 0$ for all i; quit.

(4) Compute a free resolution

$$D_{d+n}^{r_{d+1}} \xrightarrow{\psi_{d+1}} D_{d+n}^{r_d} \xrightarrow{\psi_d} \cdots \xrightarrow{\psi_2} D_{d+n}^{r_1} \xrightarrow{\psi_1} D_{d+n}^{r_0} \xrightarrow{\varphi} M \longrightarrow 0$$

of M adapted to the F_Y-filtration (cf. Proposition 2.3) and the shift vectors $\mathbf{m}_1, \dots, \mathbf{m}_{d+1}$ by using Proposition 3.11 successively, or by using Theorem 9.10, with $\mathbf{m} = \mathbf{m}_0 = 0$.

(5) Compute the induced complex

$$\frac{F_Y^{k_1}[\mathbf{m}_{d+1}](D_n[\partial_t]^{r_{d+1}})}{F_Y^{k_0-1}[\mathbf{m}_{d+1}](D_n[\partial_t]^{r_{d+1}})} \xrightarrow{\overline{\psi}_{d+1}} \cdots \xrightarrow{\overline{\psi}_2} \frac{F_Y^{k_1}[\mathbf{m}_1](D_n[\partial_t]^{r_0})}{F_Y^{k_0-1}[\mathbf{m}_1](D_n[\partial_t]^{r_0})} \xrightarrow{\overline{\psi}_1} \frac{F_Y^{k_1}(D_n[\partial_t]^{r_0})}{F_Y^{k_0-1}(D_n[\partial_t]^{r_0})} \longrightarrow 0$$

as a complex of finitely generated free left D_n-modules. Put $\overline{\psi}_0 := 0$.

(6) Via Gröbner bases of modules over D_n, compute the $-i$-th cohomology group $\operatorname{Ker} \overline{\psi}_i / \operatorname{Im} \overline{\psi}_{i+1}$ of the above complex in the form $D_n^{l_i}/I_i$ with a left D_n-module I_i for $i = 0, \dots, d$.

Note that in the step (4) of the above algorithm, only $\psi_1, \dots, \psi_{i_0+1}$ are needed if one wants to compute only the $-i$-th cohomology groups for $i = 0, \dots, i_0$. In particular, one does not need the free resolution to compute only the 0-th cohomology.

As a direct application of the algorithm above, we obtain an algorithm to compute the cohomology groups with coefficients in the formal power series solutions of $\mathcal{M}$:

$$\operatorname{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, K[[x]]) \quad (i = 0, \dots, n)$$

under the assumption that $\mathcal{M}$ is specializable along $Y := \{0\}$. In fact, we can easily verify that there exists an isomorphism (see e.g. [23, p.428] for the case $K = \mathbf{C}$)

$$\operatorname{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, K[[x]]) \simeq \operatorname{Ext}^i_K(\mathcal{M}^\bullet_Y, K) \simeq H^{-i}(\mathcal{M}^\bullet_Y).$$

If $K = \mathbf{C}$ and $\mathcal{M}$ is Fuchsian along Y in the sense of [23] (this condition holds if $\mathcal{M}$ is regular holonomic in the sense of [20]), then by virtue of the comparison theorem ([23, Théorème 2.3.1]), we have also an isomorphism

$$\operatorname{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}\{x\}) \simeq H^{-i}(\mathcal{M}^\bullet_Y),$$

where $\mathbf{C}\{x\}$ denotes the ring of convergent power series in x. Note that Kashiwara's index theorem ([17, p.127]) gives the local index

$$\sum_{i \geq 0} (-1)^i \dim_{\mathbf{C}} \operatorname{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}\{x\})$$

at $0 \in X$ in terms of some topological quantity associated with the characteristic cycle of $\mathcal{M}$.

Example 5.5 Let us consider $M := D_4/I$, where I is the left ideal of D_4 (with $K = \mathbf{C}$) generated by

$$x_3\partial_3 + x_4\partial_4 - a_1, \quad x_1\partial_1 + x_3\partial_3 - a_2, \quad x_2\partial_2 + x_4\partial_4 - a_3, \quad \partial_1\partial_4 - \partial_2\partial_3,$$

where $a_1, a_2, a_3 \in \mathbf{C}$ are parameters. Put $\mathcal{M} := \mathcal{D}_X \otimes_{D_4} M$ with $X = \mathbf{C}^4$ and $Y_1 := \{(x_1, x_2, x_3, x_4) \in X \mid x_1 = x_2 = x_3 = 0\}$. The global b-function of $\mathcal{M}$ along Y_1 is $(s - a_2)(s + a_1 - a_2 - a_3)$. Hence the cohomology groups of the restriciton of $\mathcal{M}$ to Y_1 all vanish unless a_2 or $a_1 - a_2 - a_3$ is an integer. If $a_1 = a_2 = a_3 = 0$, we have by Algorithm 5.4

$$\mathcal{H}^i(\mathcal{M}^\bullet_{Y_1}) = \begin{cases} \mathcal{D}_{Y_1}/\mathcal{D}_{Y_1}x_4\partial_4 & (i = 0), \\ (\mathcal{D}_{Y_1}/\mathcal{D}_{Y_1}x_4\partial_4)^2 & (i = -1), \\ \mathcal{D}_{Y_1}/\mathcal{D}_{Y_1}x_4\partial_4 & (i = -2), \\ 0 & (i \leq -3). \end{cases}$$

The b-function of $\mathcal{M}$ along the point $Y_0 := \{(0,0,0,0)\}$ is $s - a_2 - a_3$. Suppose $a_1 = a_2 = a_3 = 0$. Then the cohomology groups $\mathcal{H}^i(\mathcal{M}^\bullet_{Y_0})$ of the restriction of $\mathcal{M}$ to Y_0 are $\mathbf{C}, \mathbf{C}^3, \mathbf{C}^3, \mathbf{C}, 0$ for $i = 0, -1, -2, -3, -4$ respectively. Since $\mathcal{M}$ is regular holonomic, this implies that $\operatorname{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}\{x\})$ is $\mathbf{C}, \mathbf{C}^3, \mathbf{C}^3, \mathbf{C}, 0$ for $i = 0, 1, 2, 3, 4$ respectively.

Example 5.6 Put $X := \mathbf{C}^2 \ni (x, y)$ and $\mathcal{M} := \mathcal{D}_X \otimes_{D_2} (D_2/I)$ with I being the left ideal generated by

$$x\partial_x - x(x\partial_x + y\partial_y + a)(x\partial_x + b_1), \quad y\partial_y - y(x\partial_x + y\partial_y + a)(y\partial_y + b_2).$$

Then by the computation of the restriction of $\mathcal{M}$ to $(0, 0)$, we get

$$\mathcal{E}xt^{\,i}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}[[x, y]]) = \begin{cases} \mathbf{C} & (i = 0), \\ \mathbf{C}^2 & (i = 1), \\ \mathbf{C} & (i = 2). \end{cases}$$

for generic parameters a, b_1, b_2 (this means that we perform the computation over the coefficient field $K := \mathbf{Q}(a, b_1, b_2)$). In particular, we have $\sum_{i=0}^{2}(-1)^i \dim_{\mathbf{C}} \mathcal{E}xt^{\,i}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}[[x, y]]) = 0$. On the other hand, the characteristic cycle of $\mathcal{M}$ is

$$3\{\xi = \eta = 0\} + 4\{x = \eta = 0\} + 4\{y = \xi = 0\} + \{x - y = \xi + \eta = 0\} + 7\{x = y = 0\}$$

as a cycle in the cotangent bundle $T^*X = \{(x, y, \xi, \eta)\}$. Thus, by Kashiwara's index theorem we have

$$\sum_{i=0}^{2}(-1)^i \dim_{\mathbf{C}} \mathcal{E}xt^{\,i}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}\{x, y\}) = 3 - (4 + 4 + 1) + 7 = 1.$$

Hence $\mathcal{M}$ is not regular at $(0, 0)$.

$\dim_{\mathbf{C}} \mathcal{E}xt^{\,0}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{C}[[x, y]]) = 1$ implies that the system $\mathcal{M}$ admits one dimensional space of formal power series solutions at the origin. In fact, the (divergent) formal series

$$\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(1)_m(1)_n} x^m y^n, \quad (c)_m := c(c + 1) \cdots (c + m - 1)$$

spans the solution space.

6 Tensor product and localization

In this and subsequent sections, we denote by X the affine space K^n. First let us describe an algorithm to compute the tensor product and the torsion groups of two holonomic $\mathcal{D}_X$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$. We suppose that left D_n-modules N_1 and N_2 are given so that

$$M_i := \Gamma(X, \mathcal{M}_i) = D_n^{r_i}/N_i \quad (i = 1, 2).$$

Let $\pi_1, \pi_2 : X \times X \to X$ be the projections to the first and the second component respectively and put

$$\mathcal{D}'_{X\times X} := \pi_1^{-1}\mathcal{D}_X \otimes_K \pi_2^{-1}\mathcal{D}_X.$$

Then the exterior tensor product is defined by

$$\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2 := \mathcal{D}_{X\times X} \otimes_{\mathcal{D}'_{X\times X}} (\pi_1^{-1}\mathcal{M}_1 \otimes_K \pi_2^{-1}\mathcal{M}_2).$$

First, let us describe this exterior tensor product more concretely. Let $u_1, \dots, u_{r_1}$ be the residue classes of the unit vectors $e_1, \dots, e_{r_1}$ of $D_n^{r_1}$, and $v_1, \dots, v_{r_2}$ the residue classes of the unit vectors $e'_1, \dots, e'_{r_2}$ of $D_n^{r_2}$. Then as a $\mathcal{D}_{X\times X}$-module, $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is generated by $u_i \otimes v_j$ ($1 \le i \le r_1$, $1 \le j \le r_2$). Let us denote by (x, y) the coordinate system of $X \times X$. For $P = (P_1, \dots, P_{r_1}) \in \mathcal{D}_X^{r_1}$ and $Q = (Q_1, \dots, Q_{r_2}) \in \mathcal{D}_X^{r_2}$ we write

$$P \otimes Q := (P_i(x, \partial_x) Q_j(y, \partial_y))_{ij} \in \mathcal{D}_{X\times X}^{r_1 r_2}.$$

Let $\mathcal{I}'$ be the left $\mathcal{D}'_{X\times X}$-submodule of $(\mathcal{D}'_{X\times X})^{r_1 r_2}$ generated by the set

$$\{P \otimes e'_j \mid P \in N_1,\ 1 \le j \le r_2\} \cup \{e_i \otimes Q \mid Q \in N_2,\ 1 \le i \le r_1\}$$

and put $\mathcal{I} := \mathcal{D}_{X\times X} \otimes_{\mathcal{D}'_{X\times X}} \mathcal{I}'$.

Lemma 6.1 *Under the above notation, there is an isomorphism*

$$\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2 \ \simeq\ \mathcal{D}_{X\times X}^{r_1 r_2} / \mathcal{I}.$$

Proof: Put

$$\mathcal{K}' := \{(P_{ij})_{ij} \in (\mathcal{D}'_{X\times X})^{r_1 r_2} \mid \sum_{i,j} P_{ij}(u_i \otimes v_j) = 0 \text{ in } \pi_1^{-1}\mathcal{M}_1 \otimes_K \pi_2^{-1}\mathcal{M}_2\}.$$

Then it is easy to see that $\mathcal{I}' \subset \mathcal{K}'$. Hence we have a commutative diagram

$$\begin{array}{ccc} \pi_1^{-1}\mathcal{M}_1 \times \pi_2^{-1}\mathcal{M}_2 & \xrightarrow{\Phi} & (\mathcal{D}'_{X\times X})^{r_1 r_2}/\mathcal{I}' \\ & \searrow & \downarrow \\ & & \pi_1^{-1}\mathcal{M}_1 \otimes_K \pi_2^{-1}\mathcal{M}_2 \quad \simeq (\mathcal{D}'_{X\times X})^{r_1 r_2}/\mathcal{K}', \end{array}$$

where Φ is a K-bilinear map defined by

$$\Phi(\sum_i P_i u_i, \sum_j Q_j v_j) = P \otimes Q \mod \mathcal{I}',$$

which is well-defined by the definition of $\mathcal{I}'$. In view of the universal property of the tensor product, we know that the vertical map above is an isomorphism. This completes the proof. $\square$

Hence $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is computable with N_1 and N_2 being given. Put $\Delta := \{(x, y) \in X \times X \mid x = y\}$ and identify Δ and X by the map π_1. Then by [15, Proposition 4.7], which obviously applies to algebraic $\mathcal{D}$-modules as well, we have

$$\begin{aligned} \mathcal{M}_1 \otimes^L_{\mathcal{O}_X} \mathcal{M}_2 &\simeq \mathcal{D}_{\Delta \to X\times X} \otimes^L_{\mathcal{D}_{X\times X}} (\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2) \\ &= (\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2)^{\bullet}_{\Delta}. \end{aligned}$$

Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are holonomic. Then it is easy to see that $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is a holonomic $\mathcal{D}_{X\times X}$-module since its characteristic variety is contained in the Cartesian product of those of $\mathcal{M}_1$ and $\mathcal{M}_2$. Hence $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is specializable along Δ and the following algorithm is correct:

Algorithm 6.2 (The tensor product and torsion groups of two $\mathcal{D}_X$-modules)
Input: Holonomic systems $\mathcal{M}_i = \mathcal{D}_X \otimes_{D_n} M_i$, where $M_i = D_n^{r_i}/N_i$ with a D_n-submodule N_i of $D_n^{r_i}$ for $i = 1, 2$.
Output: $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2) = \mathcal{L}_k$ for $k = 0, \dots, n$.

(1) From sets G_i of generators of N_i, compute

$$G_3 := \{P \otimes e'_j \mid P \in G_1,\ 1 \le j \le r_2\} \cup \{e_i \otimes Q \mid Q \in G_2,\ 1 \le i \le r_1\}.$$

(2) Let G_4 be the result of the substitution $y_i = x_i + t_i$ $(i = 1, \dots, n)$ for each element of G_3; let I_4 be the left submodule of $D_{2n}^{r_1 r_2}$ with $D_{2n} = K[t, x]\langle \partial_t, \partial_x \rangle$ generated by G_4.

(3) Apply Algorithm 5.4 with $D_{2n}^{r_1 r_2}/I_4$ as input and $d = n$ to obtain

$$\mathcal{L}_k = \mathcal{H}^{-k}((\mathcal{D}_{X \times X} \otimes_{D_{2n}} (D_{2n}^{r_1 r_2}/I_4))^{\bullet}_{\{0\} \times X})$$

for $k = 0, \dots, n$.

Example 6.3 Put $X := K$ and

$$\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X x \partial_x, \quad \mathcal{N} := \mathcal{D}_X/\mathcal{D}_X x.$$

First, the exterior tensor product is given by

$$\mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{D}_{X \times X}/(\mathcal{D}_{X \times X} x \partial_x + \mathcal{D}_{X \times X} y)$$

with $(x, y) \in X \times X$. Its global b-function along the diagonal is s, and by restricting $\mathcal{M} \hat{\otimes} \mathcal{N}$ to the diagonal we get

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = \mathcal{D}_X/\mathcal{D}_X x \quad (i = 0, 1).$$

In the same way, we get

$$\begin{aligned} \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{D}_X/\mathcal{D}_X x, \mathcal{D}_X/\mathcal{D}_X x) &= \begin{cases} 0 & (i = 0), \\ \mathcal{D}_X/\mathcal{D}_X x & (i = 1). \end{cases} \\ \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{D}_X/\mathcal{D}_X(x\partial_x + 1), \mathcal{D}_X/\mathcal{D}_X x) &= 0 \quad (i = 0, 1). \end{aligned}$$

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let $f \in K[x]$ be an arbitrary non-constant polynomial. Then we immediately obtain an algorithm to compute the localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$ by combining this algorithm with that of computing $\mathcal{O}[1/f]$ given in [31] since $\mathcal{O}_X[f^{-1}]$ is holonomic ([15, Theorem 1.3]). Since $\mathcal{O}_X[f^{-1}]$ is flat over $\mathcal{O}_X$, the higher torsion groups vanish.

Algorithm 6.4 (The localization $\mathcal{M}[f^{-1}]$)
Input: A holonomic system $\mathcal{M} = \mathcal{D}_X \otimes_{D_n} (D_n^r/N)$ and a non-constant polynomial $f \in K[x]$.
Output: $\mathcal{M}[f^{-1}]$.

(1) Compute the global Bernstein-Sato polynomial $b_f(s)$ of f as follows (cf. [30]), where s is a single indeterminate:

   (a) Letting t be a single variable and let I be the left ideal of D_{n+1} generated by $t - f(x)$ and $\partial_{x_i} + (\partial f/\partial x_i)\partial_t$ for $i = 1, \dots, n$.

   (b) Let $b(\theta)$ be that in the step 3 of Algorithm 4.6 with D_{n+1}/I as input and $d = 1$. Put $b_f(s) := b(-s-1)$.

(2) Compute a set of generators of $J_f := \{P(s, x, \partial_x) \mid D_n[s] \mid Pf^s = 0\}$ by [31, Theorem 19].

(3) Let ν be the minimum integer root of $b_f(s) = 0$ and put $J_f(\nu) := \{P(\nu, x, \partial_x) \mid P(s, x, \partial_x) \in J_f\}$. Then $\mathcal{O}_X[f^{-1}] \simeq \mathcal{D}_X \otimes_{D_n} (D_n/J_f(\nu))$.

(4) Compute $\mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$, which is obtained as the output of Algorithm 6.2 with $D_n/J_f(\nu)$ and D_n^r/N as input and $i = 0$.

7 Algebraic local cohomology groups

Let $f_1, \dots, f_d \in K[x]$ be arbitrary polynomials and put $Y := \{x \in X \mid f_1(x) = \dots = f_d(x) = 0\}$ with $X := K^n$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Our purpose is to compute the algebraic local cohomology groups $\mathcal{H}^i_{[Y]}(\mathcal{M})$ with support Y defined by Grothendieck as $\mathcal{D}_X$-modules. Recall that $\mathcal{H}^i_{[Y]}(\mathcal{M})$ is defined as the k-th derived functor of the functor

$$\Gamma_{[Y]}(\mathcal{M}) := \varinjlim_{m \to \infty} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_Y^m; \mathcal{M}),$$

where $\mathcal{J}_Y$ is the defining ideal of Y and the inductive limit is taken as m tends to infinity. Note that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then so is $\mathcal{H}^i_{[Y]}(\mathcal{M})$ (cf. [15, Theorem 1.4]).

Put $\widetilde{X} := K^d \times X$ and identify X with the linear subvariety $\{0\} \times X$ of $\widetilde{X}$. We set

$$Z := \{(t, x) \in \widetilde{X} \mid t_i = f_i(x) \ (i = 1, \dots, d)\}.$$

Then $\mathcal{B}_{[Z]} := \mathcal{H}^d_{[Z]}(\mathcal{O}_{\widetilde{X}})$ is isomorphic to $\mathcal{D}_{\widetilde{X}}/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by

$$t_j - f_j(x) \quad (j = 1, \dots, d), \qquad \partial_{x_i} + \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_i} \partial_{t_j} \quad (i = 1, \dots, n). \tag{7.1}$$

Let $\pi : \widetilde{X} \longrightarrow X$ be the projection and put

$$\Delta' := \{(t, x, y) \in \widetilde{X} \times X \mid x = y\}.$$

Then we can identify $\widetilde{X}$ with Δ' by π. In the same way as [32, Lemma 6.3] we get

Lemma 7.1

$$\mathcal{T}or_i^{\pi^{-1}\mathcal{O}_X}(\mathcal{B}_{[Z]}, \pi^{-1}\mathcal{M}) \simeq \begin{cases} \mathcal{H}^0((\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M})^\bullet_{\Delta'}) & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$

with

$$\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M} := \mathcal{D}_{\widetilde{X}\times X} \otimes_{p_1^{-1}\mathcal{D}_{\widetilde{X}}\otimes p_2^{-1}\mathcal{D}_X} (p_1^{-1}\mathcal{B}_{[Z]} \otimes_K p_2^{-1}\mathcal{M}),$$

where p_1 and p_2 are the projections of $\widetilde{X} \times X$ to $\widetilde{X}$ and to X respectively.

In fact, this lemma follows from the fact that Δ' is non-characteristic with respect to $\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M}$. The proof of [32, Theorem 6.4] yields the following:

Proposition 7.2 *For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have an isomorphism*

$$\mathcal{H}^i_{[Y]}(\mathcal{M}) \simeq \mathcal{H}^{i-d}((\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M})^\bullet_{\{0\}\times X})$$

as left $\mathcal{D}_X$-module for any $i \geq 0$.

Algorithm 7.3 (Algebraic local cohomology groups $\mathcal{H}^i_{[Y]}(\mathcal{M})$)
Input: Polynomials $f_1, \ldots, f_d \in K[x]$ and a holonomic $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X \otimes_{D_n} (D_n^r/N)$ with an D_n-submodule N of D_n^r generated by G.
Output: $\mathcal{H}^i_{[Y]}(\mathcal{M}) = \mathcal{L}_{i-d}$ for $i = 0, \ldots, d$ with $Y := \{x \in K^n \mid f_1(x) = \ldots = f_d(x) = 0\}$.

(1) Let I be the left D_{d+2n}-submodule of D^r_{d+2n} generated by

$$\begin{aligned} G_1 &:= \{(t_j - f_j(x))e_k \mid 1 \leq j \leq d,\ 1 \leq k \leq r\} \\ &\cup\ (\partial_{x_i} + \sum_{j=1}^d \frac{\partial f_j}{\partial x_i}\partial_{t_j})e_k \mid 1 \leq i \leq n,\ 1 \leq k \leq r\} \\ &\cup\ \{P(y, \partial_y) \mid P \in G\}, \end{aligned}$$

where $e_1, \ldots, e_r$ are the unit vectors of D_n^r.

(2) Apply substitution $y_i = x_i + z_i$ for $i = 1, \ldots, n$ to G_1 and let the result be G_2. Let J_2 be the submodule of D^r_{d+2n} generated by G_2.

(3) Compute the 0-th cohomology of the restriction of D^r_{d+2n}/J_2 to $\{(t, x, z) \mid z = 0\}$ in the form $\mathcal{D}_{\widetilde{X}} \otimes_{D_{d+n}} (D^l_{d+n}/J_3)$ by Algorithm 5.4. Here we can assume $k_0 = k_1 = 0$ skipping the steps 1–3 of Algorithm 5.4.

(4) Compute $\mathcal{L}_i := \mathcal{H}^i((\mathcal{D}_{\widetilde{X}} \otimes_{D_{d+n}} (D^l_{d+n}/J_3))^\bullet_{\{0\}\times X})$ for $-d \leq i \leq 0$.

Finally, assume that Y is non-singular of codimension d and let $\iota : Y \to X$ be the embedding. Then for a coherent $\mathcal{D}_Y$-module $\mathcal{N}$,

$$\iota_+\mathcal{N} := \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}$$

is a coherent $\mathcal{D}_X$-module with support in Y; here we put $\mathcal{D}_{X\leftarrow Y} := \mathcal{D}_X/(\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_d)$, which has a structure of $(\mathcal{D}_X, \mathcal{D}_Y)$-bimodule. Moreover, the functor ι_+ gives an equivalence between the category of coherent $\mathcal{D}_Y$-modules and that of coherent $\mathcal{D}_X$-modules supported by Y ([15, Proposition 4.2]). In terms of this equivalence, we can compute the cohomology groups of the restriction of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ to Y by using Algorithm 7.3 and the following isomorphism:

Proposition 7.4 ([15, Proposition 4.3])

$$\iota_+ \mathcal{H}^i(\mathcal{M}_Y^\bullet) = \mathcal{H}^{i+d}_{[Y]}(\mathcal{M}).$$

Example 7.5 Put $K := \mathbf{C}^3 \ni (x, y, z)$ and

$$\mathcal{M} := \mathcal{D}_X/(\mathcal{D}_X\partial_x + \mathcal{D}_X\partial_y + \mathcal{D}_X(z^3\partial_z + z)).$$

Then the local cohomology groups of $\mathcal{M}$ with support $Y := \{(x, y, z) \in X \mid xz = yz = 0\}$ are given by

$$\mathcal{H}^i_{[Y]}(\mathcal{M}) = \begin{cases} \mathcal{D}_X/(\mathcal{D}_X x + \mathcal{D}_X y + \mathcal{D}_X(z^2\partial_z + 2z + 1)) & (i = 2), \\ 0 & (i = 1), \\ \mathcal{D}_X/(\mathcal{D}_X\partial_x^2 + \mathcal{D}_X(x\partial_x - 1) + \mathcal{D}_X\partial_y + \mathcal{D}_X z) & (i = 0). \end{cases}$$

Moreover, $\mathcal{H}^0_{[Y]}(\mathcal{M}) = \mathcal{D}_X u$ is also isomorphic to

$$\mathcal{D}_X/(\mathcal{D}_X\partial_x + \mathcal{D}_X\partial_y + \mathcal{D}_X z) = \mathcal{D}_X v$$

by the correspondence $v \mapsto \partial_x u$ and $u \mapsto xv$, where u and v denote the residue classes of $1 \in \mathcal{D}_X$ in respective modules.

The localization of $\mathcal{M}$ by z is given by

$$\mathcal{M}[1/z] = \mathcal{D}_X/(\mathcal{D}_X\partial_x + \mathcal{D}_X\partial_y + \mathcal{D}_X(z^2\partial_z + z + 1)).$$

Note that the methods of [32] or [42] cannot be applied since $z : \mathcal{M} \to \mathcal{M}$ is not injective.

8 Functors in the analytic category

The functors studied in the preceding sections have analytic counterparts (cf. [15],[25]). Throughout this section, we assume $K = \mathbf{C}$ and denote by $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{D}_X^{\mathrm{an}}$ the sheaves on X of holomorphic functions and of rings of differential operators with holomorphic coefficients respectively. For a left $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X^r/\mathcal{N}$, we put $\mathcal{M}^{\mathrm{an}} := \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}$. For $k \in \mathbf{Z}$, put

$$F_Y^k(\mathcal{D}_X^{\mathrm{an}}) := \{P \in \mathcal{D}_X^{\mathrm{an}}|_Y \mid P(\mathcal{J}_Y^{\mathrm{an}})^j \in (\mathcal{J}_Y^{\mathrm{an}})^{j-k} \quad \text{for any } j \geq k\},$$

where $\mathcal{J}_Y^{\mathrm{an}}$ is the defining ideal of Y in $\mathcal{O}_X^{\mathrm{an}}$. Then for $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbf{Z}^r$, the $F_Y[\mathbf{m}]$-filtrations are defined by

$$\begin{aligned} F_Y^k[\mathbf{m}]((\mathcal{D}_X^{\mathrm{an}})^r) &:= \bigoplus_{i=1}^r F_Y^{k-m_i}(\mathcal{D}_X^{\mathrm{an}}), \\ F_Y^k[\mathbf{m}](\mathcal{M}^{\mathrm{an}}) &:= F_Y^{k-m_1}(\mathcal{D}_X^{\mathrm{an}})u_1 + \cdots + F_Y^{k-m_r}(\mathcal{D}_X^{\mathrm{an}})u_r, \end{aligned}$$

where $u_1, \dots, u_r$ are the residue classes of the unit vectors of $(\mathcal{D}_X^{\mathrm{an}})^r$. The graded modules are defined in the same way as in Section 2. Put $\mathcal{D}_{Y\to X}^{\mathrm{an}} := \mathcal{O}_Y^{\mathrm{an}} \otimes_{\mathcal{O}_X^{\mathrm{an}}} \mathcal{D}_X^{\mathrm{an}}$, which is a $(\mathcal{D}_Y^{\mathrm{an}}, \mathcal{D}_X^{\mathrm{an}})$-bimodule. Then the restriction of $\mathcal{M}^{\mathrm{an}}$ to Y is defined by

$$(\mathcal{M}^{\mathrm{an}})_Y^\bullet := \mathcal{D}_{Y\to X}^{\mathrm{an}} \otimes^L_{\mathcal{D}_X^{\mathrm{an}}} \mathcal{M}^{\mathrm{an}}.$$

The b-function of $\mathcal{M}^{\mathrm{an}}$ along Y at $p \in Y$ is defined to be the generator of the ideal

$$\{b(\theta) \in \mathbf{C}[\theta] \mid b(\vartheta)\mathrm{gr}_Y^0[\mathbf{m}](\mathcal{M}^{\mathrm{an}}) = 0\},$$

where ϑ is defined as in Section 4.

Lemma 8.1 *Let $\mathcal{N}$ be a coherent left $\mathcal{D}_X$-submodule of $\mathcal{D}_X^r$. Define the $F_Y[\mathbf{m}]$-filtrations on $\mathcal{N}$ and on $\mathcal{N}^{\mathrm{an}}$ as in Section 2 with a shift vector $\mathbf{m}$. Then we have*

$$\begin{aligned}\mathrm{gr}_Y^0[\mathbf{m}](\mathcal{N}^{\mathrm{an}}) &= \mathrm{gr}_Y^0(\mathcal{D}_X^{\mathrm{an}}) \otimes_{\mathrm{gr}_Y^0(\mathcal{D}_X)} \mathrm{gr}_Y^0[\mathbf{m}](\mathcal{N}) \\ &= \mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} \mathrm{gr}_Y^0[\mathbf{m}](\mathcal{N}).\end{aligned}$$

Proof: We can prove the first equality by the same method (considering syzygies in the graded module) as [30, Theorem 3.16] (cf. also [1, Lemma 1.1.2]), where the case of $r = d = 1$ is treated; the argument applies to this case with trivial modifications. The second equality follows from

$$\mathrm{gr}_Y^0(\mathcal{D}_X) = \mathcal{D}_Y[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}], \qquad \mathrm{gr}_Y^0(\mathcal{D}_X^{\mathrm{an}}) = \mathcal{D}_Y^{\mathrm{an}}[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}].$$

□

Proposition 8.2 *The b-function $b^{\mathrm{an}}(\theta, p)$ of $\mathcal{M}^{\mathrm{an}}$ and the b-function $b(\theta, p)$ of $\mathcal{M}$ with the same shift vector $\mathbf{m}$ coincide for any $p \in Y$.*

Proof: This follows from Lemma 8.1 and the faithful flatness of $\mathcal{O}_Y^{\mathrm{an}}$ over $\mathcal{O}_Y$ (cf. [30, Lemma 4.4]). □

Proposition 8.3 *We have for any $i \in \mathbf{Z}$,*

$$\mathcal{H}^i((\mathcal{M}^{\mathrm{an}})_Y^\bullet) = \mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} \mathcal{H}^i(\mathcal{M}_Y^\bullet).$$

Proof: Since we can regard

$$\mathcal{D}_{Y\to X}^{\mathrm{an}} = \{\sum_{\nu,\beta} a_{\nu\beta}(x)\partial_t^\nu\partial_x^\beta \in \mathcal{D}_X^{\mathrm{an}} \mid a_{\nu\beta}(x) \in \mathcal{O}_Y^{\mathrm{an}}\},$$

we have an isomorphism $\mathcal{D}_{Y\to X}^{\mathrm{an}} \simeq \mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ as $(\mathcal{D}_Y^{\mathrm{an}}, \mathcal{D}_X)$-bimodules. Combining this with the flatness of $\mathcal{D}_X^{\mathrm{an}}$ over $\mathcal{D}_X$, and that of $\mathcal{D}_{Y\to X}$ over $\mathcal{D}_Y$, we get

$$\begin{aligned}(\mathcal{M}^{\mathrm{an}})_Y^\bullet &= \mathcal{D}_{Y\to X}^{\mathrm{an}} \otimes^L_{\mathcal{D}_X^{\mathrm{an}}} (\mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}) \\ &= \mathcal{D}_{Y\to X}^{\mathrm{an}} \otimes^L_{\mathcal{D}_X} \mathcal{M} \\ &= (\mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}) \otimes^L_{\mathcal{D}_X} \mathcal{M} \\ &= \mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} (\mathcal{D}_{Y\to X} \otimes^L_{\mathcal{D}_X} \mathcal{M}) \\ &= \mathcal{D}_Y^{\mathrm{an}} \otimes_{\mathcal{D}_Y} \mathcal{M}_Y^\bullet.\end{aligned}$$

This implies the assertion since $\mathcal{D}_Y^{\mathrm{an}}$ is flat over $\mathcal{D}_Y$. □

Proposition 8.4 *For $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, we have*

$$\mathcal{T}or_i^{\mathcal{O}_X^{\rm an}}(\mathcal{M}^{\rm an}, \mathcal{N}^{\rm an}) = \mathcal{D}_X^{\rm an} \otimes_{\mathcal{D}_X} \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

Proof: The assertion follows from Proposition 8.3 and [15, Proposition 4.7]. □

For an algebraic set Y of X, the algebraic local cohomology groups of $\mathcal{M}^{\rm an}$ are defined to be the derived functors of the functor

$$\Gamma_{[Y]}(\mathcal{M}^{\rm an}) := \lim_{\substack{\longrightarrow \\ m \to \infty}} \mathcal{H}om_{\mathcal{O}_X^{\rm an}}(\mathcal{O}_X^{\rm an}/(\mathcal{J}_Y^{\rm an})^m; \mathcal{M}^{\rm an}).$$

Proposition 8.5 *For a left $\mathcal{D}_X$-module $\mathcal{M}$, a polynomial $f \in \mathbf{C}[x]$, and an algebraic set Y, we have*

$$\begin{aligned} \mathcal{M}^{\rm an}[f^{-1}] &= \mathcal{D}_X^{\rm an} \otimes_{\mathcal{D}_X} \mathcal{M}[f^{-1}], \\ \mathcal{H}^i_{[Y]}(\mathcal{M}^{\rm an}) &= \mathcal{D}_X^{\rm an} \otimes_{\mathcal{D}_X} \mathcal{H}^i_{[Y]}(\mathcal{M}). \end{aligned}$$

Proof: The first equality is an immediate consequence of Proposition 8.4. The second equality follows from Proposition 8.3, and Proposition 7.2 together with its analytic counterpart. □

9 Homogenized Weyl algebra and Schreyer's method for adapted free resolution

In this and the following sections, we work in a framework more general than is needed in the preceding sections. Let $D_n = K[x]\langle\partial\rangle$ be the Weyl algebra over a field K of characteristic zero with $x = (x_1, \dots, x_n)$, $\partial = (\partial_1, \dots, \partial_n)$, $\partial_i = \partial/\partial x_i$. We introduce a vector $w = (w_1, \dots, w_n; w_{n+1}, \dots, w_{2n}) \in \mathbf{Z}^{2n} \setminus \{0\}$ that satisfies $w_i + w_{n+i} \geq 0$ for $i = 1, \dots, n$. We call such w *an admissible weight vector* for D_n. For each integer $\nu \in \mathbf{Z}$, we put

$$F_w^\nu(D_n) := \{P = \sum_{\alpha,\beta \in \mathbf{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta \in D_n \mid a_{\alpha\beta} = 0 \text{ if } \sum_{i=1}^n w_i\alpha_i + \sum_{i=1}^n w_{n+i}\beta_i > \nu\},$$

where $a_{\alpha\beta} \in K$ and the sum with respect to $\alpha, \beta \in \mathbf{N}^n$ is finite. For a nonzero $P \in D_n$, let $\mathrm{ord}_w(P)$ be the minimum integer k such that $P \in F_w^k(D_n)$. It is easy to see that $\mathrm{ord}_w(PQ) = \mathrm{ord}_w(P) + \mathrm{ord}_w(Q)$ holds for nonzero $P, Q \in D_n$. More generally, for a shift vector $\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Z}^r$, we define a filtration $F_w[\mathbf{m}]$ of D_n^r by

$$F_w^k[\mathbf{m}](D_n^r) := \bigoplus_{i=1}^r F_w^{k-m_i}(D_n)e_i,$$

where $e_1, \dots, e_r$ are the canonical generators of D_n^r. For a nonzero $P \in D_n^r$, we put $\mathrm{ord}_w[\mathbf{m}](P) := \min\{k \in \mathbf{Z} \mid P \in F_w^k[\mathbf{m}](D_n^r)\}$.

Now we introduce the homogenized Weyl algebra, which was introduced in the second version (1994) of `kan/sm1` [41] and independently in [8]:

Definition 9.1 Let $D_n^{(h)}$ be the algebra over K generated by h, $x = (x_1, \dots, x_n)$, and $\partial = (\partial_1, \dots, \partial_n)$ which satisfy the relations

$$\begin{array}{lll} x_i x_j - x_j x_i = 0, & \partial_i \partial_j - \partial_j \partial_i = 0, & x_i \partial_j - \partial_j x_i = -\delta_{ij} h^2, \\ h x_i - x_i h = 0, & h \partial_i - \partial_i h = 0 & (i, j = 1, \dots, n), \end{array}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. We call $D_n^{(h)}$ the *homogenized Weyl algebra.* The substitution $h = 1$ defines a K-algebra homomorphism

$$\rho : D_n^{(h)} \ni P \longmapsto P|_{h=1} \in D_n.$$

An element P of $D_n^{(h)}$ is uniquely expressed as a finite sum

$$P = \sum_{\lambda \in \mathbf{N}, \alpha, \beta \in \mathbf{N}^n} a_{\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta$$

with $a_{\lambda\alpha\beta} \in K$. The total degree of $P \in D_n^{(h)}$ is defined by

$$\deg(P) := \max\{\lambda + |\alpha| + |\beta| \mid a_{\lambda\alpha\beta} \neq 0\}$$

if $P \neq 0$, and $\deg(P) = -\infty$ if $P = 0$.

Now let us take another vector $\mathbf{n} = (n_1, \dots, n_r) \in \mathbf{N}^r$, which describes the shift with respect to the total degree. For $P = (P_1, \dots, P_r) \in (D_n^{(h)})^r$, we put

$$\deg[\mathbf{n}](P) := \max_{1 \le i \le r} (\deg(P_i) + n_i).$$

Definition 9.2 (1) An element $P = \sum_{i=1}^r \sum_{\lambda \in \mathbf{N}, (\alpha,\beta) \in L} a_{\lambda\alpha\beta i} h^\lambda x^\alpha \partial^\beta e_i$ of $(D_n^{(h)})^r$ is said to be $h[\mathbf{n}]$-homogeneous if there exists $k \in \mathbf{Z}$ so that $a_{\lambda\alpha\beta i} = 0$ unless $\lambda + |\alpha| + |\beta| + n_i = k$.

(2) For $P = \sum_{i=1}^r \sum_{(\alpha,\beta) \in L} a_{\alpha\beta i} x^\alpha \partial^\beta e_i \in D_n^r$, we define the $h[\mathbf{n}]$-homogenization $h[\mathbf{n}](P) \in (D_n^{(h)})^r$ by

$$h[\mathbf{n}](P) := \sum_{i=1}^r \sum_{(\alpha,\beta) \in L} a_{\alpha\beta i} h^{k - |\alpha| - |\beta| - n_i} x^\alpha \partial^\beta e_i$$

with $k := \max\{|\alpha| + |\beta| + n_i \mid a_{\alpha\beta i} \neq 0\}$. This is $h[\mathbf{n}]$-homogeneous.

If $\mathbf{n}$ is the zero vector, we denote $h[\mathbf{n}](P)$ simply by $h(P)$.

Lemma 9.3 *For $P \in D_n^r$ and $Q \in D_n$, we have $\rho(h[\mathbf{n}](P)) = P$ and $h[\mathbf{n}](QP) = h(Q) h[\mathbf{n}](P)$.*

Definition 9.4 Let $\prec$ be a monomial order (i.e. an order satisfying (3.1)) on $L \times \{1, \dots, r\}$ with $L := \mathbf{N}^{2n}$. We denote by $\mathrm{lexp}_\prec(P)$ the leading exponent of $P \in D_n^r$ with respect to $\prec$. Then $\prec$ is said to be *adapted to* the filtration $F_w[\mathbf{m}]$ if $\langle w, \alpha \rangle + m_i < \langle w, \beta \rangle + m_j$ implies $(\alpha, i) \prec (\beta, j)$ for $\alpha, \beta \in L$ and $i, j \in \{1, \dots, r\}$, and if $\mathrm{lexp}_\prec(e_i) \prec \mathrm{lexp}_\prec(x_j \partial_j e_i)$ for any $1 \le i \le r$ and $1 \le j \le n$; here we write $\langle w, \alpha \rangle = \sum_{i=1}^{2n} w_i \alpha_i$ for $\alpha = (\alpha_1, \dots, \alpha_{2n})$.

We fix a monomial order $\prec$ on $L \times \{1, \ldots, r\}$ that is adapted to the $F_w[\mathbf{m}]$-filtration. Then we define an order $\prec_{h[\mathbf{n}]}$ on $\mathbf{N} \times L \times \{1, \ldots, r\}$ by

$$\begin{aligned}&(\lambda, \alpha, i) \prec_{h[\mathbf{n}]} (\mu, \beta, j) \text{ if and only if}\\&\lambda + |\alpha| + n_i < \mu + |\beta| + n_j \text{ or else}\\&\lambda + |\alpha| + n_i = \mu + |\beta| + n_j,\ (\alpha, i) \prec (\beta, j)\end{aligned}$$

for $\lambda, \lambda' \in \mathbf{N}, \alpha, \beta \in L$ and $i, j \in \{1, ..., r\}$. Then it is easy to see that $\prec_{h[\mathbf{n}]}$ is a well-order. For a nonzero element

$$P = \sum_{i=1}^{r} \sum_{\lambda,\alpha,\beta} a_{\lambda\alpha\beta i} h^{\lambda} x^{\alpha} \partial^{\beta} e_i \tag{9.1}$$

of $(D_n^{(h)})^r$, its leading exponent $(\lambda_0, \alpha_0, \beta_0, i_0) = \mathrm{lexp}_{h[\mathbf{n}]}(P) \in \mathbf{N} \times L \times \{1, \ldots, r\}$ is defined as the maximum element of $\{(\lambda, \alpha, \beta, i) \mid a_{\lambda\alpha\beta i} \neq 0\}$ with respect to $\prec_{h[\mathbf{n}]}$. Then the leading position $\mathrm{lp}_{h[\mathbf{n}]}(P)$ and the leading coefficient $\mathrm{lcoef}_{h[\mathbf{n}]}(P)$ are defined to be i_0 and $a_{\lambda_0,\alpha_0,\beta_0,i_0}$ respectively. We denote them simply by $\mathrm{lexp}(P)$, $\mathrm{lp}(P)$, and $\mathrm{lcoef}(P)$ if there is no fear of confusion. The following lemmas follow easily from Definition 9.4 and the definitions of $D_n^{(h)}$ and $\prec_{h[\mathbf{n}]}$:

Lemma 9.5 *For $P \in (D_n^{(h)})^r$ and $Q \in D_n^{(h)}$, we have $\mathrm{lexp}(QP) = \mathrm{lexp}(Qe_k) + \mathrm{lexp}(P)$ with $k = \mathrm{lp}(P)$.*

Lemma 9.6 *If $P \in (D_n^{(h)})^r$ is $h[\mathbf{n}]$-homogeneous and $Q \in D_n^{(h)}$ is $h[0]$-homogeneous, then QP is $h[\mathbf{n}]$-homogeneous.*

Lemma 9.7 *Let $\varpi : \mathbf{N} \times L \times \{1, \ldots, r\} \to L \times \{1, \ldots, r\}$ be the projection. Then $\mathrm{lexp}_{\prec}(\rho(P)) = \varpi(\mathrm{lexp}_{h[\mathbf{n}]}(P))$ holds if P is $h[\mathbf{n}]$-homogeneous.*

In view of the above lemmas, we can define the notion of Gröbner basis in the homogenized Weyl algebra and can employ the Buchberger algorithm, which preserves the $h[\mathbf{n}]$-homogeneity:

Definition 9.8 Let N be a left $D_n^{(h)}$-submodule of $(D_n^{(h)})^r$. Then a finite subset G of N is called a Gröbner basis of N with respect to $\prec_{h[\mathbf{n}]}$ if

$$E(N) := \{\mathrm{lexp}(P) \mid P \in N \setminus \{0\}\} = \bigcup_{P \in G} (\mathrm{lexp}(P) + L).$$

Note that G generates N if G is a Gröbner basis of N since $\prec_{h[\mathbf{n}]}$ is a well-order.

Proposition 9.9 *Let N be a left D_n-submodule of D_n^r generated by $P_1, \ldots, P_k$. Let $h[\mathbf{n}](N)$ be the left $D_n^{(h)}$-submodule of $(D_n^{(h)})^r$ generated by $h[\mathbf{n}](P_i)$ for $i = 1, \ldots, k$. Let $G = \{Q_1, \ldots, Q_s\}$ be a Gröbner basis of $h[\mathbf{n}](N)$ with respect to $\prec_{h[\mathbf{n}]}$. Then for any $P \in N$, there exist $U_j \in D_n$ such that $P = U_1\rho(Q_1) + \cdots + U_s\rho(Q_s)$ and $\mathrm{ord}_w[\mathbf{m}](U_j\rho(Q_j)) \leq \mathrm{ord}_w[\mathbf{m}](P)$ for $j = 1, ..., s$.*

Proof: There exists $\nu \in \mathbf{N}$ such that $h^\nu h[\mathbf{n}](P)$ belongs to $h[\mathbf{n}](N)$. By the division algorithm in $(D_n^{(h)})^r$, we can find $h[\mathbf{n}]$-homogeneous $U_1, \dots, U_s \in (D_n^{(h)})^r$ such that $h^\nu h[\mathbf{n}](P) = U_1 Q_1 + \cdots + U_s Q_s$ and $\mathrm{lexp}(U_k Q_k) \preceq_{h[\mathbf{n}]} \mathrm{lexp}(h^\nu h[\mathbf{n}](P))$ for $k = 1, \dots, s$. Applying the ring homomorphism ρ, we get $P = \rho(U_1)\rho(Q_1) + \cdots + \rho(U_s)\rho(Q_s)$ and $\mathrm{lexp}_\prec(\rho(U_k)\rho(Q_k)) = \varpi(\mathrm{lexp}(U_k Q_k)) \preceq \varpi(\mathrm{lexp}(h^\nu h[\mathbf{n}](P))) = \mathrm{lexp}_\prec(P)$. Since U_i and Q_i are $h[\mathbf{n}]$-homogeneous, this implies $\mathrm{ord}_w[\mathbf{m}](\rho(U_i Q_i)) \le \mathrm{ord}_w[\mathbf{m}](P)$. □

We use the same notation as in the preceding proposition. Put $\Lambda := \{(i,j) \mid 1 \le i < j \le s,\ \mathrm{lp}(P_i) = \mathrm{lp}(P_j)\}$. Then for $(i,j) \in \Lambda$, let $S_{ij}, S_{ji} \in D_n^{(h)}$ be monomials such that

$$\mathrm{lexp}(S_{ji}P_i) = \mathrm{lexp}(S_{ij}P_j) = \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j), \quad \mathrm{lcoef}(S_{ji}P_i) = \mathrm{lcoef}(S_{ij}P_j).$$

By the Buchberger algorithm, we can find $h[\mathbf{n}]$-homogeneous $U_{ijk} \in D_n^{(h)}$ such that

$$S_{ji}P_i - S_{ij}P_j = \sum_{k=1}^{s} U_{ijk} P_k \tag{9.2}$$

and either $U_{ijk} \neq 0$ or

$$\mathrm{lexp}(U_{ijk}P_k) \prec_{h[\mathbf{n}]} \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$$

for each $k = 1, ..., s$. The following is an analogue of F.O. Schreyer's theorem for the syzygies in the polynomial ring:

Theorem 9.10 *In the notation above, let $\prec'$ be the order on $\mathbf{N} \times L \times \{1, \dots, s\}$ defined by*

$$\begin{aligned} &(\alpha, \mu) \prec' (\beta, \nu) \quad \textit{if and only if} \quad \mathrm{lexp}(P_\mu) + \alpha \prec_{h[\mathbf{n}]} \mathrm{lexp}(P_\nu) + \beta \\ &\textit{or else} \quad \mathrm{lexp}(P_\mu) + \alpha = \mathrm{lexp}(P_\nu) + \beta \ \textit{and} \ \mu > \nu \end{aligned}$$

for $\alpha, \beta \in \mathbf{N} \times L$ and $\mu, \nu \in \{1, \dots, s\}$. Put

$$\mathbf{m}' := (\mathrm{ord}_w[\mathbf{m}](P_1), \dots, \mathrm{ord}_w[\mathbf{m}](P_s)), \quad \mathbf{n}' = (\deg[\mathbf{n}](P_1), \dots, \deg[\mathbf{n}](P_s)).$$

Then $\prec'$ is a well-order called the Schreyer order induced by $\prec_{h[\mathbf{n}]}$.

(1) For $(i,j) \in \Lambda$,

$$V_{ij} := (0, \dots, \overset{(i)}{S_{ji}}, \dots, -\overset{(j)}{S_{ij}}, \dots, 0) - (U_{ij1}, \dots, U_{ijs})$$

is $h[\mathbf{n}']$-homogeneous and $\{V_{ij} \mid (i,j) \in \Lambda\}$ is a Gröbner basis with respect to $\prec'$ of the module

$$\mathrm{Syz}(P_1, \dots, P_s) := \{(U_1, ..., U_s) \in (D_n^{(h)})^s \mid U_1P_1 + \cdots + U_sP_s = 0\}.$$

(2) Put

$$\mathrm{Syz}(\rho(P_1), \dots, \rho(P_s)) := \{(U_1, ..., U_s) \in D_n^s \mid U_1\rho(P_1) + \cdots + U_s\rho(P_s) = 0\}.$$

Then for any $P \in \mathrm{Syz}(\rho(P_1), \dots, \rho(P_s)) \cap F_w^\nu[\mathbf{m}'](D_n^s)$ with $\nu \in \mathbf{Z}$, there exist $Q_{ij} \in D_n$ such that $P = \sum_{(i,j)\in\Lambda} Q_{ij}\rho(V_{ij})$ and that $Q_{ij}\rho(V_{ij}) \in F_w^\nu[\mathbf{m}'](D_n^s)$.

Proof: For a nonzero $P \in (D_n^{(h)})^r$ of the form (9.1), we define its initial term by

$$\mathrm{in}(P) := a_{\lambda\alpha\beta i} h^\lambda x^\alpha \xi^\beta e_i \ \in K[h,x,\xi]^r$$

with $(\lambda,\alpha,\beta,i) = \mathrm{lexp}(P)$, where $\xi = (\xi_1,\dots,\xi_n)$ are commutative indeterminates. Let s_{ij} be the monomial in $K[h,x,\xi]$ obtained by substituting ξ for ∂ in S_{ij}. Then we have $s_{ji}\mathrm{in}(P_i) - s_{ij}\mathrm{in}(P_j) = 0$ for $(i,j) \in \Lambda$. Now suppose $U = (U_1,\dots,U_s) \in \mathrm{Syz}(P_1,\dots,P_s)$ and put $(\lambda_0,\alpha_0,\beta_0,i_0) := \max_{1\le j\le s} \mathrm{lexp}(U_jP_j)$. Define $u_j \in K[h,x,\xi]$ by

$$u_j e_{i_0} = \mathrm{in}(U_j e_{i_0}) \quad \text{if } \mathrm{lexp}(U_jP_j) = (\lambda_0,\alpha_0,\beta_0,i_0),$$

and put $u_j = 0$ otherwise. Then $\sum_{j=1}^s u_j \mathrm{in}(P_j) = 0$ holds. By the definition of $\prec'$ we have

$$\mathrm{lexp}_{\prec'}(V_{ij}) = (\mathrm{lexp}(s_{ji}), i), \quad \mathrm{lexp}_{\prec'}(U) = \mathrm{lexp}_{\prec'}((u_1,\dots,u_s)). \tag{9.3}$$

On the other hand, since $(0,\dots,s_{ji},0,\dots,-s_{ij},0,\dots,0)$ for $(i,j)\in\Lambda$ constitute a Gröbner basis with respect to $\prec'$ of the syzygies on $\mathrm{in}(P_1),\dots,\mathrm{in}(P_s)$ by virtue of Schreyer's theorem for the polynomial ring (cf. [10, Theorem 15.10]), we know that

$$\mathrm{lexp}_{\prec'}((u_1,\dots,u_s)) \in \bigcup_{(i,j)\in\Lambda} ((\mathrm{lexp}(s_{ji}), i) + (\mathbf{N}\times L)).$$

This completes the proof of the first assertion in view of (9.3).

The second assertion follows from the first and Proposition 9.9 since the order $\prec'$ is adapted to the $F_w[\mathbf{m}']$-filtration restricted to $h[\mathbf{n}']$-homogeneous operators. ▯

Let N be a left D_n-submodule of D_n^r generated by $P_1,\dots,P_k$. Let $h(N)$ be the left $D_n^{(h)}$-submodule of $(D_n^{(h)})^r$ generated by $h(P_1),\dots,h(P_k)$ (the homogenizations with $\mathbf{n}=0$). Starting with $h(N)$ and $\mathbf{n}=\mathbf{m}=0$, apply the first part of Theorem 9.10 repeatedely. Then we get an exact sequence

$$\cdots \xrightarrow{\psi_3} (D_n^{(h)})^{r_2} \xrightarrow{\psi_2} (D_n^{(h)})^{r_1} \xrightarrow{\psi_1} (D_n^{(h)})^{r_0} \xrightarrow{\varphi} (D_n^{(h)})^r/h(N) \longrightarrow 0 \tag{9.4}$$

with $r_0 := r$. Put $\mathbf{m}_0 = \mathbf{n}_0 = 0$ and

$$\begin{aligned}
\mathbf{m}_i &:= (\mathrm{ord}_w[\mathbf{m}_{i-1}](\rho(\psi_i(1,0,\dots,0))),\dots,\mathrm{ord}_w[\mathbf{m}_{i-1}](\rho(\psi_i(0,\dots,0,1)))) \quad \in \mathbf{Z}^{r_i},\\
\mathbf{n}_i &:= (\deg[\mathbf{n}_{i-1}](\psi_i(1,0,\dots,0)),\dots,\deg[\mathbf{n}_{i-1}](\psi_i(0,\dots,0,1))) \quad \in \mathbf{N}^{r_i}.
\end{aligned}$$

Applying the homomorphism ρ to (9.4), we get an exact sequence

$$\cdots \xrightarrow{\rho(\psi_3)} D_n^{r_2} \xrightarrow{\rho(\psi_2)} D_n^{r_1} \xrightarrow{\rho(\psi_1)} D_n^{r_0} \xrightarrow{\rho(\varphi)} D_n^r/N \longrightarrow 0. \tag{9.5}$$

Moreover, in view of the second part of Theorem 9.10, the sequence

$$\cdots \xrightarrow{\rho(\psi_2)} F_w^k[\mathbf{m}_1](D_n^{r_1}) \xrightarrow{\rho(\psi_1)} F_w^k[\mathbf{m}_0](D_n^{r_0}) \xrightarrow{\rho(\varphi)} F_w^k[0](D_n^r)/(N\cap F_w^k[0](D_n^r)) \longrightarrow 0$$

is exact for any $k\in\mathbf{Z}$. Hence the resolution (9.5) is adapted to the F_w-filtration. Furthermore, we can prove the following in the same way as its counterpart in the polynomial ring ([10, Corollary 15.11])

Theorem 9.11 *By arranging the Gröbner bases appropriately, we can construct a free resolution (9.4) so that* $\psi_{2n+2} = 0$.

It seems an open problem whether there exists an adapted free resolution of length less than $2n+1$.

Remark 9.12 (1) If each element of the weight w is non-negative and the order $\prec$ adapted to $F_w[\mathbf{m}]$ is a well-order, then the above construction can be done directly in D_n^r without the homogenized Weyl algebra.

(2) If the weight w satisfies $w_i + w_{n+i} = 0$ for $i = 1, \ldots, n$, then we can work in $D_n[x_0]^r$ as in Section 3 instead of the homogenized Weyl algebra. Then Theorem 9.10 holds with the $h[\mathbf{n}]$-homogenization replaced by the $F_w[\mathbf{m}]$-homogenization defined by

$$h[\mathbf{m}](P) := \sum_{i=1}^{r} \sum_{\alpha,\beta\in\mathbf{N}^n} a_{\alpha\beta i} {x_0}^{\langle w,(\alpha,\beta)\rangle + m_i - k} x^\alpha \partial^\beta e_i$$

with $k := \min\{\langle w, (\alpha,\beta)\rangle + m_i \mid a_{\alpha\beta i} \neq 0\}$ for $P = \sum_{i=1}^r \sum_{\alpha,\beta\in\mathbf{N}^n} a_{\alpha\beta i} x^\alpha \partial^\beta e_i$.

Theorem 9.11 also holds in this case.

In the computation of the free resolution described in this section, we can employ the method of La Scala and Stillman [21], which computes the 'Schreyer frame' (initial terms of the resolution) first, then computes the resolution by a selection strategy or in parallel. We have implemented this algorithm in `kan/sm1`. The examples presented so far have been computed by using this implementation. For examples of Schreyer resolutions, see Examples 1.10 and 1.11 of [33]. By using a criterion for an adapted resolution (Theorems 2.5, 10.7) which will be established in the next section, there is a possiblity of obtaining a smaller adapted resolution.

10 Criterion of an adapted resolution

We use the same notation as in the preceding section. In particular, let $w \in \mathbf{Z}^{2n}$ be an admissible weight vector for the Weyl algebra D_n. For a left D_n-submodule N of D_n^r and a shift vector $\mathbf{m} \in \mathbf{Z}^r$, we put

$$F_w^k[\mathbf{m}](N) := F_w^k[\mathbf{m}](D_n^r) \cap N, \quad \mathrm{gr}_w^k[\mathbf{m}](N) := F_w^k[\mathbf{m}](N)/F_w^{k-1}[\mathbf{m}](N)$$

and $\mathrm{gr}_w[\mathbf{m}](N) := \oplus_{k\in\mathbf{Z}} \mathrm{gr}_w^k[\mathbf{m}](N)$. We abbreviate $[\mathbf{m}]$ if $r = 1$ and $\mathbf{m} = 0$. Then $\mathrm{gr}_w(D_n)$ becomes a ring (more precisely, a K-algebra) and $\mathrm{gr}_w[\mathbf{m}](N)$ has a natural structure of left $\mathrm{gr}_w(D_n)$-module. For $P \in F_w^k[\mathbf{m}](D_n^r) \setminus F_w^{k-1}[\mathbf{m}](D_n^r)$, we define $\sigma_w[\mathbf{m}](P) \in \mathrm{gr}_w^k[\mathbf{m}](D_n^r)$ to be the residue class of P.

Our purpose is to give a criterion of an adapted resolution in terms of the notion of involutive base.

Definition 10.1 Let N be a D_n-submodule of D_n^r and take a shift vector $\mathbf{m} \in \mathbf{Z}^r$. Then a subset G of N is said to be an $F_w[\mathbf{m}]$-*involutive base* of N if G generates N and $\{\sigma_w[\mathbf{m}](P) \mid P \in G\}$ generates $\mathrm{gr}_w[\mathbf{m}](N)$ in $\mathrm{gr}_w[\mathbf{m}](D_n^r)$.

The formal completion of D_n with respect to the F_w-filtration is defined as the projective limit

$$\widehat{D}_n^{(w)} := \lim_{\substack{\longleftarrow \\ k\to -\infty}} D_n/F_w^k(D_n).$$

Then $\widehat{D}_n^{(w)}$ can be regarded as a ring which contains D_n as a subring. An element P of $\widehat{D}_n^{(w)}$ is uniquely written in an infinite sum

$$P = \sum_{k\le m} P_k$$

with some $m \in \mathbf{Z}$, where $P_k \in D_n$ is homogeneous of order k with respect to w; i.e., P_k is written in the form

$$P_k = \sum_{\langle(\alpha,\beta),w\rangle = k} a_{k\alpha\beta} x^\alpha \partial^\beta.$$

More generally, $\sum_{k\le m} P_k$ defines an element of $\widehat{D}_n^{(w)}$ if $P_k \in F_w^k(D_n)$ for any $k \le m$. Note that $\widehat{D}_n^{(w)} = D_n$ holds if and only if each component of w is non-negative.

Lemma 10.2 *Under the notation and assumptions of Theorem 9.10 put*

$$\overline{S} := \{(U_1, ..., U_s) \in \mathrm{gr}_w[\mathbf{m}'](D_n^s) \mid U_1\sigma_w[\mathbf{m}](\rho(P_1)) + \cdots + U_s\sigma_w[\mathbf{m}](\rho(P_s)) = 0\}.$$

Then for any $\overline{P} \in \overline{S} \cap \mathrm{gr}_w^\nu[\mathbf{m}'](D_n^s)$ *with* $\nu \in \mathbf{Z}$, *there exist* $\overline{Q}_{ij} \in \mathrm{gr}_w^{\nu - m_{ij}}(D_n)$ *such that* $\overline{P} = \sum_{(i,j)\in\Lambda} \overline{Q}_{ij}\sigma_w[\mathbf{m}'](\rho(V_{ij}))$, *where* $m_{ij} := \mathrm{ord}_w[\mathbf{m}'](\rho(V_{ij}))$.

Proof: The $F_w[\mathbf{m}]$-filtration on $(D_n^{(h)})^r$, the associated graded module $\mathrm{gr}_w[\mathbf{m}]((D_n^{(h)})^r)$, $\mathrm{ord}_w[\mathbf{m}](P)$, and $\sigma_w[\mathbf{m}](P) \in \mathrm{gr}_w[\mathbf{m}]((D_n^{(h)})^r)$ are naturally defined, where we assign weight 0 to the indeterminate h. Then from (9.2) we get

$$\sigma_w(S_{ji}) \cdot \sigma_w[\mathbf{m}](P_i) - \sigma_w(S_{ij}) \cdot \sigma_w[\mathbf{m}](P_j) = \sum_{k=1}^s \overline{U}_{ijk} \cdot \sigma_w[\mathbf{m}](P_k)$$

with $\overline{U}_{ijk} := \sigma_w(U_{ijk})$ if $\mathrm{ord}_w[\mathbf{m}](U_{ijk}P_k) = \mathrm{ord}_w[\mathbf{m}](S_{ji}P_i)$, and $\overline{U}_{ijk} := 0$ otherwise. This implies that $\{\sigma_w[\mathbf{m}](P_1), \ldots, \sigma_w[\mathbf{m}](P_s)\}$ is a Gröbner basis in $\mathrm{gr}_w(D_n^{(h)})^r$ with respect to the order $\prec$. (Note that the arguments of Section 9 apply also to $\mathrm{gr}_w(D_n^{(h)})^r$ instead of $(D_n^{(h)})^r$.) Hence we get the assertion in the same way as the proof of Theorem 9.10. $\square$

Proposition 10.3 $\widehat{D}_n^{(w)}$ *is flat over* D_n *as a right* D_n*-module.*

Proof: Let I be a left ideal of D_n. It suffices to show that the natural homomorphism

$$\iota : \widehat{D}_n^{(w)} \otimes_{D_n} I \longrightarrow \widehat{D}_n^{(w)}$$

is injective. For that purpose, let $P_1, \ldots, P_s$ be a Gröbner basis of the left ideal $I^{(h)}$ of $D_n^{(h)}$ generated by $\{h[\mathbf{n}](P) \mid P \in I\}$ with respect to an order adapted to the F_w-filtration with $\mathbf{n} = 0$. Then $\rho(P_1), \ldots, \rho(P_s)$ are generators of I. Assume that $Q_1, \ldots, Q_s \in \widehat{D}_n^{(w)}$ satisfy

$$\iota(\sum_{i=1}^{s} Q_i \otimes \rho(P_i)) = \sum_{i=1}^{s} Q_i \rho(P_i) = (Q_1, \ldots, Q_s) \cdot (\rho(P_1), \ldots, \rho(P_s)) = 0.$$

Put $m_i := \mathrm{ord}_w(\rho(P_i))$ and consider the filtration $F_w^k[\mathbf{m}'](D_n^s)$ with $\mathbf{m}' := (m_1, \ldots, m_s)$. Then we have

$$\sigma_w[\mathbf{m}']((Q_1, \ldots, Q_s)) \cdot (\sigma_w(\rho(P_1)), \ldots, \sigma_w(\rho(P_s))) = 0.$$

Let $V_{ij} \in (D_n)^s$ be as in Theorem 9.10. Put $k = \mathrm{ord}_w[\mathbf{m}'](Q_1, \ldots, Q_s)$ and $m_{ij} = \mathrm{ord}_w[\mathbf{m}'](\rho(V_{ij}))$. Then by Lemma 10.2, there exist $U_{ij}^0 \in D_n$ so that

$$\sigma_w[\mathbf{m}']((Q_1, \ldots, Q_s)) = \sum_{i<j} \sigma_w(U_{ij}^0) \sigma_w[\mathbf{m}'](\rho(V_{ij}))$$

holds in $\mathrm{gr}_w[\mathbf{m}'](D_n^s)$ with

$$\mathrm{ord}_w(U_{ij}^0) + m_{ij} \leq k.$$

Put $Q = (Q_1, \ldots, Q_s)$ and

$$Q^{(1)} := (Q_1, \ldots, Q_s) - \sum_{i<j} U_{ij}^0 \rho(V_{ij}).$$

Then $Q^{(1)} \in (\widehat{D}_n^{(w)})^s$ satisfies

$$Q^{(1)} \cdot (\rho(P_1), \ldots, \rho(P_s)) = 0$$

and $\mathrm{ord}_w[\mathbf{m}'](Q^{(1)}) < k$. Continuing this process infinitely, we can find $U_{ij}^{(\nu)} \in (\widehat{D}_n^{(w)})^s$ such that

$$Q = \sum_{i<j} \sum_{\nu=0}^{\infty} U_{ij}^{(\nu)} \rho(V_{ij})$$

in $(\widehat{D}_n^{(w)})^s$ with $\mathrm{ord}_w(U_{ij}^{(\nu)}) \leq k - m_{ij} - \nu$. In particular, $U_{ij} := \sum_{\nu=0}^{\infty} U_{ij}^{(\nu)}$ defines an element of $\widehat{D}_n^{(w)}$ and satisfies

$$\begin{aligned} \sum_{k=1}^{s} Q_k \otimes \rho(P_k) &= \sum_{k=1}^{s} \sum_{i<j} U_{ij} \rho(V_{ij})_k \otimes \rho(P_k) \\ &= \sum_{i<j} U_{ij} \otimes \sum_{k=1}^{s} \rho(V_{ij})_k \rho(P_k) \\ &= 0, \end{aligned}$$

where $\rho(V_{ij})_k$ denotes the k-th component of $\rho(V_{ij})$. Hence ι is injective. This completes the proof. ∎

Lemma 10.4 *Let $\psi : D_n^{r_i} \longrightarrow D_n^{r_{i-1}}$ be a homomorphism of left D_n-modules for $i = 1, 2$ and assume that the sequence*

$$D_n^{r_2} \xrightarrow{\psi_2} D_n^{r_1} \xrightarrow{\psi_1} D_n^{r_0} \tag{10.6}$$

is exact. Assume moreover that

$$\psi_i(F_w^k[\mathbf{m}_i](D_n^{r_i})) \subset F_w^k[\mathbf{m}_{i-1}](D_n^{r_{i-1}})$$

for any $k \in \mathbf{Z}$ and $i = 1, 2$ with shift vectors $\mathbf{m}_i \in \mathbf{Z}^{r_i}$. Then (10.6) induces a complex

$$\mathrm{gr}_w[\mathbf{m}_2](D_n^{r_2}) \xrightarrow{\overline{\psi}_2} \mathrm{gr}_w[\mathbf{m}_1](D_n^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}_w[\mathbf{m}_0](D_n^{r_0}) \tag{10.7}$$

of left $\mathrm{gr}_w(D_n)$-modules. Under these assumptions, the complex (10.7) is exact if and only if the equalities

$$\mathrm{Im}\,\overline{\psi}_2 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Im}\,\psi_2), \quad \mathrm{Ker}\,\overline{\psi}_1 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1)$$

hold in $\mathrm{gr}_w[\mathbf{m}_1](D_n^{r_1})$.

Proof: In general, we have inclusions

$$\mathrm{Im}\,\overline{\psi}_2 \subset \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Im}\,\psi_2), \quad \mathrm{Ker}\,\overline{\psi}_1 \supset \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1)$$

while $\mathrm{gr}_w[\mathbf{m}_1](\mathrm{Im}\,\psi_2) = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1)$ holds since (10.6) is exact. Hence we have $\mathrm{Im}\,\overline{\psi}_2 = \mathrm{Ker}\,\overline{\psi}_1$ if and only if

$$\mathrm{Im}\,\overline{\psi}_2 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Im}\,\psi_2), \quad \mathrm{Ker}\,\overline{\psi}_1 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1).$$

This completes the proof. []

Lemma 10.5 *Let N be a left D_n-submodule of D_n^r and let $P_1, \dots, P_s$ be an $F_w[\mathbf{m}]$-involutive base of N with $\mathbf{m} \in \mathbf{Z}^r$. Then for any $P \in \widehat{D}_n^{(w)} N$, there exist $Q_1, \dots, Q_s \in \widehat{D}_n^{(w)}$ such that $P = \sum_{i=1}^s Q_i P_i$ and $\mathrm{ord}_w[\mathbf{m}](Q_i P_i) \le \mathrm{ord}_w[\mathbf{m}](P)$ for each i.*

Proof: Let P be an element of $\widehat{D}_n^{(w)} N$. Then there exist $Q_1, \dots, Q_s \in \widehat{D}_n^{(w)}$ such that $P = \sum_{i=1}^s Q_i P_i$. Put $k := \mathrm{ord}_w[\mathbf{m}](P)$, $m_i := \mathrm{ord}_w[\mathbf{m}](P_i)$. Choose $Q_i' \in D_n$ such that $Q_i - Q_i' \in F_w^{k-m_i-1}(\widehat{D}_n^{(w)})$ and put $P' := \sum_{i=1}^s Q_i' P_i$. Then P' belongs to N and hence we have $\sigma_w[\mathbf{m}](P) = \sigma_w[\mathbf{m}](P') \in \mathrm{gr}_w^k[\mathbf{m}](N)$. Since $P_1, \dots, P_s$ is an $F_w[\mathbf{m}]$-involutive base of N, there exist $Q_1^{(0)}, \dots, Q_s^{(0)} \in D_n$ with $\mathrm{ord}_w[\mathbf{m}](Q_i) \le k - m_i$ such that

$$P' - \sum_{i=1}^s Q_i^{(0)} P_i \ \in F_w^{k-1}[\mathbf{m}](D_n^r).$$

This implies

$$P - \sum_{i=1}^s Q_i^{(0)} P_i \ \in F_w^{k-1}[\mathbf{m}]((\widehat{D}_n^{(w)})^r) \cap \widehat{D}_n^{(w)} N.$$

Repeating this procedure, we can find $Q_i^{(\nu)} \in D_n$ with $\mathrm{ord}_w[\mathbf{m}](Q_i^{(\nu)}) \le k - m_i - \nu$ so that

$$P = \sum_{i=1}^s \sum_{\nu=0}^\infty Q_i^{(\nu)} P_i$$

holds in $(\widehat{D}_n^{(w)})^r$. This completes the proof. []

Theorem 10.6 *Let N be a left D_n-submodule of D_n^r and let $P_1, \dots, P_s$ be an $F_w[\mathbf{m}]$-involutive base of N with $\mathbf{m} \in \mathbf{Z}^r$. Then for any $P \in N$, there exist $Q_1, \dots, Q_s \in D_n$ such that $P = \sum_{i=1}^s Q_i P_i$ and $\mathrm{ord}_w[\mathbf{m}](Q_i P_i) \le \mathrm{ord}_w[\mathbf{m}](P)$ or else $Q_i = 0$ for each i.*

Proof: Put $r_0 := r$, $r_1 := s$, $\mathbf{m}_0 := \mathbf{m}$ and define a homomorphism $\psi_1 : D_n^{r_1} \longrightarrow D_n^{r_0}$ by $\psi_1(Q_1, \dots, Q_{r_1}) = \sum_{i=1}^s Q_i P_i$. Then we have $N = \mathrm{Im}\,\psi_1$. Let $P_1^{(1)}, \dots, P_{r_2}^{(1)}$ be an $F_w[\mathbf{m}_1]$-involutive base of $\mathrm{Ker}\,\psi_1$ with

$$\mathbf{m}_1 := (\mathrm{ord}_w[\mathbf{m}_0](P_1), \dots, \mathrm{ord}_w[\mathbf{m}_0](P_{r_1})).$$

Put

$$\mathbf{m}_2 := (\mathrm{ord}_w[\mathbf{m}_1](P_1^{(1)}), \dots, \mathrm{ord}_w[\mathbf{m}_1](P_{r_2}^{(1)}))$$

and define a homomorphism $\psi_2 : D_n^{r_1} \to D_n^{r_2}$ by

$$\psi_2(Q_1, \dots, Q_{r_2}) = \sum_{i=1}^{r_2} Q_i P_i^{(1)}.$$

Then the sequence

$$D_n^{r_2} \xrightarrow{\psi_2} D_n^{r_1} \xrightarrow{\psi_1} D_n^{r_0} \tag{10.8}$$

is exact and satisfies the assumption of Lemma 10.4. Since $\widehat{D}_n^{(w)}$ is flat over D_n, the sequence (10.8) induces an exact sequence

$$(\widehat{D}_n^{(w)})^{r_2} \xrightarrow{\hat{\psi}_2} (\widehat{D}_n^{(w)})^{r_1} \xrightarrow{\hat{\psi}_1} (\widehat{D}_n^{(w)})^{r_0}.$$

Put $\widehat{M} := (\widehat{D}_n^{(w)})^{r_0} / \widehat{D}_n^{(w)} N$ and let $\hat{\varphi} : (\widehat{D}_n^{(w)})^{r_0} \to \widehat{M}$ be the canonical homomorphism. We define a filtration on $\widehat{M}$ by

$$F_w^k[\mathbf{m}_0](\widehat{M}) := \hat{\varphi}(F_w^k[\mathbf{m}_0]((\widehat{D}_n^{(w)})^{r_0})).$$

We also put $\widehat{K} := \mathrm{Ker}\,\hat{\psi}_2$ and define a filtration on $\widehat{K}$ by

$$F_w^k[\mathbf{m}_2](\widehat{K}) := \widehat{K} \cap F_w^k[\mathbf{m}_2]((\widehat{D}_n^{(w)})^{r_2}).$$

Then we have an exact sequence

$$0 \longrightarrow \widehat{K} \longrightarrow (\widehat{D}_n^{(w)})^{r_2} \xrightarrow{\hat{\psi}_2} (\widehat{D}_n^{(w)})^{r_1} \xrightarrow{\hat{\psi}_1} (\widehat{D}_n^{(w)})^{r_0} \xrightarrow{\hat{\varphi}} \widehat{M} \longrightarrow 0. \tag{10.9}$$

Let us show that this induces an exact sequence

$$\begin{aligned} 0 \longrightarrow F_w^k[\mathbf{m}_2](\widehat{K}) \longrightarrow F_w^k[\mathbf{m}_2]((\widehat{D}_n^{(w)})^{r_2}) &\xrightarrow{\hat{\psi}_2} F_w^k[\mathbf{m}_1]((\widehat{D}_n^{(w)})^{r_1}) \\ &\xrightarrow{\hat{\psi}_1} F_w^k[\mathbf{m}_0]((\widehat{D}_n^{(w)})^{r_0}) \xrightarrow{\hat{\varphi}} F_w^k[\mathbf{m}_0](\widehat{M}) \longrightarrow 0 \end{aligned} \tag{10.10}$$

for any $k \in \mathbf{Z}$. In fact, assume $P \in F_w^k[\mathbf{m}_1]((\widehat{D}_n^{(w)})^{r_1})$ satisfies $\hat{\psi}_1(P) = 0$. Then by the exactness of (10.9), P belongs to the image of $\hat{\psi}_2$. Hence by Lemma 10.5, P belongs to

$\hat\psi_2(F_w^k[\mathbf{m}_2](\widehat{D}_n^{(w)})^{r_2})$. Moreover, in view of Lemma 10.5 and the flatness of $\widehat{D}_n^{(w)}$ over D_n, we have

$$\operatorname{Ker}\hat\varphi \cap F_w^k[\mathbf{m}_0]((\widehat{D}_n^{(w)})^{r_0}) = \widehat{D}_n^{(w)} N \cap F_w^k[\mathbf{m}_0]((\widehat{D}_n^{(w)})^{r_0}) = \hat\psi_1(F_w^k[\mathbf{m}_1]((\widehat{D}_n^{(w)})^{r_1}))$$

since $P_1, \dots, P_{r_1}$ is an $F_w[\mathbf{m}_0]$-involutive base of N. Hence (10.10) is exact. This implies that the induced sequence

$$\operatorname{gr}_w[\mathbf{m}_2]((\widehat{D}_n^{(w)})^{r_2}) \xrightarrow{\overline{\psi}_2} \operatorname{gr}_w[\mathbf{m}_1]((\widehat{D}_n^{(w)})^{r_1}) \xrightarrow{\overline{\psi}_1} \operatorname{gr}_w[\mathbf{m}_0]((\widehat{D}_n^{(w)})^{r_0})$$

is exact. This sequence coincides with (10.7) since $\operatorname{gr}_w[\mathbf{m}_i]((\widehat{D}_n^{(w)})^{r_i}) = \operatorname{gr}_w[\mathbf{m}_i](D_n^{r_i})$. Thus $\operatorname{Ker}\overline{\psi}_1 = \operatorname{gr}_w[\mathbf{m}_1](\operatorname{Ker}\psi_1)$ holds in $\operatorname{gr}_w[\mathbf{m}_1](D_n^{r_1})$ in view of Lemma 10.4.

Now let P be an element of N and put $k := \operatorname{ord}_w[\mathbf{m}](P)$. Then there exist $Q_1, \dots, Q_s \in D_n$ such that $P = \sum_{i=1}^s Q_i P_i$. Put $m_i := \operatorname{ord}_w[\mathbf{m}](P_i)$. Suppose $m := \max\{\operatorname{ord}_w(Q_i) + m_i \mid i = 1, \dots, s\} > k$. Put $Q_i^{(0)} := \sigma_w(Q_i)$ if $\operatorname{ord}_w(Q_i) + m_i = m$ and $Q_i^{(0)} := 0$ otherwise. Then we have

$$\sum_{i=1}^s Q_i^{(0)} \sigma_w[\mathbf{m}](P_i) = 0$$

in $\operatorname{gr}_w[\mathbf{m}](D_n^r)$. This means that $Q^{(0)} := (Q_1^{(0)}, \dots, Q_s^{(0)})$ belongs to $\operatorname{Ker}\overline{\psi}_1 = \operatorname{gr}_w[\mathbf{m}_1](\operatorname{Ker}\psi_1)$. Hence there exist $Q' = (Q_1', \dots, Q_s') \in \operatorname{Ker}\psi_1$ such that $\sigma_w[\mathbf{m}_1](Q') = Q^{(0)} = \sigma_w[\mathbf{m}_1](Q)$ with $Q := (Q_1, \dots, Q_s)$. Then we have

$$P = \sum_{i=1}^s (Q_i - Q_i')P_i, \quad \operatorname{ord}_w[\mathbf{m}_1](Q - Q') < \operatorname{ord}_w[\mathbf{m}_1](Q).$$

Continuing this process at most $m - k$ times, we can find $Q'' = (Q_1'', \dots, Q_s'') \in D_n^s$ such that $P = \sum_{i=1}^s Q_i'' P_i$ and $\operatorname{ord}_w[\mathbf{m}_1](Q'') \le k$. This completes the proof. ▯

Theorem 10.7 *Let $\psi : D_n^{r_i} \longrightarrow D_n^{r_{i-1}}$ be a homomorphism of left D_n-modules for $i = 1, 2$ and assume that the sequence*

$$D_n^{r_2} \xrightarrow{\psi_2} D_n^{r_1} \xrightarrow{\psi_1} D_n^{r_0} \tag{10.11}$$

is exact. Assume moreover that

$$\psi_i(F_w^k[\mathbf{m}_i](D_n^{r_i})) \subset F_w^k[\mathbf{m}_{i-1}](D_n^{r_{i-1}})$$

for any $k \in \mathbf{Z}$ and $i = 1, 2$ with shift vectors $\mathbf{m}_i \in \mathbf{Z}^{r_i}$. Let $e_1, \dots, e_{r_1}$ be the canonical generators of $D_n^{r_1}$ and $e_1', \dots, e_{r_2}'$ be those of $D_n^{r_2}$. Then the following three conditions are equivalent:

(1) The sequence

$$F_w^k[\mathbf{m}_2](D_n^{r_2}) \xrightarrow{\psi_2} F_w^k[\mathbf{m}_1](D_n^{r_1}) \xrightarrow{\psi_1} F_w^k[\mathbf{m}_0](D_n^{r_0})$$

is exact for any $k \in \mathbf{Z}$, and $\{\psi_1(e_1), \dots, \psi_1(e_{r_1})\}$ is an $F_w[\mathbf{m}_0]$-involutive base of $\operatorname{Im}\psi_1$.

(2) *The sequence*

$$\mathrm{gr}_w[\mathbf{m}_2](D_n^{r_2}) \xrightarrow{\overline{\psi}_2} \mathrm{gr}_w[\mathbf{m}_1](D_n^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}_w[\mathbf{m}_0](D_n^{r_0})$$

is exact.

(3) $\{\psi_1(e_1), \dots, \psi_1(e_{r_1})\}$ *is an* $F_w[\mathbf{m}_0]$*-involutive base of the module it generates, and* $\{\psi_2(e'_1), \dots, \psi_2(e'_{r_2})\}$ *is an* $F_w[\mathbf{m}_1]$*-involutive base of* $\mathrm{Ker}\,\psi_1$.

Proof: By extending (10.11) to an exact sequence

$$0 \longrightarrow M' \longrightarrow D_n^{r_2} \xrightarrow{\psi_2} D_n^{r_1} \xrightarrow{\psi_1} D_n^{r_0} \longrightarrow M \longrightarrow 0$$

and considering the induced filtrations on M and M', we easily see in view of Theorem 10.6 that (1) implies the exactness of the sequence

$$0 \to F_w[\mathbf{m}_2](M') \to F_w^k[\mathbf{m}_2](D_n^{r_2}) \xrightarrow{\psi_2} F_w^k[\mathbf{m}_1](D_n^{r_1}) \xrightarrow{\psi_1} F_w^k[\mathbf{m}_0](D_n^{r_0}) \to F_w^k[\mathbf{m}_0](M) \to 0$$

for any $k \in \mathbf{Z}$. Hence (1) implies (2). (3) implies (1) by virtue of Theorem 10.6. Hence we have only to show that (2) implies (3). By Lemma 10.4, (2) is equivalent to

$$\mathrm{Im}\,\overline{\psi}_2 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Im}\,\psi_2), \quad \mathrm{Ker}\,\overline{\psi}_1 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1).$$

The first equality is equivalent to the fact that $\psi_2(e'_1), \dots, \psi_2(e'_{r_2})$ are an $F_w[\mathbf{m}_1]$-involutive base, and they generate $\mathrm{Ker}\,\psi_1$ by the assumption.

Let $P = \sum_{i=1}^{r_1} Q_i\psi_1(e_i)$ be an arbitrary element of $\mathrm{Im}\,\psi_1$ with $Q_1, \dots, Q_{r_1} \in D_n$. Suppose $k := \mathrm{ord}_w[\mathbf{m}_0](P) < \mathrm{ord}_w[\mathbf{m}_1]((Q_1, \dots, Q_{r_1}))$. Then the equality $\mathrm{Ker}\,\overline{\psi}_1 = \mathrm{gr}_w[\mathbf{m}_1](\mathrm{Ker}\,\psi_1)$ assures the existence of $Q' = (Q'_1, \dots, Q'_{r_1}) \in \mathrm{Ker}\,\psi_1$ such that $\sigma_w[\mathbf{m}_1](Q') = \sigma_w[\mathbf{m}_1](Q)$ with $Q := (Q_1, \dots, Q_{r_1})$. Hence we have

$$P = \sum_{i=1}^{r_1}(Q_i - Q'_i)\psi_1(e_i)$$

with $\mathrm{ord}_w[\mathbf{m}_1](Q - Q') < \mathrm{ord}_w[\mathbf{m}_1](Q)$. Repeating this procedure for a finite number of times, we can find $Q'' = (Q''_1, \dots, Q''_{r_1}) \in D_n^{r_1}$ such that

$$P = \sum_{i=1}^{r_1} Q''_i\psi_1(e_i)$$

with $\mathrm{ord}_w[\mathbf{m}_1](Q'') \le k$. This implies, in particular, that $\psi_1(e_1), \dots, \psi_1(e_{r_2})$ are an $F_w[\mathbf{m}_0]$-involutive base. This completes the proof. []

Theorem 2.5 follows from this theorem (and its proof).

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