An algorithm to compute the differential equations for the logarithm of a polynomial

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Abstract

We present an algorithm to compute the annihilator of (i.e., the linear differential equations for) the multi-valued analytic function $f^\lambda(\log f)^m$ in the Weyl algebra D_n for a given non-constant polynomial f, a non-negative integer m, and a complex number λ. This algorithm essentially consists of the differentiation with respect to s of the annihilator of f^s in the ring $D_n[s]$ and ideal quotient computation in D_n. The obtained differential equations constitute what is called a holonomic system in D-module theory. Hence combined with the integration algorithm for D-modules, this enables us to compute a holonomic system for the integral of a function involving the logarithm of a polynomial with respect to some variables.

1 Introduction

For a given function u, it is an interesting problem both in theory and in practice to determine the differential equations which u satisfies. Let us restrict our attention to linear differential equations with polynomial coefficients. Then our problem can be formulated as follows: Let D_n be the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients in the variables $x = (x_1, \dots, x_n)$. An element P of D_n is expressed as a finite sum

$$P = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta \tag{1}$$

with $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$ and $a_{\alpha,\beta} \in \mathbb{C}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ are multi-indices with $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\partial_i = \partial/\partial x_i$ $(i = 1, \dots, n)$ denote derivations. The *annihilator*, or the annihilating ideal, of u in D_n is defined to be

$$\mathrm{Ann}_{D_n} u = \{P \in D_n \mid Pu = 0\},$$

which is a left ideal of D_n. Since D_n is a non-commutative Noetherian ring, there exist a finite number of operators $P_1, \dots, P_N \in D_n$ which generate $\mathrm{Ann}_{D_n} u$ as left ideal. Thus we can regard the system

$$P_1 u = \cdots = P_N u = 0$$

of linear differential equations as a maximal one that u satisfies.

As to systems of linear differential equations, there is a notion of holonomicity, or being *holonomic*, which plays a central role in D-module theory. See Appendix A for a precise definition. A holonomic system of linear differential equations admits only a finite number of linearly independent solutions although it is not a sufficient condition for holonomicity. A *holonomic function* is by definition a function which satisfies a holonomic system.

The importance of the holonomicity lies in, in addition to the finiteness property above, the fact that it is preserved under basic operations on functions such as sum, product, restriction and integration. Hence starting from some basic holonomic functions we can construct various holonomic functions by using such operations.

As one of basic holonomic functions, let us consider f^λ with a non-constant polynomial f in $x = (x_1, \ldots, x_n)$ and a complex number λ. Then the function f^λ is holonomic and there are algorithms to compute its annihilator strictly ([10],[3],[16]).

Our purpose is to give an algorithm to compute the annihilator of $f^\lambda(\log f)^m$ with a positive integer m, or more generally a function of the form

$$u = e^h f^\lambda (g_0 + g_1 \log f + \cdots + g_m (\log f)^m)$$

with polynomials $f, h, g_0, \ldots, g_m$ in x, precisely and to prove that u is a holonomic function. This is achieved by differentiation with respect to the parameter s of the annihilator of f^s in $D_n[s]$. This method can be extended to functions of the form $f_1^{\lambda_1} \cdots f_N^{\lambda_N} (\log f_1)^{m_1} \cdots (\log f_N)^{m_N}$ for polynomials f_k, complex numbers λ_k and nonnegative integers m_k.

Since the algorithm yields a holonomic system, we can apply the integration algorithm for D-modules (see [12], [16], [11]) to get a holonomic system for the integral of a function involving the logarithm of a polynomial.

2 Annihilator with a parameter

Let f be a non-constant polynomial in n variables $x = (x_1, \ldots, x_n)$ with coefficients in the field $\mathbb{C}$ of the complex numbers. From an algorithmic viewpoint, we assume that the coefficients of f belong to a computable subfield of $\mathbb{C}$.

First, we consider formal functions of the form $f^s(\log f)^k$ with an indeterminate s. More precisely, for a non-negative integer m, we introduce the module

$$\mathcal{L}(f, m) := \bigoplus_{k=0}^{m} \mathbb{C}[x, f^{-1}, s] f^s (\log f)^k,$$

of which $f^s(\log f)^k$ are regarded as a free basis over the ring $\mathbb{C}[x, f^{-1}, s]$. Then $\mathcal{L}(f, m)$ has a natural structure of left $D_n[s]$-module, which is induced by the action of the derivation $\partial_j = \partial/\partial x_j$ defined by, for $a \in \mathbb{C}[x, f^{-1}, s]$,

$$\partial_j\{a f^s (\log f)^k\} = \left(\frac{\partial a}{\partial x_j} + s a f^{-1} \frac{\partial f}{\partial x_j}\right) f^s (\log f)^k + k a f^{-1} \frac{\partial f}{\partial x_j} f^s (\log f)^{k-1} \quad (j = 1, \ldots, n)$$

if $k \geq 1$ and

$$\partial_j(af^s) = \left(\frac{\partial a}{\partial x_j} + saf^{-1}\frac{\partial f}{\partial x_j}\right) f^s \quad (j = 1, \ldots, n).$$

In view of this action, it is easy to see that $\mathcal{L}(f, m)/\mathcal{L}(f, m-1)$ is isomorphic to $\mathcal{L}(f, 0) = \mathbb{C}[x, f^{-1}, s]f^s$ as a left $D_n[s]$-module.

Consider the left $D_n[s]$-submodule

$$\mathcal{N}(f, m) := D_n[s]f^s + \cdots + D_n[s](f^s(\log f)^m)$$

of $\mathcal{L}(f, m)$. Our purpose is to determine the annihilating module

$$\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m) := \Big\{P = (P_0, P_1, \ldots, P_m) \in D_n[s]^{m+1} \mid \sum_{k=0}^{m} P_k(f^s(\log f)^k) = 0\Big\}$$

and the annihilating ideal

$$\mathrm{Ann}_{D_n[s]} f^s(\log f)^m := \{P \in D_n[s] \mid P(f^s(\log f)^m) = 0\}.$$

Note that there are isomorphisms

$$\mathcal{N}(f, m) \simeq D_n[s]^{m+1}/\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m),$$
$$D_n[s](f^s(\log f)^m) \simeq D_n[s]/\mathrm{Ann}_{D_n[s]}(f^s(\log f)^m).$$

Now let us regard $f^s(\log f)^m$ as a multi-valued analytic function in (x, s) on $\{(x, s) \in \mathbb{C}^{n+1} \mid f(x) \neq 0\}$. The following lemma is well-known; see, e.g., [4] for a differential-algebraic proof.

Lemma 1 *Let $f \in \mathbb{C}[x]$ be a non-constant polynomial. Then for $a_i(x) \in \mathbb{C}[x]$,*

$$\sum_{i=0}^{m} a_i(x)(\log f)^i = 0$$

holds as analytic function if and only if $a_i(x) = 0$ for all i.

Proof: We argue by induction on m. In view of the uniqueness of analytic continuation, we have only to show that each $a_i(x)$ vanishes near x_0. Let $x_0 \in \mathbb{C}^n$ be a non-singular point of the hypersurface $f(x) = 0$. By an analytic local coordinate transformation, we may suppose that $a_i(x)$ are analytic near $x_0 = 0$ and $f(x) = x_1$. That is,

$$a_0(x) + a_1(x)\log x_1 + \cdots + a_m(x)(\log x_1)^m = 0 \tag{2}$$

holds on a neighborhood U of 0. Fix a point $x = (x_1, \ldots, x_n)$ in U such that $x_1 \neq 0$. By analytic continuation along a circle $(e^{\sqrt{-1}t}x_1, x_2, \ldots, x_n)$ with $0 \leq t \leq 2\pi$, the identity (2) is transformed to

$$a_0(x) + a_1(x)(\log x_1 + 2\pi\sqrt{-1}) + \cdots + a_m(x)(\log x_1 + 2\pi\sqrt{-1})^m = 0.$$

By subtraction, we get an identity of the form

$$b_0(x) + b_1(x)\log x_1 + \cdots b_{m-1}(x)(\log x_1)^{m-1} = 0$$

with

$$b_{m-1}(x) = 2m\pi\sqrt{-1}a_m(x).$$

From the induction hypothesis it follows that $b_0(x) = \cdots = b_{m-1}(x) = 0$, which implies $a_m(x) = 0$. We are done by induction on m. □

3 Computation of annihilator

Now let us describe an algorithm for computing the annihilator of $f^s(\log f)^m$.

Algorithm 1 Input: a non-constant polynomial f in the variables $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of $\mathbb{C}$, and a non-negative integer m.

1. Let $G = \{P_1(s), \dots, P_k(s)\}$ be a generating set of the left ideal $\mathrm{Ann}_{D_n[s]} f^s := \{P(s) \in D_n[s] \mid P(s)f^s = 0\}$ by using an algorithm of [10] or [3] (see also [6]).
2. Let
$$e_0 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)$$
be the canonical unit vectors of $\mathbb{C}^{m+1}$. For each $i = 1, \dots, k$ and $j = 0, 1, \dots, m$, set
$$P_i(s)^{(j)} := \sum_{\nu=0}^{j} \binom{j}{\nu} \frac{\partial^{j-\nu} P_i(s)}{\partial s^{j-\nu}} e_\nu.$$

Output: $G' := \{P_i(s)^{(j)} \mid 1 \le i \le k,\ 0 \le j \le m\}$ generates $\mathrm{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m)$.

Algorithm 2 Input: a non-constant polynomial f in the variables $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of $\mathbb{C}$, and a non-negative integer m.

1. Let G' be the output of Algorithm 1.
2. Compute a Gröbner basis G'' of the module generated by G' with respect to a term order $\prec$ for $(D_n[s])^{m+1}$ such that $Me_j \prec M'e_k$ for any monomials M and M' if $k < j$. Let G_0 be the set of the last component of each element of G''.

Output: G_0 generates $\mathrm{Ann}_{D_n[s]} f^s(\log f)^m$.

Lemma 2 *Let I be a left ideal of $D_n[s]$ generated by $\{P_1(s), \dots, P_k(s)\}$. For $P(s) \in D_n[s]$ and $j \in \mathbb{N}$, set*

$$P(s)^{(j)} := \sum_{\nu=0}^{j} \binom{j}{\nu} \frac{\partial^{j-\nu} P(s)}{\partial s^{j-\nu}} e_\nu.$$

Then the left $D_n[s]$-submodule of $(D_n[s])^{m+1}$ which is generated by $\{P(s)^{(j)} \mid P(s) \in I,\ 0 \le j \le m\}$ coincides with the one which is generated by $\{P_i(s)^{(j)} \mid 1 \le i \le k,\ 0 \le j \le m\}$ for any integer $m \ge 0$.

Proof: Let $\mathcal{N}$ be the left $D_n[s]$-module generated by $\{P_i(s)^{(j)} \mid 1 \leq i \leq k,\ 0 \leq j \leq m\}$ and $P(s)$ be a nonzero element of I. Then there exist

$$Q_i(s) = \sum_{l=0}^{m_i} Q_{il} s^l \quad (Q_{il} \in D_n)$$

such that $P(s) = \sum_{i=1}^{k} Q_i(s) P_i(s)$. Then we have

$$P(s)^{(j)} = \sum_{i=1}^{k} \sum_{l=0}^{m_i} Q_{il} (s^l P_i(s))^{(j)}.$$

Hence we have only to show that $(s^l P_i(s))^{(j)}$ belongs to $\mathcal{N}$. This can be done as follows:

$$\begin{aligned}
(s^l P_i(s))^{(j)} &= \sum_{\nu=0}^{j} \binom{j}{\nu} \left(\frac{\partial}{\partial s}\right)^{j-\nu} (s^l P_i(s)) e_\nu \\
&= \sum_{\nu=0}^{j} \binom{j}{\nu} \sum_{\mu=0}^{\min\{j-\nu, l\}} \binom{j-\nu}{\mu} (l)_\mu s^{l-\mu} \\
&\quad \times \left(\frac{\partial}{\partial s}\right)^{j-\nu-\mu} P_i(s) e_\nu \\
&= \sum_{\mu=0}^{\min\{j,l\}} \binom{j}{\mu} (l)_\mu s^{l-\mu} \sum_{\nu=0}^{j-\mu} \binom{j-\mu}{\nu} \\
&\quad \times \left(\frac{\partial}{\partial s}\right)^{j-\mu-\nu} P_i(s) e_\nu \\
&= \sum_{\mu=0}^{\min\{j,l\}} \binom{j}{\mu} (l)_\mu s^{l-\mu} P_i(s)^{(j-\mu)},
\end{aligned}$$

where $(l)_\mu := l(l-1)\cdots(l-\mu+1)$. □

Theorem 1 *The output of Algorithm 1 coincides with* $\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m)$.

Proof: Let $P(s)$ belong to $\mathrm{Ann}_{D_n[s]} f^s$. Differentiating the equation $P(s) f^s = 0$ with respect to s, we get

$$\sum_{\nu=0}^{j} \binom{j}{\nu} \frac{\partial^{j-\nu} P_i(s)}{\partial s^{j-\nu}} (f^s (\log f)^\nu) = 0$$

for $0 \leq j \leq m$. This shows that each $P_i(s)^{(j)}$ annihilates $(f^s, \ldots, f^s(\log f)^m)$.

Set $\mathcal{M} := \mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m)$. Let $\mathcal{N}$ be the left $D_n[s]$-module generated by the output G' of Algorithm 1. The argument above shows that $\mathcal{N}$ is a left D_n-submodule of $\mathcal{M}$. Hence we have only to prove $\mathcal{N} = \mathcal{M}$. For this purpose let $\mathcal{N}_j$ be the left $D_n[s]$-module generated by $\{P_i(s)^{(\nu)} \mid 1 \leq i \leq k,\ 0 \leq \nu \leq j\}$ and set

$$\mathcal{M}_j := \{(Q_0, Q_1, \ldots, Q_m) \in \mathcal{M} \mid Q_\nu = 0 \text{ if } \nu > j\}.$$

Let $Q(s) = (Q_0(s), \ldots, Q_j(s), 0, \ldots, 0)$ be an element of $\mathcal{M}_j$. Then

$$\sum_{\nu=0}^{j} Q_\nu(s)(f^s(\log f)^\nu) = 0$$

holds. In view of the action of $D_n[s]$ on $\mathcal{L}(f, j)$ noted in Section 2, this implies $Q_j(s)f^s = 0$. Hence $Q_j(s)^{(j)}$ belongs to $\mathcal{N}_j$ by Lemma 2. It is easy to see that $Q(s) - Q_j(s)^{(j)}$ belongs to $\mathcal{M}_{j-1}$. This means $\mathcal{M}_j = \mathcal{N}_j + \mathcal{M}_{j-1}$ for $1 \leq j \leq m$. Then we can show that $\mathcal{N}_j = \mathcal{M}_j$ holds for $1 \leq j \leq m$ by induction on j noting $\mathcal{N}_0 = \mathcal{M}_0$. □

The correctness of Algorithm 2 follows immediately.

Remark 1 If f is weighted homogeneous, i.e., if there exist rational numbers w_i such that $\sum_{i=1}^n w_i x_i \partial_i(f) = f$, then $D_n[s]f^s(\log f)^m$ is isomorphic to $\mathcal{N}(f, m)$ as left $D_n[s]$-module. Namely, the homomorphism of D_n to $(D_n)^{m+1}$ which sends 1 to e_m induces an isomorphism

$$D_n[s]/\mathrm{Ann}_{D_n[s]} f^s(\log f)^m \simeq (D_n[s])^{m+1}/\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m)$$

of left $D_n[s]$-module. In fact, this follows from the relations

$$\left(\sum_{i=1}^n w_i x_i \partial_i - s\right)(f^s(\log f)^k) = k f^s(\log f)^{k-1} \qquad (k \geq 1).$$

4 Specialization of parameter

Let us fix a complex number λ. (From an algorithmic viewpoint, we assume λ lies in a computable subfield of the field $\mathbb{C}$.) We set

$$\mathcal{L}(f, m, \lambda) := \sum_{k=0}^{m} \mathbb{C}[x, f^{-1}] f^\lambda (\log f)^k,$$

where $f^\lambda(\log f)^k$ constitute a free basis over $\mathbb{C}[x, f^{-1}]$ in view of Lemma 1. Substituting λ for s gives $\mathcal{L}(f, m, \lambda)$ a natural structure of left D_n-module. In fact, one has

$$\partial_j\{af^\lambda(\log f)^k\} = \left(\frac{\partial a}{\partial x_j} + \lambda a f^{-1}\frac{\partial f}{\partial x_j}\right) f^\lambda(\log f)^k + kaf^{-1}\frac{\partial f}{\partial x_j} f^\lambda(\log f)^{k-1} \quad (j = 1, \ldots, n)$$

for $k \geq 1$ and

$$\partial_j(af^\lambda) = \left(\frac{\partial a}{\partial x_j} + \lambda a f^{-1}\frac{\partial f}{\partial x_j}\right) f^\lambda \quad (j = 1, \ldots, n)$$

with $a \in \mathbb{C}[x, f^{-1}]$. This implies that $\mathcal{L}(f, m, \lambda)/\mathcal{L}(f, m-1, \lambda)$ is isomorphic to $\mathcal{L}(f, 0, \lambda) = D_n f^\lambda$ as a left D_n-module.

Set

$$\mathcal{N}(f,m,\lambda) := D_n f^\lambda + \cdots + D_n(f^\lambda(\log f)^m).$$

We define the annihilators of $(f^\lambda, \ldots, f^\lambda(\log f)^m)$ and of $f^\lambda(\log f)^m$ to be

$$\mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^k) := \{P = (P_0, P_1, \ldots, P_m) \in (D_n)^{m+1} \mid \sum_{k=0}^{m} P_k(f^\lambda(\log f)^k) = 0\},$$

$$\mathrm{Ann}_{D_n} f^\lambda(\log f)^m := \{P \in D_n \mid P(f^\lambda(\log f)^m) = 0\},$$

respectively. Then we have isomorphisms

$$\mathcal{N}(f,m,\lambda) \simeq (D_n)^{m+1}/\mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^k),$$
$$D_n(f^\lambda(\log f)^m) \simeq D_n/\mathrm{Ann}_{D_n}(f^\lambda(\log f)^m).$$

In the sequel, we need information on some roots of the *Bernstein-Sato polynomial* or the *b-function* of f, which is, by definition, the monic polynomial $b_f(s)$ of the least degree such that a formal functional equation

$$P(s)f^{s+1} = b_f(s)f^s \tag{3}$$

holds with some $P(s) \in D_n[s]$. The existence of such a functional equation was proved by Bernstein [1]. It was proved by Kashiwara [5] that the roots of $b_f(s) = 0$ are negative rational numbers. An algorithm to compute $b_f(s)$ and an associated operator $P(s)$ was given in [9]. The following theorem generalizes a result of Kashiwara [5, Proposition 6.2]:

Theorem 2 *Let λ be a complex number such that $b_f(\lambda - \nu) \neq 0$ for any positive integer ν. Then we have*

$$\mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^k) = \{P(\lambda) \mid P(s) \in \mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^k)\},$$

$$\mathrm{Ann}_{D_n} f^\lambda(\log f)^m = \{P(\lambda) \mid P(s) \in \mathrm{Ann}_{D_n[s]} f^s(\log f)^m\}.$$

Proof: We have only to show the first equality. Assume that $\sum_{k=0}^m P_k(f^\lambda(\log f)^k) = 0$ holds with $P_k \in D_n$. Then there exist a non-negative integer $l \geq 0$ and polynomials $a_k(x,s) \in \mathbb{C}[x,s]$ such that

$$\sum_{k=0}^{m} P_k(f^s(\log f)^k) = (s-\lambda)\sum_{k=0}^{m} a_k(x,s) f^{s-l}(\log f)^k.$$

By using the functional equation (3), we can find an operator $Q(s) \in D_n[s]$ such that

$$b_f(s-1)\cdots b_f(s-l)f^{s-l} = Q(s)f^s.$$

In view of the action of $D_n[s]$ on $\mathcal{L}(f, m)$, there exist a polynomial $a'_k(x, s)$ in x, s and a non-negative integer l_1 such that

$$b_f(s-1)\cdots b_f(s-l)f^{s-l}(\log f)^m = Q(s)\{f^s(\log f)^m\} + \sum_{k=0}^{m-1} a'_k(x,s)f^{s-l_1}(\log f)^k.$$

Proceeding inductively, we conclude that there exist a polynomial $b(s) \in \mathbb{C}[s]$ which is a product (possibly with multiplicities) of $b_f(s-j)$ with $j \geq 1$ and operators $\widetilde{Q}_k(s) \in D_n[s]$ such that

$$b(s)\sum_{k=0}^{m} a_k(x,s)f^{s-l}(\log f)^k = \sum_{k=0}^{m} \widetilde{Q}_k(s)\{f^s(\log f)^k\}.$$

Hence

$$\widetilde{P}(s) := b(s)\sum_{k=0}^{m} P_k e_k - (s-\lambda)\sum_{k=0}^{m} \widetilde{Q}_k(s)e_k$$

belongs to $\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^k)$ and

$$b(\lambda)\sum_{k=0}^{m} P_k e_k = \widetilde{P}(\lambda).$$

This completes the proof since $b(\lambda) \neq 0$ by the assumption. $\Box$

If $b_f(\lambda - \nu) = 0$ for some positive integer ν, then set

$$\nu_0 := \max\{\nu \in \mathbb{Z} \mid \nu > 0,\, b_f(\lambda-\nu) = 0\}$$

and $\lambda_0 := \lambda - \nu_0$. Then λ_0 satisfies the condition of Theorem 2. Hence for $(P_0, \ldots, P_m) \in (D_n)^{m+1}$, we have

$$\begin{aligned} &(P_0, \ldots, P_m) \in \mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^k) \\ \Leftrightarrow\quad &(P_0 f^{\nu_0}, \ldots, P_m f^{\nu_0}) \in \mathrm{Ann}_{D_n}(f^{\lambda_0}, \ldots, f^{\lambda_0}(\log f)^k) \\ \Leftrightarrow\quad &(P_0, \ldots, P_m) \in \mathrm{Ann}_{D_n}(f^{\lambda_0}, \ldots, f^{\lambda_0}(\log f)^k) : f^{\nu_0}. \end{aligned}$$

To give an algorithm for the module quotient in a more general setting, let us define the componentwise product of two elements $P = (P_0, \ldots, P_m)$ and $Q = (Q_0, \ldots, Q_m)$ of $(D_n)^{m+1}$ to be $PQ := (P_0Q_0, \ldots, P_mQ_m)$. Let N be a left D_n-submodule of $(D_n)^{m+1}$ and $P = (P_0, \ldots, P_m) \in (D_n)^{m+1}$ with $P_i \neq 0$ for $0 \leq i \leq m$. Then the module quotient $N : P$ is defined to be

$$N : P := \{Q \in (D_n)^{m+1} \mid QP \in N\},$$

which is a left D_n-submodule of $(D_n)^{m+1}$.

Algorithm 3 Input: A set G_1 of generators of a left D_n-submodule N of $(D_n)^{m+1}$ and $P = (P_0, \ldots, P_m) \in (D_n)^{m+1}$ with $P_i \neq 0$ for $0 \leq i \leq m$.

1. Introducing a new variable t, compute a Gröbner basis G_2 of the left $D_n[t]$-module of $(D_n[t])^{m+1}$ which is generated by $\{(1-t)P_k e_k \mid 0 \le k \le m\} \cup \{tQ \mid Q \in G_1\}$ with respect to a term order $\prec$ such that $x^\alpha \partial^\beta e_j \prec te_k$ for any $j,k \in \{0,1,\ldots,m\}$ and $\alpha, \beta \in \mathbb{N}^n$.

2. $G_3 := G_2 \cap (D_n)^{m+1}$.

3. $G_4 := \{Q/P \mid Q \in G_3\}$, where Q/P denotes the element in $(D_n)^{m+1}$ such that $(Q/P)P = Q$ in the sense of componentwise product.

Output: G_4 generates the module quotient $N : P$.

In fact, we can show in the same way as in the commutative case that G_3 generates the left module $N \cap (D_n)^{m+1}P$. In particular, for each $Q \in G_3$, there exists a unique $Q' \in (D_n)^{m+1}$ such that $Q = Q'P$. Let us denote this Q' by Q/P. Then Q' belongs to the quotient module $N : P$. Conversely, if Q' belongs to $N : P$, then $Q'P$ belongs to $N \cap (D_n)^{m+1}P$. Hence Q' belongs to the module generated by G_4. In our experiments, Algorithm 3 outperforms an alternative method based on syzygy computation.

Summed up, the annihilators of $(f^\lambda((\log f)^k)_{0\le k\le m}$ and of $(f^\lambda(\log f)^m$ are computed as follows:

Algorithm 4 Input: a non-constant polynomial f in the variables $x = (x_1, \ldots, x_n)$ with coefficients in a computable subfield of $\mathbb{C}$, a number λ which belongs to a computable subfield of $\mathbb{C}$, and a non-negative integer m.

1. Compute a set G_1 of generators of
$$\mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s((\log f)^k)$$
by Algorithm 1.

2. Compute $b_f(s)$ ([9], [10], [3]) and let ν_0 be the largest positive integer ν such that $b_f(\lambda-\nu) = 0$ if there is any such ν. Set $\nu_0 = 0$ if none.

3. Set $\lambda_0 := \lambda - \nu_0$ and $G_2 := G_1|_{s=\lambda_0}$ (substitute λ_0 for s in each element of G_1).

4. If $\nu_0 > 0$, then let G_3 be a set of generators of the module quotient $\langle G_2\rangle : f^{\nu_0} = \langle G_2\rangle : (f^{\nu_0}, \ldots, f^{\nu_0})$, where $\langle G_2\rangle$ denotes the left module generated by G_2.

5. If $\nu_0 = 0$, then set $G_3 := G_2$.

6. Compute a Gröbner basis G_4 of the module generated by G_3 with respect to a term order $\prec$ for $(D_n)^{m+1}$ such that $Me_j \prec M'e_k$ for any monomials M and M' if $k < j$. Let G_5 be the set of the last component of each element of G_4.

Output: G_3 generates $\mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^m)$; G_5 generates $\mathrm{Ann}_{D_n} f^\lambda(\log f)^m$.

Remark 2 In the step 2 of Algorithm 4, we need information only on the roots of the b-function of the form $\lambda - \nu$ with $\nu \in \mathbb{N}$. Hence without computing $b_f(s)$ itself, one can employ a method of [6] to check if a given number is a root of the b-function by virtue of the fact that the roots of $b_f(s)$ are greater than $-n$ ([15]).

Algorithm 5 Input: a non-constant polynomial f and polynomials $g_0, \dots, g_m, h$ in $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of $\mathbb{C}$, a number λ which belongs to a computable subfield of $\mathbb{C}$, and $m \in \mathbb{N}$.

1. Compute a set of generators of

$$N := \mathrm{Ann}_{D_n}(f^\lambda, \dots, f^\lambda (\log f)^m)$$

   by using Algorithm 4.

2. Compute a set G_1 of generators of the left ideal $I := \{P \in D_n \mid P(g_0, \dots, g_m) \in N\}$ by a module quotient computation similar to Algorithm 3.

3. Set $G_2 := \{e^h P e^{-h} \mid P \in G_1\}$, where $e^h P e^{-h} \in D_n$ is obtained by substituting $\partial_i - \partial h/\partial x_i$ for ∂_i $(1 \leq i \leq n)$ in P.

Output: G_2 generates $\mathrm{Ann}_{D_n} u$ with

$$u := e^h f^\lambda (g_0 + g_1 \log f + \cdots + g_m (\log f)^m).$$

Theorem 3 *In the preceding algorithm, $D_n u = D_n/I$ is holonomic.*

Proof: Since $D_n \ni P \mapsto e^h P e^{-h}$ induces an isomorphism of the characteristic varieties (see Appendix A), we may assume $h = 0$. It is well-known that $D_n f^\lambda$ is holonomic ([1],[2]). Since $\mathcal{L}(f, m, \lambda)/\mathcal{L}(f, m-1, \lambda)$ is isomorphic to $\mathcal{L}(f, 0, \lambda) = D_n f^\lambda$ as left D_n-module, $\mathcal{L}(f, m, \lambda)$ is also holonomic by induction on m. Hence $D_n u$ is holonomic as a submodule of $\mathcal{L}(f, m, \lambda)$. □

5 Logarithm as distribution

So far, we have studied $f^\lambda (\log f)^m$ as a multi-valued analytic function outside $f = 0$. Hence, in the real domain $\mathbb{R}^n$, the computation in the preceding section is valid only on $U_f := \{x \in \mathbb{R}^n \mid f(x) > 0\}$. Otherwise, we can regard it as a distribution (a generalized function of L. Schwartz) $f_+^\lambda (\log f_+)^m$ defined on the whole $\mathbb{R}^n$ through the 'paring'

$$\langle f_+^\lambda (\log f_+)^m, \varphi \rangle = \int_{U_f} f(x)^\lambda (\log f(x))^m \varphi(x)\, dx$$

with an arbitrary C^∞ function φ which satisfies

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \quad (\forall \alpha, \beta \in \mathbb{N}^n).$$

The integral above is well-defined only for $\mathrm{Re}\, \lambda \geq 0$, but as a distribution it can be analytically continued to $\mathbb{C}$ with respect to λ with possible poles. It is easy to show the following:

Proposition 1 *If $P(s) \in D_n[s]$ annihilates $f^s (\log f)^m$, then for $\lambda \in \mathbb{C}$, $P(\lambda)$ annihilates $f_+^\lambda (\log f_+)^m$ as long as f_+^λ is well-defined as distribution. Moreover, $D_n f_+^\lambda (\log f_+)^m$ is holonomic.*

6 Implementation and examples

By using the algorithms presented so far, we can compute the annihilator of $f^\lambda(\log f)^m$ following the diagram

$$\begin{array}{ccc} \mathrm{Ann}_{D_n[s]} f^s & & \\ \downarrow \text{(a)} & & \\ \mathrm{Ann}_{D_n[s]}(f^s, \ldots, f^s(\log f)^m) & \overset{\text{(c1)}}{\to} & \mathrm{Ann}_{D_n[s]} f^s(\log f)^m \\ \downarrow \text{(b1)} & & \downarrow \text{(c2)} \\ \mathrm{Ann}_{D_n}(f^\lambda, \ldots, f^\lambda(\log f)^m) & \overset{\text{(b2)}}{\to} & \mathrm{Ann}_{D_n} f^\lambda(\log f)^m \end{array}$$

We have implemented the algorithms in a computer algebra system Risa/Asir [7], which is capable of Gröbner basis computation of modules over the ring of differential operators as well as over the ring of polynomials. The algorithm as a whole largely depends on the computation of $\mathrm{Ann}_{D_n[s]} f^s$ and (some roots of) the b-function. Once they are computed, the rest of the computation consists of elimination ((b2) or (c1)) and quotient ((b1) or (c2)) computation for a module over, or an ideal of, the Weyl algebra, which can be the heavier part if m is large. Note that step (a) is straightforward. There are two paths to reach $\mathrm{Ann}_{D_n} f^\lambda(\log f)^m$: (a)→(b1)→(b2) or (a)→(c1)→(c2). Which path is the more efficient seems to depend on the input.

Example 1 Let f be a square-free polynomial in one variable x with complex coefficients. Since $\mathrm{Ann}_{D_1[s]} f^s$ is generated by $f\partial - sf'$, $\mathrm{Ann}_{D_1[s]}(f^s, \ldots, f^s(\log f)^m)$ is generated by $m+1$ elements

$$(f\partial_x - sf', 0, \cdots, 0), \quad (-f', f\partial_x - sf', 0, \cdots, 0), \quad \cdots, \quad (0, \cdots, 0, -mf', f\partial_x - sf')$$

with $\partial_x = d/dx$ and $f' = df/dx$.

Since $b_f(s) = s+1$, the substitution $s = \lambda$ gives generators of $\mathrm{Ann}_{D_1}(f^\lambda, \ldots, f^\lambda(\log f)^m)$ if $\lambda \neq 0, 1, 2, \ldots$. In particular,

$$\mathrm{Ann}_{D_1}(f^{-1}, \ldots, f^{-1}(\log f)^m)$$

is generated by

$$(f\partial_x + f', 0, \cdots, 0), \quad (-f', f\partial_x + f', 0, \cdots, 0), \quad \cdots, \quad (0, \cdots, 0, -mf', f\partial_x + f').$$

Since f and f' are relatively prime, it is easy to see that

$$\mathrm{Ann}_{D_1}(1, \ldots, (\log f)^m) = \mathrm{Ann}_{D_1}(f^{-1}, \ldots, f^{-1}(\log f)^m) : f$$

is generated by

$$(\partial_x, 0, \cdots, 0), \quad (-f', f\partial_x, 0, \cdots, 0), \quad \cdots, \quad (0, \cdots, 0, -mf', f\partial_x).$$

Explicit generators of $\mathrm{Ann}_{D_1}(\log f)^m$ for $m \geq 1$ would be complicated: For example, if $f = x^2+1$ and $m = 2$, step (b2) gives three generators

$$\begin{aligned}
P_1 &= x^2(x^2+1)^2\partial_x^3 + (3x^5-3x)\partial_x^2 + (x^4+3)\partial_x, \\
P_2 &= x(x^2+1)^2\partial_x^4 + (9x^4+8x^2-1)\partial_x^3 + 16x^3\partial_x^2 + 4x^2\partial_x, \\
P_3 &= (x^2+1)^2\partial_x^5 + 14x(x^2+1)\partial_x^4 + (52x^2+16)\partial_x^3 \\
&\quad + 52x\partial_x^2 + 8\partial_x
\end{aligned}$$

of $\mathrm{Ann}_{D_1}(\log f)^2$, which is not generated by a single element. If one works in the ring of differential operators with rational function coefficients, then the annihilator of $(\log f)^2$ is generated by P_1 since we have

$$xP_2 = \partial_x P_1, \qquad x^3P_3 = (x\partial_x^2 - \partial_x)P_1.$$

Example 2 Set

$$f = xy^2 + z^2$$

with $n = 3$, $(x_1, x_2, x_3) = (x, y, z)$, $\partial_x = \partial/\partial x$ and so on. First $\mathrm{Ann}_{D_3[s]} f^s$ is generated by

$$\begin{aligned}
&y\partial_y + z\partial_z - 2s, && -2x\partial_x + y\partial_y, \\
&2z\partial_x - y^2\partial_z, && z\partial_y - xy\partial_z,
\end{aligned}$$

from which we obtain a set of generators of

$$\mathrm{Ann}_{D_3[s]}(f^s, f^s\log f, f^s(\log f)^2).$$

Since the Bernstein-Sato polynomial of f is $b_f(s) = (s+1)^2(2s+3)$, the substitution $s = -1$ gives a set of generators of $\mathrm{Ann}_{D_3}(f^{-1}, f^{-1}\log f, f^{-1}(\log f)^2)$. Then by module quotient computation and elimination in the free module $(D_3)^3$, we get a set of generators

$$\begin{aligned}
&2x\partial_x - y\partial_y, \quad 2z\partial_x - y^2\partial_z, \quad z\partial_y - yx\partial_z, \\
&y\partial_y^3 + (2z\partial_z+3)\partial_y^2 + zx\partial_z^3 + x\partial_z^2
\end{aligned}$$

of $\mathrm{Ann}_{D_3}(\log f)^2$.

The table below shows timing data for the computation of the annihilator of $\mathrm{Ann}_{D_n}(\log f)^m$ measured on a computer equipped with 1.7 GHz Intel Core i5 processor and 4 GB memory. The computation is done along the path (a)→(b1)→(b2), in which (b2) is the most time-consuming part.

f	$m=2$	$m=4$	$m=8$	$m=16$
xy^2+z^2	0.02s	0.04s	0.14s	2.1s
xy^2+z^2+1	0.04s	0.31s	20.8s	–
$x^3+xy^2+z^2$	0.04s	0.12s	1.6s	586s

7 Examples of integrals

Example 3 Consider the integral

$$u(t) := \int_{-\infty}^{\infty} e^{-tx^2+x} (\log(x^2+1))^2 \, dx$$

for $t > 0$. It is easy to compute the annihilator of the integrand from that of $(\log(x^2+1))^2$, which is generated by P_1, P_2, P_3 of Example 1. Then executing the D-module theoretic integration algorithm (Algorithm 6 in Appendix B) by using a library file '`nk_restriction`' of Risa/Asir, we get a differential equation $Pu(t) = 0$ with a differential operator P of order 7. If we use only P_1 as annihilator of $(\log(x^2+1))^2$, then we get $Qu(t) = 0$ with a differential operator Q of order 9. The equation $Qu(t) = 0$ is weaker than $Pu(t) = 0$. This shows an advantage of working with differential operators with polynomial coefficients not with rational function coefficients.

An alternative way to compute this integral is to first compute differential equations for the integral

$$v(s,t) := \int_{-\infty}^{\infty} e^{-tx^2+x} (x^2+1)^s \, dx$$

with a parameter s. By using an algorithm described in [11], we get $P(s)v(s,t) = 0$ with

$$\begin{aligned} P(s) = 4t^2\partial_t^3 + (-8t^2 + (8s+18)t + 1)\partial_t^2 \\ + (4t^2 + (-8s-20)t + 4s^2 + 14s + 10)\partial_t + 2t - 2s - 3. \end{aligned}$$

Differentiation with respect to s and substitution $s = 0$ yield

$$\begin{aligned} P(0)v(0,t) = P'(0)v(0,t) + P(0)v'(0,t) \\ = P''(0)v(0,t) + 2P'(0)v'(0,t) + P(0)v''(0,t) = 0, \end{aligned}$$

where $'$ denotes differentiation with respect to s. Then by eliminating $v(0,t)$ and $v'(0,t)$, we get a differential equation $Rv''(t,0) = Ru(t) = 0$ with a differential operator R of order 9, which is weaker than $Pu(t) = 0$ above but essentially different from $Qu(t) = 0$.

Example 4 Set

$$u(x) := \int_{\mathbb{R}^2} e^{-y^2-z^2} (\log(xy^2 + z^2 + 1))^2 \, dydz$$

for $x > 0$. Then by the integration algorithm, we get a differential equation $Pu(x) = 0$ with

$$\begin{aligned} P = & 16x^4(x-1)^2\partial_x^7 + 16x^2(x-1)(29x^2 - 17x - 2)\partial_x^6 \\ & + (4504x^4 - 5336x^3 + 896x^2 + 240x + 16)\partial_x^5 \\ & + (17712x^3 - 14220x^2 + 540x + 288)\partial_x^4 \\ & + (27153x^2 - 12348x - 441)\partial_x^3 \\ & + (12915x - 2205)\partial_x^2 + 945\partial_x. \end{aligned}$$

The computation of the integral takes about 4.3 seconds.

Example 5 Set

$$u(x) := \int_{\mathbb{R}^2} e^{-y^2-z^2} (\log(xy^2 + z^2))^2 \, dydz$$

for $x > 0$. Since $f := xy^2 + z^2$ vanishes if $y = z = 0$, we must regard $(\log f)^2$ as a distribution on $\mathbb{R}^2$ with respect to (y, z) with a parameter x. A holonomic system for $(\log f)^2$ regarded as such is obtained by the substitution $s = 0$ from the annihilator of $f^s(\log f)^2$ in $D_3[s]$, which is weaker than the annihilator of $(\log f)^2$ as analytic function. From this we get a differential equation $Pu(x) = 0$ with

$$\begin{aligned} P = {} & 8x^3(x-1)^3\partial_x^6 + 12x^2(x-1)^2(13x-7)\partial_x^5 \\ & + (926x^4 - 1926x^3 + 1218x^2 - 218x)\partial_x^4 \\ & + (1911x^3 - 3107x^2 + 1369x - 125)\partial_x^3 \\ & + (1155x^2 - 1360x + 325)\partial_x^2 + (105x - 75)\partial_x. \end{aligned}$$

The computation of the integral takes about 0.51 seconds.

Example 6 Let us consider the integral

$$u(x) := \int_0^1 (\log(x^2 + xy + y^2))^2 \, dy$$

for $x \neq 0$. Following an algorithm presented in [11], we rewrite the integral as

$$u(x) = \int_{-\infty}^{\infty} Y(y)Y(1-y)(\log(x^2 + xy + y^2))^2 \, dy$$

by using the Heaviside function $Y(t)$, which takes the value 1 if $t > 0$ and 0 otherwise. Then by the integration algorithm for D-modules we get $Pu(x) = 0$ with

$$\begin{aligned} P = {} & x^2(x^2+x+1)^2 \\ & \times (8x^7 + 24x^6 + 56x^5 + 76x^4 + 86x^3 + 43x^2 + 6x - 2)\partial_x^5 \\ & + (\text{terms of order} \leq 4). \end{aligned}$$

The whole computation takes about 2.0 seconds.

A Holonomic systems

Let us recall the definition of a holonomic system introduced by [17] (see also [2]). Let P be a nonzero differential operator written in the form (1) and define its order by

$$m := \mathrm{ord}(P) := \max\{|\beta| \mid a_{\alpha,\beta} \neq 0 \, (\exists \alpha)\}.$$

Then the *principal symbol* of P is the polynomial defined by

$$\sigma(P)(x, \xi) = \sum_{|\beta|=m} \sum_{\alpha} a_{\alpha,\beta} x^\alpha \xi^\beta,$$

where $\xi = (\xi_1, \ldots, \xi_n)$ are the commutative variables corresponding to the derivations $\partial = (\partial_1, \ldots, \partial_n)$.

Set $M := D_n/I$ with a left ideal I of D_n. The left D_n-module M represents a system

$$Pu = 0 \quad (\forall P \in I)$$

of differential equations for an unknown function u. The *characteristic variety* of M is defined to be the algebraic set

$$\mathrm{Char}(M) := \{(x, \xi) \in \mathbb{C}^{2n} \mid \sigma(P)(x, \xi) = 0 \quad \text{for any } P \in I \setminus \{0\}\}$$

of $\mathbb{C}^{2n}$. It was proved in [17] that the dimension of every irreducible component of $\mathrm{Char}(M)$ is not less than n unless $M = 0$. Hence M is called *holonomic* if and only if the dimension of $\mathrm{Char}(M)$ is n or else $M = 0$. The characteristic variety can be computed via a Gröbner basis with respect to a term order which is compatible with the $(\mathbf{0}, \mathbf{1})$-weight with respect to (x, ∂) (cf. [8]). Once the defining ideal $\sigma(I) \subset \mathbb{C}[x, \xi]$ of the characteristic variety is computed, its dimension is known by the Hilbert function of $\mathbb{C}[x, \xi]/\sigma(I)$.

Holonomicity is preserved under operations such as sum, product, restriction to affine subvarieties, and integration with respect to some of the variables (cf. [1], [2]) and they are computable (see [13], [16], [11]). Let $R_n := \mathbb{C}(x)\langle \partial_1, \ldots, \partial_n \rangle$ be the ring of differential operators with *rational function coefficients*. A D_n-module M is said to be of finite rank and the dimension is called the rank of M if $R_n M$ is a finite dimensional vector space over $\mathbb{C}(x)$. A holonomic D_n-module M is of finite rank but the converse is not true in general.

Example 7 Set $f = xy^2 + z^2$ and consider the function f^{-1}. It is easy to see that the operators

$$\begin{aligned} f\partial_x + f_x &= (xy^2 + z^2)\partial_x + y^2, \\ f\partial_y + f_y &= (xy^2 + z^2)\partial_y + 2xy, \\ f\partial_z + f_z &= (xy^2 + z^2)\partial_z + 2z \end{aligned}$$

annihilate f^{-1} with $f_x = \partial f/\partial x$, etc. The left ideal J generated by these three operators coincides with the annihilator of f^{-1} in R_3 but D_3/J is not holonomic since the characteristic variety of D_3/J is

$$\mathrm{Char}(D_3/J) = \{(x, y, z; \xi, \eta, \zeta) \in \mathbb{C}^6 \mid xy^2 + z^2 = 0\} \cup \{\xi = \eta = \zeta = 0\},$$

which is of codimension one. The true annihilator I of f^{-1} is generated by

$$\begin{aligned} &y\partial_y + z\partial_z + 2, \quad -2x\partial_x + y\partial_y, \\ &2z\partial_x - y^2 x\partial_z, \quad z\partial_y - xy\partial_z, \end{aligned}$$

and the characteristic variety of D_3/I is given by

$$\begin{aligned}\mathrm{Char}(D_3/I) &= \{x = y = z = 0\} \cup \{y = z = \xi = 0\} \\ &\cup \{\xi = \eta = \zeta = 0\} \cup \{y^2x + z^2 = z\eta - yx\zeta = 2z\xi - y^2\zeta \\ &= \eta^2 + x\zeta^2 = y\eta + z\zeta = 2x\xi + z\zeta = \eta^2 + x\zeta^2 = 0\},\end{aligned}$$

which implies that D_3/I is holonomic.

There is an algorithm for a given D_n-module M of finite rank to construct a holonomic D_n-module $\widetilde{M}$ and a surjective D_n-homomorphism of M to $\widetilde{M}$ ([14],[18]). In particular, if the annihilator $I = \mathrm{Ann}_{R_n} u$ of an analytic function u in R_n is known and R_n/I is of finite dimension over $\mathbb{C}(x)$, then the annihilator $\mathrm{Ann}_{D_n} u$ is computable by using the 'Weyl closure algorithm' of [18]. For example, one can compute the annihilator $\mathrm{Ann}_{D_n} u$ of $u := e^{f/g}$ for arbitrary polynomials f, g starting from a finite rank system

$$(g^2\partial_i - f_i g + f g_i)u = 0 \quad (1 \le i \le n)$$

with $f_i = \partial f/\partial x_i$, $g_i = \partial g/\partial x_i$, which constitutes the annihilator of u in R_n.

B Integration algorithm

We shall briefly recall the D-module theoretic integration algorithm. Let I be a left ideal of D_{n+d} which annihilates a function $u(x, t)$ in $(x, t) = (x_1, \dots, x_n, t_1, \dots, t_d)$, where D_{n+d} denotes the Weyl algebra in (x, t). (We keep the notation D_n for the Weyl algebra in x.) Suppose that the integral

$$v(x) := \int_{\mathbb{R}^d} u(x, t)\, dt_1 \cdots dt_d$$

is well-defined for $x \in U$ with some open set U of $\mathbb{R}^n$. In the sequel, we use the notation $\partial_{x_i} = \partial/\partial x_i$, $\partial_{t_j} = \partial/\partial t_j$ and

$$\partial_x = (\partial_{x_1}, \dots, \partial_{x_n}), \quad \partial_t = (\partial_{t_1}, \dots, \partial_{t_d}).$$

The *integration ideal* of a left ideal I of D_{n+d} is defined to be the left ideal

$$\textstyle\int I\, dt := (\partial_{t_1} D_{n+d} + \cdots + \partial_{t_d} D_{n+d} + I) \cap D_n$$

of D_n. It is easy to see that $Pv(x) = 0$ holds for all $x \in U$ and $P \in \int I\, dt$. Moreover, it follows from the D-module theory (see e.g., [2]) that $D_n/\int I\, dt$ is holonomic if D_{n+d}/I is so. The following algorithm was essentially given in [12], [13], [16]:

Algorithm 6 Input: A set G_0 of generators of a left ideal I of D_{n+d} such that D_{n+d}/I is holonomic.

1. Compute a Gröbner basis G_1 of I with respect to a monomial order which is compatible with the weight vector $w = (0, \dots, 0, 1, \dots, 1; 0, \dots, 0, -1, \dots, -1)$ for the variables $(x, t, \partial_x, \partial_t)$.

2. Compute the b-function of I with respect to w, which is a nonzero univariate polynomial $b(s)$ of the minimum degree such that $b(-\partial_{t_1}t_1 - \cdots - \partial_{t_d}t_d) + P$ belongs to I with some $P \in D_{n+d}$ of order ≤ -1 with respect to the weight vector w. This can be done by computing the intersection of the left ideal of D_{n+d} generated by the w-initial parts of G_1, with the subring $\mathbb{C}[\partial_{t_1}t_1 + \cdots + \partial_{t_d}t_d]$.

3. Let k_1 be the maximum integral root of $b(s) = 0$ if any; if there is none or else $k_1 < 0$, then set $G := \{1\}$ and quit.

4. For $P \in G_1$ and $\alpha \in \mathbb{N}^d$ such that $\mathrm{ord}_w(P) + |\alpha| \leq k_1$, one has an expression of the form

$$t^\alpha P = \sum_{j=1}^{d} \partial_{t_j} Q_j + \sum_{|\beta| \leq k_1} R_\beta t^\beta$$

with $Q_j \in D_{n+d}$ and unique $R_\beta \in D_n$. Set $\chi(t^\alpha P) := \sum_{|\beta| \leq k_1} R_\beta t^\beta$. Let N be the left D_n-submodule of $\bigoplus_{|\beta| \leq k_1} D_n t^\beta$ generated by

$$\{\chi(t^\alpha P) \mid P \in G_1,\ |\alpha| + \mathrm{ord}_w(P) \leq k_1\}.$$

5. Compute a set G of generators of the ideal $N \cap D_n$.

Output: G generates $\int I\, dt$ and $D_n / \int I\, dt$ is holonomic.

In this algorithm, steps 1 and 2 often cause a bottleneck. If we use an arbitrary subset G_1 of I instead of the step 1 and an arbitrary integer $k_1 \geq 0$ irrespective of the b-function, then this algorithm outputs a subideal of the integration ideal. This provides us with a heuristic integration algorithm.

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