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Operational calculus for holonomic distributions in the framework of D-module theory

Dedicated to Professor H. Komatsu and Professor T. Kawai

By

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Abstract

Let f be a real polynomial of $x = (x_1, \ldots, x_n)$ and φ be a locally integrable function of x which satisfies a holonomic system of linear differential equations. We study the distribution $f_+^\lambda \varphi$ with a meromorphic parameter λ, especially its Laurent expansion and integration, from an algorithmic viewpoint in the framework of D-module theory.

§ 1. Introduction

Let f be a non-constant real polynomial in $x = (x_1, \ldots, x_n)$ and φ be a locally integrable function on an open subset U of $\mathbb{R}^n$. Then φ can be regarded as a distribution (generalized function in the sense of L. Schwartz) on U. We assume that there exists a left ideal I of the ring D_n of differential operators with polynomial coefficients in x which annihilates φ on $U_f := \{x \in U \mid f(x) \neq 0\}$, i.e., $P\varphi$ vanishes on U_f for any $P \in I$. Moreover, we assume that $M := D_n/I$ is a holonomic D_n-module. In this situation, φ is called a (locally integrable) holonomic function or a holonomic distribution.

Let us consider the distribution $f_+^\lambda \varphi$ on U with a holomorphic parameter λ. This distribution can be analytically extended to a distribution-valued meromorphic function of λ on the complex plane $\mathbb{C}$. Such a distribution was systematically studied by Kashiwara and Kawai in [2] with f being, more generally, a real-valued real analytic function. Their investigation was focused on a special case where M has regular singularities but most of the arguments work without this assumption.

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The main purpose of this article is to give algorithms to compute

1. A holonomic system for the distribution $f_+^{\lambda_0}\varphi$ with λ_0 not being a pole of $f_+^\lambda\varphi$.
2. A holonomic system for each coefficient of the Laurent series of $f_+^\lambda\varphi$ about an arbitrary point.
3. Difference equations for the local zeta function $Z(\lambda) = \int_{\mathbb{R}^n} f_+^\lambda\varphi\,dx$.

As was pointed out in [2], an answer to the first problem provides us with an algorithm to compute a holonomic system for the product of two locally L^2 holonomic functions. Note that the product does not necessarily satisfies the tensor product of the two holonomic systems for both functions.

In Section 2, we review the theoretical properties of $f_+^\lambda\varphi$ mostly following Kashiwara [1] and Kashiwara and Kawai [2] in the analytic category; i.e, under a weaker assumption that f is a real-valued real analytic function and that φ satisfies a holonomic system of linear differential equations with analytic coefficients.

In Section 3, we give algorithms to computes holonomic systems considered in Section 2. As a byproduct, we obtain an algorithm to compute difference equations for the local zeta function, which was outlined in [4].

§ 2. Theoretical background

Let $\mathcal{D}_{\mathbb{C}^n}$ be the sheaf on $\mathbb{C}^n$ of linear partial differential operators with holomorphic coefficients, which is generated by the derivations $\partial_j = \partial_{x_j} = \partial/\partial x_j$ $(j = 1,\dots,n)$ over the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of rings of holomorphic functions on $\mathbb{C}^n$, with the coordinate system $x = (x_1,\dots,x_n)$ of $\mathbb{C}^n$.

We denote by $\mathcal{D}b$ the sheaf on $\mathbb{R}^n$ of the Schwartz distributions. Assume that $f = f(x)$ is a nonzero real-valued real analytic function defined on an open connected set U of $\mathbb{R}^n$. Let φ be a locally integrable function on U. Then $f_+^\lambda\varphi$ is also locally integrable on U for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$, where $f_+(x) = \max\{f(x), 0\}$.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module defined on an open set Ω of $\mathbb{C}^n$ such that $U \subset \Omega \cap \mathbb{R}^n$. We also assume that f is holomorphic on Ω. We say that a distribution φ is a solution of $\mathcal{M}$ on U if there exist a section u of $\mathcal{M}$ on U and a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism $\Phi : \mathcal{D}_{\mathbb{C}^n}u \to \mathcal{D}b$ defined on U such that $\Phi(u) = \varphi$. As a matter of fact, we have only to assume that φ is a solution of $\mathcal{M}$ on $U_f := \{x \in U \mid f(x) \neq 0\}$ and that $\mathcal{M}$ is holonomic on $\Omega_f := \{x \in \Omega \mid f(x) \neq 0\}$.

§ 2.1. Fundamental lemmas

Under the assumptions above, $f_+^\lambda\varphi$ is a $\mathcal{D}b(U)$-valued holomorphic function of λ on the right half-plane

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}.$$

In other words, let $\mathcal{ODb}$ be the sheaf on $\mathbb{C} \times \mathbb{R}^n \ni (\lambda, x)$ of distributions with a holomorphic parameter λ. Then $f_+^\lambda \varphi$ belongs to

$$\mathcal{ODb}(\mathbb{C}_+ \times U) = \Big\{ v(\lambda, x) \in \mathcal{Db}(\mathbb{C}_+ \times U) \mid \frac{\partial v}{\partial \overline{\lambda}} = 0 \Big\}.$$

Let s be an indeterminate corresponding to λ. The following lemma (Lemma 2.9 of [2]) plays an essential role in the following arguments.

Lemma 2.1 (Kashiwara-Kawai [2]). *Let Ω' be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega'$ is non-empty and contained in U. Assume $P(s) \in \mathcal{D}_{\mathbb{C}^n}(\Omega')[s]$ and $P(\lambda)(f_+^\lambda \varphi) = 0$ holds in $\mathcal{ODb}(\mathbb{C}_+ \times V_f)$ with $V_f := \{x \in V \mid f(x) \neq 0\}$. Then $P(\lambda)(f_+^\lambda \varphi) = 0$ holds in $\mathcal{ODb}(\mathbb{C}_+ \times V)$.*

Let us generalize this lemma slightly. For a positive integer m, let us define a section $f_+^\lambda (\log f_+)^m \varphi$ of the sheaf $\mathcal{ODb}$ on $\mathbb{C}_+ \times U$ by

$$\langle f_+^\lambda (\log f_+)^m \varphi, \psi \rangle = \int_{\{x \in U \mid f(x) > 0\}} \varphi(x) f(x)^\lambda (\log f(x))^m \varphi(x) \psi(x)\, dx \quad (\forall \psi \in C_0^\infty(U)),$$

where $C_0^\infty(U)$ denotes the space of C^∞ functions on U with compact supports. In fact, $f_+^\lambda (\log f_+)^m \varphi$ is the m-th derivative of the distribution $f_+^\lambda \varphi$ with respect to λ.

Lemma 2.2. *Let Ω' be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega'$ is non-empty and contained in U. Let $\varphi_0, \dots, \varphi_m$ be locally integrable functions on V. Assume $P_k(s) \in \mathcal{D}_{\mathbb{C}^n}(\Omega')[s]$ ($k = 0, 1, \dots, m$) and*

$$\sum_{k=0}^m P_k(\lambda)(f_+^\lambda (\log f_+)^k \varphi_k) = 0 \tag{2.1}$$

holds in $\mathcal{ODb}(\mathbb{C}_+ \times V_f)$. Then (2.1) holds in $\mathcal{ODb}(\mathbb{C}_+ \times V)$.

Proof. We follow the argument of the proof of Lemma 2.9 in [2]. Let ϕ belong to $C_0^\infty(V)$ with $K := \operatorname{supp} \phi$. Let $\chi(t)$ be a C^∞ function of a variable t such that $\chi(t) = 1$ for $|t| \leq 1/2$ and $\chi(t) = 0$ for $|t| \geq 1$. Then we have

$$\begin{aligned} \Big\langle \sum_{k=0}^m P_k(\lambda)(f_+^\lambda (\log f_+)^k \varphi_k),\ \phi \Big\rangle &= \Big\langle \sum_{k=0}^m P_k(\lambda)(f_+^\lambda (\log f_+)^k \varphi_k),\ \chi(\frac{f}{\tau})\phi \Big\rangle \\ &= \sum_{k=0}^m \int_V f_+^\lambda (\log f_+)^k \varphi_k {}^t P_k(\lambda)\Big(\chi(\frac{f}{\tau})\phi\Big)\, dx \end{aligned}$$

for any $\tau > 0$, where ${}^t P_k(\lambda)$ denotes the adjoint operator of $P_k(\lambda)$. Let m_k be the order of $P_k(s)$ and d_k be the degree of $P_k(s)$ in s. Then there exist constants C_k such that

$$\sup_{x \in K} \Big| {}^t P_k(\lambda)\Big(\chi(\frac{f(x)}{\tau})\phi(x)\Big) \Big| \leq C_k (1 + |\lambda|)^{d_k} \tau^{-m_k} \qquad (0 < \forall \tau < 1).$$

Assume Re $\lambda > \max\{m_k + 1 \mid 0 \le k \le m\}$ and $0 < \tau < 1$. Then we have

$$\begin{aligned}
&\left|\int_V f_+^\lambda (\log f_+)^k \varphi_k {}^t P_k(\lambda)\Big(\chi\big(\frac{f}{\tau}\big)\phi\Big)\,dx\right| \\
&\le C_k(1+|\lambda|)^{d_k}\tau^{-m_k}\int_{\{x\in V \mid 0<f(x)\le\tau\}} |f_+^\lambda(\log f_+)^k\varphi_k(x)|\,dx \\
&\le k! C_k(1+|\lambda|)^{d_k}\tau^{\mathrm{Re}\ \lambda - m_k - 1}\int_{\{x\in V \mid 0<f(x)\le\tau\}} |\varphi_k(x)|\,dx
\end{aligned}$$

since $|\log t|^k \le k! t^{-1}$ holds for $0 < t < 1$. This implies

$$\left\langle \sum_{k=0}^m P_k(\lambda)(f_+^\lambda(\log f_+)^k\varphi_k),\ \phi \right\rangle = \lim_{\tau\to+0}\sum_{k=0}^m \int_V f_+^\lambda(\log f_+)^k\varphi_k {}^t P_k(\lambda)\Big(\chi\big(\frac{f}{\tau}\big)\phi\Big)\,dx = 0.$$

The assertion of the lemma follows from the uniqueness of analytic continuation. □

§ 2.2. Generalized b-function and analytic continuation

We assume that there exists on Ω a sheaf $\mathcal{I}$ of coherent left ideals of $\mathcal{D}_{\mathbb{C}^n}$ which annihilates φ on $U_f = \{x \in U \mid f(x) \neq 0\}$, namely, $P\varphi = 0$ holds on $W \cap U_f$ for any section P of $\mathcal{I}$ on an open set W of $\mathbb{C}^n$. We set $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n}/\mathcal{I}$ and denote by u the residue class of $1 \in \mathcal{D}_X$ modulo $\mathcal{I}$. In the sequel, we assume that $\mathcal{M}$ is holonomic on $\Omega_f = \{z \in \Omega \mid f(z) \neq 0\}$, i.e., that $\mathrm{Char}(\mathcal{M}) \cap \pi^{-1}(\Omega_f)$ is of dimension n, where $\mathrm{Char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$ and $\pi : T^*\mathbb{C}^n \to \mathbb{C}^n$ is the canonical projection.

Let $\mathcal{L} = \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s$ be the free $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]$-module generated by the symbol f^s. Then $\mathcal{L}$ has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}[s]$-module induced by the derivation $\partial_i f^s = s(\partial f/\partial x_i)f^{-1}f^s$. Let us consider the tensor product $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ of $\mathcal{O}_{\mathbb{C}^n}$-modules, which has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}[s]$-module.

Lemma 2.3. *Let v and $P(s)$ be sections of $\mathcal{M}$ and of $\mathcal{D}_{\mathbb{C}^n}[s]$ respectively on an open subset of Ω. Then $P(s)(f^s \otimes v) = 0$ holds in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ if and only if $(f^{m-s}P(s)f^s)(1 \otimes v) = 0$ holds in $\mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{M}$ for a sufficiently large $m \in \mathbb{N}$.*

Proof. Set $\mathcal{M}[s] = \mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{M}$, which has a natural structure of left module over $\mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{C}^n} = \mathcal{D}_{\mathbb{C}^n}[s]$. Then we have $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} = \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$ as left $\mathcal{D}_{\mathbb{C}^n}[s]$-module. Let v be a section of $\mathcal{M}[s]$. Since $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]$ as $\mathcal{O}_{\mathbb{C}^n}[s]$-module, $f^s \otimes v$ vanishes in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$ if and only if $1 \otimes v$ vanishes in $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$. First, let us show that this happens if and only if $f^m v = 0$ in $\mathcal{M}[s]$ with some $m \in \mathbb{N}$.

Let $\rho : \mathcal{O}_{\mathbb{C}^n}[s, t] \to \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]$ be the homomorphism defined by $\rho(h(s, t)) = h(s, f^{-1})$ for $h(s, t) \in \mathcal{O}_{\mathbb{C}^n}[s, t]$. Let $\mathcal{K}$ be the kernel of ρ. Then we have an exact sequence

$$\mathcal{K} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s] \longrightarrow \mathcal{O}_{\mathbb{C}^n}[s, t] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s] \xrightarrow{\rho\otimes\mathrm{id}} \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s] \longrightarrow 0.$$

Hence $1 \otimes v$ vanishes in $\mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$ if and only if there exists $h(s,t) = \sum_{k=0}^{m} h_k(s) t^k \in \mathcal{K}$ such that $1 \otimes v = h(s,t) \otimes v$ holds in $\mathcal{O}_{\mathbb{C}^n}[s,t] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$, which is equivalent to $h_k(s)v = \delta_{0k} v$ $(k = 0, 1, \dots, m)$ since $\mathcal{O}_{\mathbb{C}^n}[s,t]$ is free over $\mathcal{O}_{\mathbb{C}^n}[s]$. On the other hand, $\sum_{k=0}^{m} h_k(s) f^{-k} = \rho(h(s,t)) = 0$ implies

$$0 = f^m h_0(s)v + f^{m-1} h_1(s)v + \cdots + f h_{m-1}(s)v + h_m(s)v = f^m v.$$

Conversely, if $f^m v = 0$ for some $m \in \mathbb{N}$, then we have $1 \otimes v = f^{-m} \otimes f^m v = 0$ in $\mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}$.

Let $P(s)$ be a section of $\mathcal{D}_{\mathbb{C}^n}[s]$ of order m. For $i = 1, \dots, n$,

$$\partial_i (f^s \otimes v) = f^{s-1} \otimes (s f_i + f \partial_i) v = f^{s-1} \otimes (f^{1-s} \partial_i f^s) v$$

holds in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$ with $f_i = \partial f / \partial x_i$. This allows us to show that

$$P(s)(f^s \otimes v) = f^{s-m} \otimes (f^{m-s} P(s) f^s) v$$

holds in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$. (Note that $f^{m-s} P(s) f^s$ belongs to $\mathcal{D}_{\mathbb{C}^n}[s]$.) Summing up, we have shown that $P(s)(f^s \otimes v)$ vanishes in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}[s]} \mathcal{M}[s]$ if and only if $(f^{l-s} P(s) f^s) v$ vanishes in $\mathcal{M}[s]$ for some $l \geq m$. $\square$

Lemma 2.3 with $P(s) = 1$ immediately implies

Proposition 2.4. *Let $\mathcal{M}[f^{-1}] := \mathcal{O}_{\mathbb{C}^n}[f^{-1}] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ be the localization of $\mathcal{M}$ with respect to f, which has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}$-module. Then the natural homomorphism $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \to \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}[f^{-1}]$ is an isomorphism.*

Proposition 2.5. *Let $P(s)$ be a section of $\mathcal{D}_{\mathbb{C}^n}[s]$ on an open set Ω' of $\mathbb{C}^n$ and suppose $P(s)(f^s \otimes u) = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$. Set $V = U \cap \Omega'$. Then $P(\lambda)(f_+^\lambda \varphi) = 0$ holds in $\mathcal{O}\mathcal{D}b(\mathbb{C}_+ \times V)$.*

Proof. Let $\mathcal{O}_{+\infty}\mathcal{D}b$ be the sheaf on $\mathbb{R}^n$ associated with the presheaf

$$W \longmapsto \varinjlim \mathcal{O}\mathcal{D}b(\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > a\} \times W)$$

for every open set W of $\mathbb{R}^n$, where the inductive limit is taken as $a \to \infty$. The $\mathbb{C}$-bilinear sheaf homomorphism

$$\mathcal{L} \times \mathcal{M} \ni (a(s) f^{s-m}, Pu) \longmapsto (a(\lambda) f_+^{\lambda - m}) P\varphi \in \mathcal{O}_{+\infty}\mathcal{D}b$$

with $a(s) \in \mathcal{O}_X[s]$, $m \in \mathbb{N}$, $P \in \mathcal{D}_X$, which is well-defined and $\mathcal{O}_{\mathbb{C}^n}$-balanced on V_f since $f_+^{\lambda - m}$ is real analytic there, induces a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism

$$\Psi : \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \longrightarrow \mathcal{O}_{+\infty}\mathcal{D}b$$

on V_f such that $\Psi(a(s)f^{s-m}\otimes Pu) = a(\lambda)f_+^{\lambda-m}P\varphi$. In particular, if $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ satisfies $P(s)(f^s\otimes u) = 0$ in $\mathcal{L}\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{M}$, then $P(\lambda)(f_+^\lambda\varphi) = 0$ holds in $\mathcal{O}_{+\infty}\mathcal{D}b(V_f)$, hence also in $\mathcal{O}_{+\infty}\mathcal{D}b(V)$ by Lemma 2.1. Since $f_+^\lambda\varphi$ belongs to $\mathcal{O}\mathcal{D}b(\mathbb{C}_+\times V)$, it follows that $P(f_+^\lambda\varphi) = 0$ holds in $\mathcal{O}\mathcal{D}b(\mathbb{C}_+\times V)$. This completes the proof. □

Kashiwara proved in [1] (Theorem 2.7) that on a neighborhood of each point p of Ω, there exist nonzero $b(s)\in\mathbb{C}[s]$ and $P(s)\in\mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$P(s)(f^{s+1}\otimes u) = b(s)f^s\otimes u \quad \text{in } \mathcal{L}\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{M}.$$

Such $b(s)$ of the smallest degree $b(s) = b_p(s)$ is called the (generalized) b-function for f and u at p.

Assume $p\in U$. Then by the proposition above,

$$P(\lambda)(f_+^{\lambda+1}\varphi) = b(\lambda)f_+^\lambda\varphi$$

holds in $\mathcal{O}\mathcal{D}b(\mathbb{C}_+\times V)$ with an open neighborhood V of p. It follows that $f_+^\lambda\varphi$ is a $\mathcal{D}b(V)$-valued meromorphic function of λ on $\mathbb{C}$. Let us assume that U is relatively compact in Ω. The poles of $f_+^\lambda\varphi$ are contained in

$$\{\lambda - k \mid b_p(\lambda) = 0\ (\exists p\in U),\ k\in\mathbb{N}\}.$$

Proposition 2.6 (Lemma 2.10 of [2])**.** *There exists a positive real number ε such that $f_+^\lambda\varphi$ belongs to $\mathcal{O}\mathcal{D}b(\{\lambda\in\mathbb{C}\mid \mathrm{Re}\ \lambda > -\varepsilon\}\times U)$.*

Proof. Let λ_0 be an arbitrary pole of $f_+^\lambda\varphi$. There exists $\psi\in C_0^\infty(U)$ such that λ_0 is a pole of $Z(\lambda) := \langle f_+^\lambda\varphi,\psi\rangle$. In particular, $|Z(\lambda_0+t)|$ tends to infinity as $t\to+0$. On the other hand, $Z(\lambda)$ is continuous on $\{\lambda\in\mathbb{C}\mid \mathrm{Re}\ \lambda\geq 0\}$. This implies $\mathrm{Re}\ \lambda_0 < 0$. The conclusion follows since there are at most a finite number of poles of $f_+^\lambda\varphi$ in the set $\{\lambda\in\mathbb{C}|\mathrm{Re}\ \lambda > -1\}$. □

In conclusion, $f_+^\lambda\varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function on $\mathbb{C}$ whose poles are contained in $\{\lambda\in\mathbb{C}\mid\mathrm{Re}\ \lambda < 0\}$.

§ 2.3. Holonomicity of $f_+^\lambda\varphi$ and its applications

Let f, φ, $\mathcal{M} = \mathcal{D}_{\mathbb{C}^n}/\mathcal{I}$ be as in the preceding subsection. Let $\mathcal{N} = \mathcal{D}_{\mathbb{C}^n}[s](f^s\otimes u)$ be the left $\mathcal{D}_{\mathbb{C}^n}[s]$-submodule of $\mathcal{L}\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{M}$ generated by $f^s\otimes u$. Theorem 2.5 of Kashiwara [1] guarantees that $\mathcal{N}_{\lambda_0} := \mathcal{N}/(s-\lambda_0)\mathcal{N}$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module on Ω for any $\lambda_0\in\mathbb{C}$.

Proposition 2.7. *Let λ_0 be an arbitrary complex number and $f^{\lambda_0}\otimes\varphi$ the residue class of $f^s\otimes u\in\mathcal{N}$ modulo $(s-\lambda_0)\mathcal{N}$.*

1. *$\mathcal{N}_0$ is isomorphic to $\mathcal{M}$ as $\mathcal{D}_{\mathbb{C}^n}$-module on Ω_f.*

2. *If $\mathcal{M}$ is f-saturated, i.e., if $fv = 0$ with $v \in \mathcal{M}$ implies $v = 0$, then there is a surjective $\mathcal{D}_{\mathbb{C}^n}$-homomorphism $\Phi : \mathcal{N}_0 \to \mathcal{M}$ on Ω such that $\Phi(f^0 \otimes u) = u$. Moreover, Φ is an isomorphism on Ω_f.*

Proof. Since $\mathcal{M}[f^{-1}] = \mathcal{M}$ on Ω_f, we may assume that $\mathcal{M}$ is f-saturated. In view of Lemma 2.3 and the definition of $\mathcal{N}_0$, $P \in \mathcal{D}_{\mathbb{C}^n}$ annihilates $f^0 \otimes u$ if and only if there exist $Q(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ and an integer $m \geq \operatorname{ord} Q(s)$ such that $(f^{m-s}Q(s)f^s)(1 \otimes u) = 0$ in $\mathcal{M}[s]$ and $P = Q(0)$. If there exist such $Q(s)$ and m, set

$$f^{m-s}Q(s)f^s = Q_0 + Q_1 s + \cdots + Q_l s^l \quad (Q_i \in \mathcal{D}_{\mathbb{C}^n}).$$

Then $Q_i u = 0$ holds for any i. In particular, $Q_0 = f^m P$ annihilates u. This implies $Pu = 0$ since $\mathcal{M}$ is f-saturated. Hence the homomorphism Φ is well-defined.

Now assume $p \in \Omega_f$ and $Pu = 0$ in the stalk $\mathcal{M}_p$ of $\mathcal{M}$ at p. Then $Q(s) := f^s P f^{-s}$ belongs to $\mathcal{D}_{\mathbb{C}^n,p}[s]$ and annihilates $f^s \otimes u$ by Lemma 2.3. Hence $P = Q(0)$ annihilates $f^0 \otimes u$. This implies that Φ is an isomorphism on Ω_f. □

Theorem 2.8. *If λ_0 is not a pole of $f_+^\lambda\varphi$, then $f_+^{\lambda_0}\varphi$ is a solution of $\mathcal{N}_{\lambda_0}$.*

Proof. Assume that $\lambda_0 \in \mathbb{C}$ is not a pole of $f_+^\lambda\varphi$. Let P be a section of $\mathcal{D}_{\mathbb{C}^n}$ which annihilates $f^{\lambda_0} \otimes u$. Then there exist $Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$P = Q(s) + (s - \lambda_0)R(s), \quad Q(s)(f^s \otimes u) = 0 \text{ in } \mathcal{N}.$$

Proposition 2.5 implies that $Q(\lambda)(f_+^\lambda\varphi)$ vanishes as section of the sheaf $\mathcal{O}\mathcal{D}b$. In particular, $P(f_+^{\lambda_0}\varphi) = Q(\lambda_0)(f_+^{\lambda_0}\varphi) = 0$ holds as distribution. Thus the homomorphism

$$\mathcal{N}_{\lambda_0} = \mathcal{D}_{\mathbb{C}^n}(f^{\lambda_0} \otimes u) \ni P(f^{\lambda_0} \otimes u) \longmapsto P(f_+^{\lambda_0}\varphi) \in \mathcal{D}b$$

is well-defined and $\mathcal{D}_{\mathbb{C}^n}$-linear. Hence $f_+^{\lambda_0}\varphi$ is a solution of $\mathcal{N}_{\lambda_0}$. □

The following two theorems are essentially due to Kashiwara and Kawai [2] although they are stated with additional assumptions and stronger results.

Theorem 2.9. *φ is a solution of the holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{N}_0$.*

Proof. First note that $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s](-f)^s$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s$ as left $\mathcal{D}_{\mathbb{C}^n}[s]$-module since $\partial_i(-f)^s = sf_i f^{-1}(-f)^s$ holds in $\mathcal{O}_{\mathbb{C}^n}[f^{-1}, s](-f)^s$ with $f_i = \partial f/\partial x_i$. Assume that $P(f^0 \otimes u) = 0$ holds in $\mathcal{N}_0 = \mathcal{N}/s\mathcal{N}$. Then there exist $Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$P = Q(s) + sR(s), \quad Q(s)(f^s \otimes u) = 0 \text{ in } \mathcal{N}.$$

Let $\theta(t)$ be the Heaviside function; i.e., $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t \le 0$. Then we have $\theta(f) = f_+^0$ and $\theta(-f) = (-f)_+^0$. Theorem 2.8 implies that $P = Q(0)$ annihilates both $\theta(f)\varphi$ and $\theta(-f)\varphi$, and hence also $\varphi = \theta(f)\varphi + \theta(-f)\varphi$. Thus φ is a solution of $\mathcal{N}_0$. □

Theorem 2.10. *Let φ_1 and φ_2 be locally L^p and L^q functions respectively on an open set $U \subset \mathbb{R}^n$ with $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. Assume that φ_1 and φ_2 are solutions of holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively on U. Then for any point x_0 of U, there exists a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{M}$ on a neighborhood of x_0 of which the product $\varphi_1\varphi_2$ is a solution.*

Proof. There exist analytic functions f_1 and f_2 on a neighborhood V of x_0 such that the singular support (the projection of the characteristic variety minus the zero section) of $\mathcal{M}_k$ is contained in $f_k = 0$ for $k = 1, 2$. Set $f(z) = f_1(z)\overline{f_1(\overline{z})}f_2(z)\overline{f_2(\overline{z})}$. Then $f(x)$ is a real-valued real analytic function and φ_1 and φ_2 are real analytic on V_f. Then it is easy to see, in the same way as in the proof of Theorem 2.8, that $\varphi_1\varphi_2$ is a solution of $\mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2$ on V_f. To complete the proof, we have only to apply Theorem 2.9 to $\mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2$ and $\varphi_1\varphi_2$ in place of $\mathcal{M}$ and φ respectively. □

§ 2.4. Laurent coefficients of $f_+^\lambda\varphi$

Let f, φ, $\mathcal{M}$ be as in preceding subsections.

Theorem 2.11. *Let p be a point of U. Then each coefficient of the Laurent expansion of $f_+^\lambda\varphi$ about an arbitrary $\lambda_0 \in \mathbb{C}$ is a solution of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module on a common neighborhood of p.*

Proof. Fix $m \in \mathbb{N}$ such that $\mathrm{Re}\,\lambda_0 + m \ge 0$. By using the functional equation involving the generalized b-function, we can find a nonzero $b(s) \in \mathbb{C}[s]$ and a germ $P(s)$ of $\mathcal{D}_{\mathbb{C}^n}[s]$ at p such that

$$b(\lambda)f_+^\lambda\varphi = P(\lambda)(f_+^{\lambda+m}\varphi).$$

Factor $b(s)$ as $b(s) = (s - \lambda_0)^l c(s)$ with $c(s) \in \mathbb{C}[s]$ such that $c(\lambda_0) \neq 0$ and an integer $l \ge 0$. Then we have

$$(\lambda - \lambda_0)^l f_+^\lambda\varphi = \frac{1}{c(\lambda)}P(\lambda)(f_+^{\lambda+m}\varphi).$$

The right-hand side is holomorphic in λ on an neighborhood of $\lambda = \lambda_0$. Let

$$f_+^\lambda\varphi = \sum_{k=-l}^{\infty}(\lambda - \lambda_0)^k\varphi_k$$

be the Laurent expansion with $\varphi_k \in \mathcal{D}b(U)$, which is given by

$$\varphi_k = \frac{1}{(l+k)!}\lim_{\lambda\to\lambda_0}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}\left((\lambda-\lambda_0)^l f_+^\lambda\varphi\right) = \frac{1}{(l+k)!}\lim_{\lambda\to\lambda_0}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}\left(\frac{1}{c(\lambda)}P(\lambda)(f_+^{\lambda+m}\varphi)\right).$$

Hence there exist $Q_{kj} \in \mathcal{D}_{\mathbb{C}^n}$ such that

$$\varphi_k = \sum_{j=0}^{l+k} Q_{kj}(f_+^{\lambda_0+m}(\log f_+)^j\varphi). \tag{2.2}$$

First let us show that $f_+^{\lambda_0+m}(\log f_+)^j\varphi$ with $0 \le j \le k$ satisfy a holonomic system. Consider the free $\mathcal{O}_{\mathbb{C}^n}[s,f^{-1}]$-module

$$\tilde{\mathcal{L}} := \mathcal{O}_{\mathbb{C}^n}[s,f^{-1}]f^s \oplus \mathcal{O}_{\mathbb{C}^n}[s,f^{-1}]f^s\log f \oplus \mathcal{O}_{\mathbb{C}^n}[s,f^{-1}]f^s(\log f)^2 \oplus \cdots,$$

which has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}[s]$-module. Let

$$\mathcal{N}[k] := \mathcal{D}_{\mathbb{C}^n}[s](f^s\otimes u) + \mathcal{D}_{\mathbb{C}^n}[s]((f^s\log f)\otimes u) + \cdots + \mathcal{D}_{\mathbb{C}^n}[s]((f^s(\log f)^k)\otimes u)$$

be the left $\mathcal{D}_{\mathbb{C}^n}[s]$-submodule of $\tilde{\mathcal{L}}\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{M}$ generated by $(f^s(\log f)^j)\otimes u$ with $j = 0,1,\ldots,k$. It is easy to see that $\mathcal{N}[k]/\mathcal{N}[k-1]$ is isomorphic to $\mathcal{N} = \mathcal{N}[0]$ as left $\mathcal{D}_{\mathbb{C}^n}[s]$-module since

$$P(s)((f^s(\log f)^k)\otimes u) \equiv (f^{s-m}(\log f)^k)\otimes(f^{m-s}P(s)f^s)u \mod \mathcal{N}[k-1]$$

holds for any $P(s)\in\mathcal{D}_{\mathbb{C}^n}[s]$ with $m = \operatorname{ord} P(s)$. Moreover, $\mathcal{N}_{\lambda_0}[k] := \mathcal{N}[k]/(s-\lambda_0)\mathcal{N}[k]$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module since $\mathcal{N}_{\lambda_0}[k]/\mathcal{N}_{\lambda_0}[k-1]$ is isomorphic to $\mathcal{N}_{\lambda_0} = \mathcal{N}_{\lambda_0}[0]$, and hence is holonomic as left $\mathcal{D}_{\mathbb{C}^n}$-module.

Let $(f^{\lambda_0+m}(\log f)^j)\otimes u \in \mathcal{N}_{\lambda_0+m}[k]$ be the residue class of $(f^s(\log f)^j)\otimes u$ modulo $(s-\lambda_0-m)\mathcal{N}[k]$. Suppose $\sum_{j=0}^k P_j((f^{\lambda_0+m}(\log f)^j)\otimes u)$ vanishes in $\mathcal{N}_{\lambda_0+m}[k]$ with P_j being a section of $\mathcal{D}_{\mathbb{C}^n}$ on an open neighborhood of a point p of U. Then there exist $Q_j(s)\in\mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$\sum_{j=0}^k P_j((f^s(\log f)^j)\otimes u) = (s-\lambda_0-m)\sum_{j=0}^k Q_j(s)((f^s(\log f)^j)\otimes u)$$

holds in $\mathcal{N}[k]$. Then it is easy to see that

$$\sum_{j=0}^k P_j(\lambda)(f_+^\lambda(\log f_+)^j\varphi) = (\lambda-\lambda_0-m)\sum_{j=0}^k Q_j(\lambda)(f_+^\lambda(\log f_+)^j\varphi) \tag{2.3}$$

holds in $\mathcal{O}\mathcal{D}b(\mathbb{C}_+\times W_f)$ with an open neighborhood W of p. Lemma 2.2 and analytic continuation imply that (2.3) holds in $\mathcal{O}\mathcal{D}b(\mathbb{C}_+\times W)$. By Proposition 2.6, we have in $\mathcal{D}b(W)$

$$\sum_{j=0}^k P_j((f_+^{\lambda_0+m}(\log f_+)^j)\varphi) = 0.$$

In conclusion, with k replaced by $l+k$, there exists a $\mathcal{D}_{\mathbb{C}^n}$-homomorphism Φ : $\mathcal{N}_{\lambda_0+m}[l+k] \to \mathcal{D}b$ such that

$$\Phi((f^{\lambda_0+m}(\log f)^j)\otimes u) = f_+^{\lambda_0+m}(\log f_+)^j\varphi \qquad (0\le j\le l+k).$$

Set

$$w := \sum_{j=0}^{l+k} Q_{kj}((f^{\lambda_0+m}(\log f)^j)\otimes u), \qquad \mathcal{M}_k := \mathcal{D}_{\mathbb{C}^n}w.$$

Then $\mathcal{M}_k$ is a $\mathcal{D}_{\mathbb{C}^n}$-submodule of $\mathcal{N}_{\lambda_0+m}[l+k]$ and hence holonomic. Since $\Phi(w) = \varphi_k$ in view of (2.2), φ_k is a solution of $\mathcal{M}_k$. This completes the proof. □

§ 3. Algorithms

We give algorithms for computing holonomic systems introduced in the previous section assuming that f is a real polynomial and that $\mathcal{M}$ is algebraic, i.e., defined by differential operators with polynomial coefficients. Let $D_n := \mathbb{C}\langle x,\partial\rangle = \mathbb{C}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$ be the ring of differential operators with polynomial coefficients with $\partial_j = \partial/\partial x_j$. The ring D_n is also called the n-th Weyl algebra over $\mathbb{C}$.

In the sequel, let f be a non-constant real polynomial of $x=(x_1,\dots,x_n)$ and φ be a locally integrable function on an open connected set U of $\mathbb{R}^n$. We assume that there exists a left ideal I of D_n which annihilates φ on U_f, i.e., $P\varphi = 0$ holds on U_f for any $P\in I$, such that $M := D_n/I$ is a holonomic D_n-module. We denote by u the residue class of $1\in D_n$ modulo I. Let $L = \mathbb{C}[x,f^{-1},s]f^s$ be the free $\mathbb{C}[x,f^{-1},s]$-module generated by f^s, which has a natural structure of left $D_n[s]$-module. Let $N := D_n[s](f^s\otimes u)$ be the left D_n-submodule of $L\otimes_{\mathbb{C}[x]} M$ generated by $f^s\otimes u$.

As was established in the previous section, $f_+^\lambda\varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function on $\mathbb{C}$ and is a solution of N.

§ 3.1. Mellin transform

Let us assume that φ is real analytic on U_f and set

$$\tilde\varphi(x,\lambda) := \int_{-\infty}^{\infty} t_+^\lambda\delta(t-f(x))\varphi(x)\,dt.$$

This is well-defined and coincides with $f_+^\lambda\varphi$ as a distribution on $U_f\times\mathbb{C}_+$. Then we have

$$\int_{-\infty}^{\infty} t_+^\lambda t\delta(t-f(x))\varphi(x)\,dt = \tilde\varphi(x,\lambda+1),$$
$$\int_{-\infty}^{\infty} t_+^\lambda\partial_t(\delta(t-f(x))\varphi(x))\,dt = -\int_{-\infty}^{\infty}\partial_t(t_+^\lambda)\delta(t-f(x))\varphi(x)\,dt = -\lambda\tilde\varphi(x,\lambda-1).$$

Let $D_{n+1} = D_n\langle t, \partial_t\rangle$ be the $(n+1)$-th Weyl algebra with $\partial_t = \partial/\partial t$. Let us consider the ring $D_n\langle s, E_s, E_s^{-1}\rangle$ of difference-differential operators with the shift operator $E_s : s \mapsto s+1$, where s is an indeterminate corresponding to λ. In view of the identities above, let us define the ring homomorphism (Mellin transform of operators)

$$\mu : D_{n+1} \longrightarrow D_n\langle s, E_s, E_s^{-1}\rangle$$

by

$$\mu(t) = E_s, \quad \mu(\partial_t) = -sE_s^{-1}, \quad \mu(x_j) = x_j, \quad \mu(\partial_{x_j}) = \partial_{x_j}.$$

It is easy to see that μ is well-defined and injective since $[\partial_t, t] = [\mu(\partial_t), \mu(t)] = 1$. Hence we may regard D_{n+1} as a subring of $D_n\langle s, E_s, E_s^{-1}\rangle$. Since $\mu(\partial_t t) = -s$, we can also regard $D_n[s]$ as a subring of D_{n+1}. Thus we have inclusions

$$D_n[s] \ \subset \ D_{n+1} \ \subset \ D_n\langle s, E_s, E_s^{-1}\rangle$$

of rings and $L \otimes_{\mathbb{C}[x]} M$ has a structure of left $D_n\langle s, E_s, E_s^{-1}\rangle$-module compatible with that of left $D_n[s]$-module. Let $\mathcal{F}(U)$ be the $\mathbb{C}$-vector space of the $\mathcal{D}b(U)$-valued meromorphic functions on $\mathbb{C}$. Then $\mathcal{F}(U)$ has a natural structure of left $D_n\langle s, E_s, E_s^{-1}\rangle$-module, which is compatible with that of $D_n[s]$-module. In particular, we can regard $\mathcal{F}(U)$ as a left D_{n+1}-module.

§ 3.2. Computation of $N = D_n[s](f^s \otimes u)$

The inclusion $D_{n+1}f^s \subset L = \mathbb{C}[x, f^{-1}, s]f^s$ induces a natural D_{n+1}-homomorphism

$$\begin{array}{ccc} D_{n+1}f^s \otimes_{\mathbb{C}[x]} M & \xrightarrow{\ \iota\ } & L \otimes_{\mathbb{C}[x]} M \\ \cup & & \cup \\ N' & \xrightarrow{\ \iota'\ } & N \end{array}$$

where N' is the left $D_n[s]$-submodule of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$ and N is the left $D_n[s]$-submodule of $L \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$. The homomorphism ι induces a surjective $D_n[s]$-homomorphism $\iota' : N' \to N$.

Proposition 3.1. *The homomorphism ι is injective if and only if M is f-saturated; i.e., the homomorphism $f : M \to M$ is injective.*

Proof. First note that $D_{n+1}f^s$ is isomorphic to the first local cohomology group $\mathbb{C}[x, t, (t-f)^{-1}]/\mathbb{C}[x, t]$ of $\mathbb{C}[x, t]$ supported in the non-singular hypersurface $t - f(x) = 0$ since

$$(t-f)f^s = 0, \quad (\partial_{x_i} + f_i\partial_t)f^s = 0 \quad (i = 1, \ldots, n).$$

In particular, $D_{n+1}f^s$ is a free $\mathbb{C}[x]$-module generated by $\partial_t^j f^s$ with $j \geq 0$. Hence an arbitrary element w of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$ is uniquely written in the form

$$w = \sum_{j=0}^{k} (\partial_t^j f^s) \otimes u_j$$

with $u_j \in M$ and $k \in \mathbb{N}$. Then

$$\iota(w) = \sum_{j=0}^{k} (-1)^j s(s-1)\cdots(s-j+1) f^{s-j} \otimes u_j$$

vanishes if and only if $f^{s-j} \otimes u_j = 0$, which is equivalent to $f^{m_j} u_j = 0$ with some $m_j \in \mathbb{N}$ by Lemma 2.3, for all $j = 0, 1, \ldots, k$. This completes the proof. $\square$

Let $\tilde{M}$ be the left D_n-submodule of the localization $M[f^{-1}] := \mathbb{C}[x, f^{-1}] \otimes_{\mathbb{C}[x]} M$ which is generated by $1 \otimes u$. Then $\tilde{M}$ is f-saturated and the natural homomorphism

$$L \otimes_{\mathbb{C}[x]} M \longrightarrow L \otimes_{\mathbb{C}[x]} \tilde{M}$$

is an isomorphism by Lemma 2.3.

An algorithm to compute $M[f^{-1}]$ was presented in [7] under the assumption that M is holonomic on $\mathbb{C}^n \setminus \{f = 0\}$. It provides us with an algorithm to compute $\tilde{M}$, i.e., the annihilator of $1 \otimes u \in M[f^{-1}]$. Hence we may assume, from the beginning, that M is holonomic and f-saturated. Then $\iota' : N' \to N$ is an isomorphism by Proposition 3.1. The f-saturatedness is equivalent to the vanishing of the zeroth local cohomology group of M with support in $f = 0$, which can be computed by algorithms presented in [3],[8],[6].

Thus we have only to give an algorithm to compute the structure of N' assuming M to be f-saturated. We follow an argument introduced by Walther [8]. Note that we gave in [3] an algorithm based on tensor product computation which is less efficient.

Definition 3.2. For a differential operator $P = P(x, \partial) \in D_n$, set

$$\tau(P) := P\left(x, \partial_{x_1} + f_1 \partial_t, \ldots, \partial_{x_n} + f_n \partial_t\right) \in D_{n+1}$$

with $f_j = \partial f / \partial x_j$. This substitution is well-defined since the operators $\partial_{x_j} + f_j \partial_t$ commute with one another and $[\partial_{x_j} + f_j \partial_t, x_i] = \delta_{ij}$ holds.

Moreover, for a left ideal I of D_{n+1}, let $\tau(I)$ be the left ideal of D_{n+1} which is generated by the set $\{\tau(P) \mid P \in I\}$.

Lemma 3.3. *$\tau(P)(f^s \otimes v) = f^s \otimes (Pv)$ holds in $L \otimes_{\mathbb{C}[x]} M$ for any $P \in D_n$ and $v \in M$.*

Proof. By the definition of the action of D_{n+1} on $L \otimes_{\mathbb{C}[x]} M$ via the Mellin transform, we have

$$(\partial_{x_j} + f_j\partial_t)(f^s \otimes v) = sf^{-1}f_jf^s \otimes v + f^s \otimes (\partial_{x_j}v) - sf_jf^{-1}f^s \otimes v = f^s \otimes (\partial_{x_j}v).$$

This implies the conclusion of the lemma. □

Proposition 3.4. *Let I be a left ideal of D_n and set $M = D_n/I$ with $u \in M$ being the residue class of* 1 *modulo I. Let J be the left ideal of D_{n+1} which is generated by $\tau(I) \cup \{t - f(x)\}$. Then J coincides with the annihilator $\mathrm{Ann}_{D_{n+1}}(f^s \otimes u)$ of $f^s \otimes u \in D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$.*

Proof. We have only to show that for $P \in D_{n+1}$ the equivalence

$$P \in J \quad \Leftrightarrow \quad P(f^s \otimes u) = 0 \quad \text{in} \quad D_{n+1}f^s \otimes_{\mathbb{C}[x]} M.$$

Suppose Q belongs to J. Then P annihilates $f^s \otimes u$ by Lemma 3.3.

Conversely, suppose $P(f^s \otimes u) = 0$ in $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$. We can rewrite P in the form

$$P = \sum_{\alpha\in\mathbb{N}^n,\nu\in\mathbb{N}} p_{\alpha,\nu}(x)\partial_t^\nu\Big(\partial_{x_1} + \frac{\partial f}{\partial x_1}\partial_t\Big)^{\alpha_1} \cdots \Big(\partial_{x_n} + \frac{\partial f}{\partial x_n}\partial_t\Big)^{\alpha_n} + Q \cdot (t - f(x))$$

with $p_{\alpha,\nu}(x) \in \mathbb{C}[x]$ and $Q \in D_{n+1}$. Setting $P_\nu := \sum_{\alpha\in\mathbb{N}^n} p_{\alpha,\nu}(x)\partial_x^\alpha$, we get

$$0 = P(f^s \otimes u) = \sum_{\nu=0}^{\infty}(\partial_t^\nu f^s) \otimes P_\nu u \,\in\, D_{n+1}f^s \otimes_{\mathbb{C}[x]} M.$$

It follows that each P_ν belongs to I since $\{\partial_t^\nu f^s\}$ constitutes a free basis of $D_{n+1}f^s$ over $\mathbb{C}[x]$. Hence we have

$$P = \sum_{\nu=1}^{\infty} \partial_t^\nu \tau(P_\nu) + Q \cdot (t - f(x)) \,\in J.$$

This completes the proof. □

In order to compute the structure of the $D_n[s]$-submodule $N' = D_n[s](f^s \otimes u)$ of $D_{n+1}f^s \otimes_{\mathbb{C}[x]} M$, we have only to compute the annihilator

$$\mathrm{Ann}_{D_n[s]}(f^s \otimes u) = D_n[s] \cap J,$$

where we regard $D_n[s]$ as a subring of D_{n+1}. This can be done as follows:

Introducing new variables σ and τ, for $P \in D_{n+1}$, let $h(P) \in D_{n+1}[\tau]$ be the homogenization of P with respect to the weights

x_j	∂_{x_j}	t	∂_t	τ	σ
0	0	-1	1	-1	1

Let J' be the left ideal of $D_{n+1}[\sigma,\tau]$ generated by

$$\{h(P) \mid P \in \tilde{G}\} \cup \{1 - \sigma\tau\},$$

where $\tilde{G}$ is a set of generators of J.

Set $J'' = J' \cap D_{n+1}$. Since each element P of J'' is homogeneous with respect to the above weights, there exists $P'(s) \in D_n[s]$ such that $P = SP'(-\partial_t t)$ with $S = t^\nu$ or $S = \partial_t^\nu$ with some integer $\nu \geq 0$. We set $P'(s) = \psi(P)(s)$. Then $\{\psi(P) \mid P \in J''\}$ generates the left ideal $J \cap D_n[s]$ of $D_n[s]$. This procedure can be done by using a Gröbner basis in $D_{n+1}[\sigma,\tau]$. In conclusion, we have a set of generators of $J \cap D_n[s]$. Then N', and hence N also if M is f-saturated, is isomorphic to $D_n[s]/(J \cap D_n[s])$ as left $D_n[s]$-module.

The generalized b-function for f and u can be computed as the generator of the ideal

$$\mathbb{C}[s] \cap (\mathrm{Ann}_{D_n[s]} f^s \otimes u + D_n[s]f)$$

of $\mathbb{C}[s]$ by elimination via Gröbner basis computation in $D_n[s]$.

§ 3.3. Holonomic systems for the Laurent coefficients of $f_+^\lambda \varphi$

Let λ_0 be an arbitrary complex number. Our purpose is to compute a holonomic system of which each coefficient of the Laurent expansion of $f_+^\lambda \varphi$ is a solution.

Let $b_0(s)$ be the (global) b-function of f and u. We can find a $P_0(s) \in D_n[s]$ such that

$$P_0(s)(f^{s+1} \otimes u) = b_0(s) f^s \otimes u$$

holds in N by, e.g., syzygy computation. Take $m \in \mathbb{N}$ such that $\mathrm{Re}\,\lambda_0 + m \geq 0$ or $b_0(\lambda_0 + m + k) \neq 0$ ($\forall k \in \mathbb{N}$). Then $\lambda_0 + m$ is not a pole of $f_+^\lambda \varphi$.

We can find a nonzero polynomial $b(s)$ and $P(s) \in D_n[s]$ such that

$$b(\lambda) f_+^\lambda = P(\lambda) f_+^{\lambda+m}.$$

In fact, we have only to set

$$P(s) := P_0(s)P_0(s+1)\cdots P_0(s+m-1), \quad b(s) := b_0(s)b_0(s+1)\cdots b_0(s+m-1).$$

Factorize $b(s)$ as $b(s) = c(s)(s-\lambda_0)^l$ with $c(\lambda_0) \neq 0$. Then $f_+^\lambda \varphi$ has a Laurent expansion of the form

$$f_+^\lambda \varphi = \sum_{k=-l}^{\infty} (\lambda - \lambda_0)^k \varphi_k$$

around λ_0, where $\varphi_k \in \mathcal{D}b(U)$ is given by

$$\varphi_k = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \left(\frac{\partial}{\partial \lambda}\right)^{l+k} (c(\lambda)^{-1} P(\lambda) f_+^{\lambda+m}) = \sum_{j=0}^{l+k} Q_{kj} (f_+^{\lambda_0+m} (\log f)^j)$$

with

$$Q_{kj} := \frac{1}{j!(l+k-j)!} \left[\left(\frac{\partial}{\partial \lambda}\right)^{l+k-j} (c(\lambda)^{-1} P(\lambda))\right]_{\lambda=\lambda_0}.$$

Let

$$\tilde{L} = \mathbb{C}[x, f^{-1}, s] f^s \oplus \mathbb{C}[x, f^{-1}, s] f^s \log f \oplus \mathbb{C}[x, f^{-1}, s] f^s (\log f)^2 \oplus \cdots$$

be the free $\mathbb{C}[x, f^{-1}, s]$-module with a natural structure of left $D_n\langle s, \partial_s\rangle$-module. Consider the left $D_n[s]$-submodule

$$N[k] = D_n[s](f^s \otimes u) + D_n[s]((f^s \log f) \otimes u) + \cdots + D_n[s]((f^s (\log f)^k) \otimes u)$$

of $\tilde{L} \otimes_{\mathbb{C}[x]} M$. For a complex number λ_0, set

$$N_{\lambda_0}[k] = N[k]/(s-\lambda_0)N[k].$$

Let us first give an algorithm to compute the structure of $N[k]$.

Proposition 3.5. *Let G_0 be a set of generators of the annihilator $\mathrm{Ann}_{D_n[s]}(f^s \otimes u) = J \cap D_n[s]$. Let $e_1 = (1, 0, \dots, 0)$, $\cdots$, $e_{k+1} = (0, \dots, 0, 1)$ be the canonical basis of $\mathbb{Z}^{k+1}$. For each $Q(s) \in G_0$ and an integer j with $0 \le j \le k$, set*

$$Q^{(j)}(s) := \sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} e_{i+1} \in (D_n[s])^{k+1}.$$

Let J_k be the left $D_n[s]$-submodule of $(D_n[s])^{k+1}$ generated by $G_1 := \{Q^{(j)}(s) \mid Q(s) \in G_0,\ 0 \le j \le k\}$. Then $(D_n[s])^{k+1}/J_k$ is isomorphic to $N[k]$.

Proof. Let $\varpi : (D_n[s])^{k+1} \to N[k]$ be the canonical surjection. Let $Q(s)$ belong to G_0. Differentiating the equation $Q(s)(f^s \otimes u) = 0$ in $N[k]$ with respect to s, one gets

$$\sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} ((f^s (\log f)^i) \otimes u) = 0.$$

Hence J_k is contained in the kernel of ϖ. Conversely, assume that

$$\vec{Q}(s) = (Q_0(s), Q_1(s), \dots, Q_k(s))$$

belongs to the kernel of ϖ. This implies $Q_k(s)(f^s \otimes u) = 0$ since $N[k]/N[k-1]$ is isomorphic to $N = D_n[s](f^s \otimes u)$. Hence $\vec{Q}(s) - Q_k^{(k)}(s)$ belongs to the kernel of ϖ, the last component of which is zero. We conclude that $\vec{Q}(s)$ belongs to J_k by induction. □

Thus we have

$$N_{\lambda_0}[k] = (D_n)^{k+1}/J_k|_{s=\lambda_0}, \qquad J_k|_{s=\lambda_0} := \{Q(\lambda_0) \mid Q(s) \in J_k\}.$$

Set

$$w := \sum_{j=0}^{l+k} Q_{kj}((f^{\lambda_0+m}(\log f)^j) \otimes u), \qquad M_k := D_n w.$$

Then we have

$$Pw = 0 \quad \Leftrightarrow \quad P(Q_{k0}, Q_{k1}, \dots, Q_{k,l+k}) \in J_{l+k}|_{s=\lambda_0+m}.$$

Thus we can find a set of generators of $\mathrm{Ann}_{D_n} w$ by computation of syzygy or intersection. As was shown in §2.4, φ_k is a solution of the holonomic system M_k.

§ 3.4. Difference equations for the local zeta function

In the sequel, we assume that φ is a locally integrable function on $\mathbb{R}^n$. As we have seen so far, $f_+^\lambda \varphi \in \mathcal{F}(\mathbb{R}^n)$ is a solution of the holonomic D_{n+1}-module D_{n+1}/J. Hence if the local zeta function $Z(\lambda) := \int_{\mathbb{R}^n} f_+^\lambda \varphi \, dx$ is well-defined, e.g., if φ has compact support, or else is smooth on $\mathbb{R}^n$ with all its derivatives rapidly decreasing on the set $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$, then $Z(\lambda)$ is a solution of the integral module

$$D_{n+1}/(J + \partial_{x_1} D_{n+1} + \cdots + \partial_{x_n} D_{n+1})$$

of D_{n+1}/J, which is a holonomic module over $D_1 = \mathbb{C}\langle t, \partial_t \rangle$. This D_1-module can be computed by the integration algorithm which is the 'Fourier transform' of the restriction algorithm given in [6] (see [5] for the integration algorithm). Then by Mellin transform we obtain linear difference equations for $Z(\lambda)$. Thus we get

Theorem 3.6. *Under the above assumptions, $Z(\lambda)$ satisfies a non-trivial linear difference equation with polynomial coefficients in λ.*

Example 3.7. $\Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} \, dx = \int_{-\infty}^\infty x_+^\lambda e^{-x} \, dx$ satisfies the difference equation

$$(E_\lambda - (\lambda + 1))\Gamma(\lambda + 1) = 0,$$

where $E_\lambda : \lambda \mapsto \lambda + 1$ is the shift operator.

§ 3.5. Examples

Let us present some examples computed by using algorithms introduced so far and their implementation in the computer algebra system Risa/Asir.

Example 3.8. Set $f = x^3 - y^2 \in \mathbb{R}[x, y]$ and $\varphi = 1$. Since the b-function of f is $b_f(s) = (s+1)(6s+5)(6s+7)$, possible poles of f_+^λ are $-1-\nu$, $-5/6-\nu$, $-6/7-\nu$ with $\nu \in \mathbb{N}$ and they are at most simple poles. The residue $\mathrm{Res}_{\lambda=-1} f_+^\lambda$ is a solution of

$$D_2/(D_2(2x\partial_x + 3y\partial_y + 6) + D_2(2y\partial_x + 3x^2\partial_y) + D_2(x^3 - y^2)).$$

$\mathrm{Res}_{\lambda=-5/6} f_+^\lambda$ is a solution of $D_2/(D_2x + D_2y)$. Hence it is a constant multiple of the delta function $\delta(x, y) = \delta(x)\delta(y)$. $\mathrm{Res}_{\lambda=-7/6} f_+^\lambda$ is a solution of $D_2/(D_2x^2 + D_2(x\partial_x + 2) + D_2y)$. Hence it is a constant multiple of $\delta'(x)\delta(y)$.

Example 3.9. Set $f = x^3 - y^2$ and $\varphi(x, y) = \exp(-x^2 - y^2)$. Then φ is a solution of a holonomic system $M := D_2/(D_2(\partial_x + 2x) + D_2(\partial_y + 2y))$ on $\mathbb{R}^2$, which is f-saturated since it is a simple D_2-module. The generalized b-function for f and $u := [1] \in M$ is $b_f(s) = (s+1)(6s+5)(6s+7)$. The local zeta function $Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi \, dxdy$ is annihilated by the difference operator

$$\begin{aligned} &32E_s^4 + 16(4s+13)E_s^3 - 4(s+3)(27s^2 + 154s + 211)E_s^2 \\ &- 6(s+2)(s+3)(36s^2 + 162s + 173)E_s - 3(s+1)(s+2)(s+3)(6s+5)(6s+13), \end{aligned}$$

where s is an indeterminate corresponding to λ. From this we see that $-7/6$ is not a pole of $Z(\lambda)$.

Example 3.10. Set $\varphi(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi(x) = 0$ for $x \leq 0$. Then $\varphi(x)$ belongs to the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ and satisfies a holonomic system

$$M := D_1/D_1(x^2\partial_x + x^2 - 1),$$

which is x-saturated. The generalized b-function for $f = x$ and $u = [1] \in M$ is $s + 1$. The local zeta function $Z(\lambda) := \int_{\mathbb{R}} x_+^\lambda \varphi(x) \, dx$ is entire (i.e., without poles) and satisfies a difference equation

$$(E_\lambda^2 - (\lambda + 2)E_\lambda - 1)Z(\lambda) = 0.$$

This can also be deduced by integration by parts.

Example 3.11. Set $\varphi_1(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi_1(x) = 0$ for $x \leq 0$. Set $\varphi(x, y) = \varphi_1(x)e^{-y}$. Then φ satisfies a holonomic system

$$M := D_2/(D_2(x^2\partial_x + x^2 - 1) + D_2(\partial_y + 1)).$$

The generalized b-function for $f := y^3 - x^2$ and $u = [1] \in M$ is $s+1$. Moreover, we can confirm that M is f-saturated by using the localization algorithm in [7]. The local zeta function $Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi \, dxdy$ is well-defined since $f(x,y) < 0$ if $y < 0$. It is annihilated by a difference operator of the form

$$E_s^{11} + a_{10}(s)E_s^{10} + \cdots + a_1(s)E_s + a_0(s),$$
$$a_0(s) = c(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7)(s+8)(s+9),$$

where c is a positive rational number and $a_1(s), \ldots, a_{10}(s)$ are polynomials of s with rational coefficients. Possible poles of $f_+^\lambda \varphi$ are the negative integers. For example, -1 is at most a simple pole of $f_+^\lambda \varphi$ and $\mathrm{Res}_{\lambda=-1} f_+^\lambda \varphi$ is a solution of a holonomic system

$$D_2/(D_2(3x^2\partial_x + 2xy\partial_y + 3x^2 + (2y+6)x - 3) + D_2(y^3 - x^2)).$$

Example 3.12. Set $f = x^3 - y^2z^2$. The b-function of f is $(s+1)(3s+4)(3s+5)(6s+5)^2(6s+7)^2$. For example, its maximum root $-5/6$ is at most a pole of order 2 of f_+^λ. Let

$$f_+^\lambda = \left(\lambda + \frac{5}{6}\right)^{-2} \varphi_{-2} + \left(\lambda + \frac{5}{6}\right)^{-1} \varphi_{-1} + \varphi_0 + \cdots$$

be the Laurent expansion. Then φ_{-2} satisfies

$$x\varphi_{-2} = y\varphi_{-2} = z\varphi_{-2} = 0.$$

Hence φ_{-2} is a constant multiple of $\delta(x,y)$. On the other hand, φ_{-1} satisfies a holonomic system

$$x\varphi_{-1} = (y\partial_y - z\partial_z)\varphi_{-1} = yz\varphi_{-1} = (z^2\partial_z - z)\varphi_{-1} = 0.$$

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